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Short Communication

Non-linear free vibration of orthotropic circular plates at elevated temperature

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Abstract

Non-linear behaviour of an axisymmetric orthotropic circular plate at elevated temperature for both clamped movable and immovable edges has been studied, using generalised dynamical field equations (in the von Karman sense) derived in terms of displacement components. Relative time-periods for linear and non-linear vibrations are seen to depend on relative amplitudes and thermal loading parameter. Critical buckling temperatures for both the boundary conditions have been obtained in the limiting case and corresponding results for isotropic plates have been compared with known results.

Key words: Non-linear free vibrations, orthotropic circular plates, elevated temperature, critical buckling temperature.

1. Introduction

With the increased use of strong and light-weight structures, especially in aerospace engineering, and in the vibrations of machine parts, many problems of non-linear vibrations arise where complementary stresses in the middle plane of the plate must be taken into account for deriving the governing field equations of the plate.

Extensive studies on the large amplitude (non-linear) vibrations of elastic circular plates have been made by Berger's method [1-4] as well as by von Karman's method [5-7]. Berger's method has some advantages over von Karman's method since it leads to decoupled equations. However, Nowinski and Ohanabe [8] and Prathap [9] have pointed out certain inaccuracies in Berger's equations and in view of this von Karman's method should be resorted to until some alternative theory is set forth.

In the present investigation, non-linear behaviour of an axisymmetric orthotropic circular plate at elevated temperature for both clamped movable and immovable boundary conditions has been studied, using generalised dynamical field equations (in the von Karman sense) derived in terms of displacement components. Relative time-periods for linear and non-linear vibrations are seen to depend on relative amplitudes and thermal loading parameter. Critical buckling temperatures have been deduced in the limiting case and compared with known results for the isotropic plates.

2. *Governing equations*

Considering equilibrium equations of the non-linear theory for the case of an axisymmetric orthotropic circular plate subject to thermal stresses and with notations as in Nowinski [10], the basic governing equations for the dynamical analysis in terms of displacement components can be expressed in the forms

$$r^2 u_{,rr} + r u_{,r} - \frac{C_{22}}{C_{11}} u = 1/2\left(\frac{C_{12}}{C_{11}} - 1\right) r w_{,r}^{\,2} -$$

$$-r^2 w_{,r} w_{,rr} + \frac{\beta_{22} - \beta_{11}}{C_{11} h} N_T r - \frac{\beta_{11} r^2}{C_{11} h} \frac{\mathrm{d} N_T}{\mathrm{d} r} \tag{1}$$

$$\frac{C_{11} h^3}{12} \left(w_{,rrr} + \frac{2}{r} w_{,rr}\right) - \frac{C_{22} h^3}{12}\left(\frac{1}{r^2} w_{,rr} - w_{,r}/r^3\right) - \rho h w_{,tt} =$$

$$= q - 1/r \, \mathrm{d}/\mathrm{d}r \, (r \, \mathrm{d}w/\mathrm{d}r \, N_{rr}) + (\beta_{11} - \beta_{22}) \, N_T + \beta_{11} \, r \, \mathrm{d}M_T/\mathrm{d}r \tag{2}$$

3. Free vibrations

For free vibrations $q = 0$; however, it is not exactly true that $M_T = 0$; it is an assumption based on the neglect of temperature variation in depth due to compression even though Jones *et al* [11] assume $M_T = 0$. For free thermal vibrations, the temperature field should be taken to depend on the radial co-ordinate r as considered by Buckens [12]. Accordingly, M_T disappears from equation (2) and only N_T survives in equation (1).

4. Method of solution

The deflection $w(r,t)$ is expressed in the separable form

$$w(r,t) = A \left[1 + \sum_{i=2,4,\ldots}^{\infty} A_i (r/a)^i \right] F(t) \tag{3}$$

$$\approx A \left[1 + A_2 (r/a)^2 + A_4 (r/a)^4 \right] F(t) \tag{3.1}$$

where A is the maximum deflection at the centre of the plate and the constants A_2 and A_4 must be determined from the boundary conditions.

Considering equations (1) and (3.1) one gets the in-plane displacement $u(r,t)$ finite at the origin, in the form

$$u(r,t) = C_0 r^k + \frac{C_1 r^3}{9-k^2} + \frac{C_2 r^5}{25-k^2} + \frac{C_3 r^7}{49-k^2} + \psi(r) \tag{4}$$

where C_1, C_2 and C_3 are known constants and C_0 is a constant of integration to be determined from in-plane boundary conditions for movable and immovable edges, $\Psi(r)$ is the particular integral for the thermal loading terms on the right-hand side of equation (1) and $k^2 = C_{22}/C_{11}$.

