J. Indian Inst. Sci., 65 (B), May 1984, pp. 95-103

Short Communication

On some transcendental equations of astrophysical interest

S. KARANJAI
Department of Mathematics, University of North Bengal, Rajarammohunpur 734 430, W.B., India.

Received on December 31, 1983; Revised on May 3, 1984.

Abstract

In this paper, we report rapid numerical methods for computing the roots of certain commonly occurring transcendental equations of astrophysical interest. We have tabulated the roots at different values of the variable and calculated approximate values of *H*-functions with the roots of one of the transcendental equations considered here.

Key words: Radiative transfer, transcendental equations, *H*-function.

1. Introduction

We often come across certain types of transcendental equations in problems concerning radiative transfer and neutron diffusion. For example, Chandrasekhar's *H*-function, which is a very important function in radiative transfer problems, has poles at $\pm 1/k$ where k is the root of a transcendental equation. The form of this transcendental equation again differs with the different types of phase functions and scatterings. The roots of such transcendental equations are thus frequently required in the numerical calculations of *H*-functions and absorption line profiles. In this paper, we report rapid numerical methods for computing the roots of some commonly occurring transcendental equations.

2. Transcendental equations and their solutions

The equations discussed in this paper are

$$Z_0 = 1/B \tan^{-1} B \tag{1}$$

This equation, which occurs in neutron diffusion problems [1], may be rearranged to a more convenient form by putting $w = 1/Z_0$, $k = 1/B$, *viz.*

$$1 - wk \tan^{-1}(1/k) = 0 \qquad w > 1 \tag{2}$$

Another equation of common occurrence in the problems of isotropic scattering of radiation[2] is of the type

$$1 - \frac{w}{2k} \ln \frac{1+k}{1-k} = 0, \; w \leq 1 \tag{3}$$

Yet another equation is

$$1 - wk \ln \frac{1+k}{k} = 0, \; w > 1 \tag{4}$$

For a given value of w, equations (2), (3) and (4) may be written as $f(k) = 0$. In principle, the roots may be calculated to any desired degree of accuracy by Newton Raphson's method. This, however, turns out to be extremely laborious unless a good first approximation to k is used. This has been achieved by developing approximate formulae for the roots as a function of w. These approximate formulae were obtained by appropriate rearrangement of transcendental equation followed by power series expansion and neglecting higher order terms in the expansion. This may be demonstrated by the approximate root for equation (2).

The equation may be written as

$$\tan\left(\frac{1}{wk}\right) = \frac{1}{k} \tag{5}$$

Using the trigonometric identity

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \tag{5a}$$

with $x = 1/3\,wk$, together with power series expansion for $\tan x$ in terms of x and neglecting terms of the order x^5 and higher (for a large range of w, $3wk > 1$ *i.e.*, $x < 1$) we obtain

$$w - 3w\,(x^2 + 2x^4/3) = 1 - x^4/5 \tag{6}$$

Putting $\underline{z} = 1/x^2$, equation (6) transforms to a quadratic eqn. *viz.*,

$$5\,(w-1)\,Z^2 - 15\,w\underline{z} - (10\,w - 1) = 0 \tag{7}$$

This gives

$$Z = \frac{1}{x^2} = 9\,w^2 k^2 = \frac{15\,w \pm [225\,w^2 + 200\,(w-1)(w-1/10)]^{1/2}}{10\,(w-1)} \tag{7a}$$

Hence $k \simeq \left[\frac{1+[1+0.9(w-1)/w]^{1/2}}{6w\,(w-1)}\right]^{1/2}$ (8)

This equation gives k correct within ${}^{+0.0005}_{-0.0010}$ of the actual value in the range $w = 1$ to $w = 5$. Using this as the first approximation to k in the Newton Raphson's method, roots correct to 7 digits may be obtained only in 2-3 iterations.

Using similar expansions and approximations, the following expressions for approximate roots of equations (3) and (4) have also been obtained.

For equation (3)

$$k = \frac{1 - e^{-x}}{1 + e^{-x}} \tag{9}$$

where

$$x = |(1/\