Assuming the temperature distribution $\theta(r,z)$ to depend on the radial co-ordinate in the form[12] given by

$$\theta(r,z) = \tau_0(r) = T_0(1 - r/a),$$

one gets the expressions for N_T *and accordingly* $\psi(r)$ *is determined.*

We now substitute the expression for $w(r,t)$ given by equation (3.1) as well as the required expression for N_{rr} given by

$$N_{rr} = \int_{-h/2}^{h/2} \tau_{rr}\, dz = C_{11} h(u_{,r} + \tfrac{1}{2} w_{,r}^2) + C_{12} h \frac{u}{r} + \beta_{11} N_T \tag{5}$$

into equation (2) and applying Galerkin procedure one arrives, after a lengthy but simple calculation, the following time-differential equation in the form

$$C_{11} h A F(t)[A_2\{2 C_0 a^{k-1}(\frac{1}{k+1} + \frac{A_2}{k+3} + \frac{A_4}{k+5})$$

$$(k+1)(k+\frac{C_{12}}{C_{11}}) + \frac{8\, C_1 a^2}{9-k^2}(\frac{1}{4} + \frac{A_2}{6} + \frac{A_4}{8})(3+\frac{C_{12}}{C_{11}}) + \frac{12\, C_2 a^4}{25-k^2}$$

$$(\frac{1}{6} + \frac{A_2}{8} + \frac{A_4}{10})(5+\frac{C_{12}}{C_{11}}) + \frac{16\, C_3 a^6}{49-k^2}(\frac{1}{8} + \frac{A_2}{10} + \frac{A_4}{12})(7+\frac{C_{12}}{C_{11}}) +$$

$$\frac{4(\beta_{22}-\beta_{11})}{C_{11}} \times \frac{T_0}{1-k^2} \times (\frac{1}{2} + \frac{A_2}{4} + \frac{A_4}{6})(1+\frac{C_{12}}{C_{11}}) - \frac{6(\beta_{22}-2\beta_{11})}{C_{11}(4-k^2)}$$

$$T_0(\frac{1}{3} + \frac{A_2}{5} + \frac{A_4}{7})(2+\frac{C_{12}}{C_{11}})\} + A_4\{4 C_0 a^{k-1}(k+3)\times(k+\frac{C_{12}}{C_{11}})$$

$$\left(\frac{1}{k+3}+\frac{A_2}{k+5}+\frac{A_4}{k+7}\right)+\frac{24\,C_1\,a^2}{9-k^2}\left(\frac{1}{6}+\frac{A_2}{8}+\frac{A_4}{10}\right)\left(3+\frac{C_{12}}{C_{11}}\right)+\frac{32\,C_2\,a^4}{25-k^2}$$

$$\left(5+\frac{C_{12}}{C_{11}}\right)\times\left(\frac{1}{8}+\frac{A_2}{10}+\frac{A_4}{12}\right)+\frac{40\,C_3\,a^6}{49-k^2}\left(\frac{1}{10}+\frac{A_2}{12}+\frac{A_4}{14}\right)\left(7+\frac{C_{12}}{C_{11}}\right)+$$

$$\frac{16\,(\beta_{22}-\beta 11)\,T_0}{C_{11}\,(1-k^2)}\left(1+\frac{C_{12}}{C_{11}}\right)\times\left(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8}\right)-\frac{20\,(\beta_{22}-2\beta_{11})\,T_0}{C_{11}\,(4-k^2)}$$

$$\left(\frac{1}{5}+\frac{A_2}{7}+\frac{A_4}{9}\right)\left(2+\frac{C_{12}}{C_{11}}\right)\}+\frac{\beta_{11}\,T_0}{C_{11}}\left[A_2\{4\left(\frac{1}{2}+\frac{A_2}{4}+\frac{A_4}{6}\right)-6\left(\frac{1}{3}+\frac{A_2}{5}+\right.\right.$$

$$\left.+\frac{A_4}{7}\right)\}+A_4\{16\left(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8}-20\left(\frac{1}{5}+\frac{A_2}{7}+\frac{A_4}{9}\right)\}\right]]+$$

$$+\;C_{11}\,h\,A^3\,F^3(t)\left[A_2^3\frac{16}{a^2}\left(\frac{1}{4}+\frac{A_2}{6}+\frac{A_4}{8}\right)+\frac{A_2^2\,A_4}{a^2}\;144\left(\frac{1}{6}+\frac{A_2}{8}+\frac{A_4}{10}\right)\right.$$

$$\left.+\frac{A_2\,A_4^2}{a^2}\left(\frac{1}{8}+\frac{A_2}{10}+\frac{A_4}{12}\right)\times\,384+\frac{A_4^3}{a^2}\times 320\left(\frac{1}{10}+\frac{A_2}{12}+\frac{A_4}{14}\right)\right]=\frac{C_{11}h^3}{12\,a^2}$$

$$8\,A_4\left(9-\frac{C_{22}}{C_{11}}\right)\left(\frac{1}{2}+\frac{A_2}{4}+\frac{A_4}{6}\right)A\,F(t)+\;\rho h\,A\,F(t)\,a^2\;\left[\frac{1}{2}+\frac{A_2}{2}+\frac{A_2\,A_4}{4}\right.$$

$$\left.+\;\frac{A_2^2+2A_4}{6}+\frac{A_4^2}{10}\;\right] \tag{6}$$

5. Determination of the constant of integration C_0

For clamped immovable edges of the plate we have $u = 0$ at $r = a$ and for movable edges of a plate we have $N_{rr} = 0$ at $r = a$. Required expressions for the constant C_0 for the two cases are obtained by inserting the above boundary conditions in equations (4) and (5). With these values of C_0 inserted into equation (6) one gets, finally, the time-differential equation in the form

$$d^2F(t)/dt^2\;+\;\alpha F(t)\;+\;\beta F^3(t)\;=\;0 \tag{7}$$

6. Boundary conditions—clamped plate

For a plate clamped along the boundary

$$w\;=\;dw/dr\;=\;0\qquad\text{at}\qquad r=a$$

and considering equation (3.1) one gets $A_2 = -2$ and $A_4 = 1$.

Corresponding values of A_2 and A_4 for simply-supported edges can be determined by considering the conditions

$$w = 0 = M_n \text{ (moment) at } r = a.$$

7. Solution of time-differential equation

The solution of equation (7) with initial conditions

$$F(0) = 1, \quad \mathrm{d}F(0)/\mathrm{d}t = 0 \tag{8}$$

has been given by Nash and Modeer [13] with the help of Jacobian elliptic functions and hence the ratio of the non-linear and linear time-periods T^*/T is given by

$$T^*/T = \frac{2\,\Theta}{\pi}/(1+\beta/\alpha)^{1/2} \tag{9}$$

8. Numerical results and discussion

For both movable and immovable edges of the circular plate variations of non-dimensional time-periods T^*/T for different variations of non-dimensional amplitudes A/h and non-dimensional temperature $N_T^* = -\beta_{11} T_0/C_{11}$ have been computed and presented in Tables I and II considering the set of values.

$$E_{11} = 1\times 10^5,\ E_{22} = 0.5\times 10^5,\ \nu_1 = 0.5,\ \nu_2 = 0.025,\ C_{22}/C_{11} = 0.5$$
$$= E_{22}/E_{11},\ C_{12}/C_{11} = 0.025,\ k^2 = 0.5,\ \beta_{22}/\beta_{11} = 0.5,\ a/h = 15.$$

From the tables, it is observed that for both the edge conditions the effect of N_T^* is to diminish the non-dimensional time-periods. The effect of temperature on non-dimensional time-periods is more for plates with immovable edges than for plates with movable edges for the corresponding variations of non-dimensional amplitudes. As it should be, the non-linear behaviour of the plates due to elevated temperature obtained here, is similar in nature as that of the plates subjected to in-plane compressive forces [14].

Table I
Circular plate with movable edge

A/h	0	0.4	0.8	1.2	1.6	2.0
$T^*/T(N_T^* = 0)$	1	.99250	.97099	.938052	.897102	.8515666
$T^*/T(N_T^* = .05)$	1	.97998	.92645	.85391	.77608	.70157
$T^*/T(N_T^* = .075)$	1	.87421	.67602	.52176	.41694	.34451

Table II
Circular plate with immovable edge

A/h	0	0.4	0.8	1.2	1.6	2.0
$T^*/T(N^*_T = 0)$	1	.97433	.90778	.82189	.73442	.65451
$T^*/T(N^*_T = .005)$	1	.96356	.87429	.76842	.6692	.58455
$T^*/T(N^*_T = .01)$	1	.93718	.80214	.66713	.55757	.47332

9. Buckling criterion and critical buckling temperatures

Considering the foregoing set of values of elastic constants and required expressions for α and β one gets

$$\beta/\alpha \text{ (for movable edges)} = 12(A/h)^2\,(.000624)\ /\ (.079 - N_T^*) \tag{10}$$

$$\beta/\alpha \text{ (for immovable edges)} = 12(A/h)^2\,(.0004522)\ /\ (.01626 - N_T^*) \tag{11}$$

Tables I and II have been constructed for the pre-buckling state by considering values of N_T^* sufficiently near to .079 and .01626 for movable and immovable edges respectively.

Buckling occurs when

$$N_T^* = .079 \text{ (movable edges)} \tag{12}$$

$$N_T^* = .01626 \text{ (immovable edges)} \tag{13}$$

which give the critical buckling temperature for the above two cases.

10. Results for simply-supported plates

Results for the non-linear dynamic analysis of simply-supported orthotropic circular plates at elevated temperature can be obtained by considering the values of A_2 and A_4 given below

$$A_2 = -\frac{2(3-\nu_2)}{5-\nu_2} \cdot A_4 = \frac{1-\nu_2}{5-\nu_2} \tag{14}$$

where ν_2 is the Poisson's ratio in the ϕ -direction.

The analysis of the preceding section may be followed by using the same equations and expressions where the values of A_2 and A_4 should be considered from equation (14).

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References

1. PAL, M.C. — Large amplitude free vibration of circular plates subjected to aerodynamic heating, *Int. J. Solids Struct*, 1970, **6**, 301-313.
2. PAL, M.C. — Static and dynamic non-linear behaviour of heated orthotropic circular plates, *Int. J. Non-linear Mech.*, 1973, **8**, 489-504.
3. BISWAS, P. — Thermal buckling and non-linear free vibrations of polygonal plates at elevated temperature, Proc. 3rd Int. Conf. on Numerical methods in thermal problems, University of Washington, Seattle, USA, Aug. 2-5, 1983, pp. 309-319.
4. SATYAMOORTHY, M. AND PANDALAI, K.A.V. — Large amplitude vibrations of variable thickness plates, Noise, Shock and Vibration Conf., 1974, Monash University, Melbourne, pp. 99-106.
5. NOWINSKI, J.L. — Nonlinear vibrations of elastic circular plates exhibiting rectilinear orthotrophy, *ZAMP*, 1963, **14**, 112-124.
6. BISWAS, P. AND KAPOOR, P. — Non-linear free vibrations and thermal buckling of circular plates at elevated temperature, *Indian J. Pure Appl. Math.*, 1984, **15**(7), 809-812.
7. BANERJEE, B. AND DATTA, S. — Large amplitude vibrations of thin elastic plates by the method of conformal transformation, *Int. J. Mech. Sci.*, 1979, **21**, 689-696.
8. NOWINSKI, J.L. AND OHNABE, H. — On certain inconsistencies in Berger equations for large deflections of plastic plates, *Int. J. Mech. Sci.*, 1972, **14**, 165-170
9. PRATHAP, G. — On the Berger approximation: a critical re-examination, *J. Sound Vibr.*, 1979, **66**, 149-154.
10. NOWINSKI, J.L. — *Theory of thermo-elasticity with applications*, Sijthoff and Noordhoff Int. Pub., Alphen AAN Den Rijn, The Netherlands, p. 131.
11. JONES, R., MAJUMDAR, J. AND CHEUNG, Y.K. — Vibrations and buckling of elastic plates at elevated temperatures. *Int. J. Solids Struct* 1980, **16**(1), 61-70.
12. BUCKENS, P. — Vibrations of thermally-stressed plates, *J. Thermal Stresses*, 1979, **2**, 367-385.
13. NASH, W.A. AND MODEER, J.R. — Certain approximate analysis of the non-linear behaviour of plates and shallow shells. Progress at the University of Florida. Tech. Paper No. 193. Vol XIV. No 10. 1960.
14. BISWAS, P. — Non-linear vibrations of orthotropic rectangular plates under in-plane forces, *J. Aeronaut. Soc. India*, 1981, **33**(3-4), 103-105.

Authors' Note

This paper has been condensed as suggested by the Editor. As a result many equations and expressions have been omitted. Interested readers may write to the authors for additional information.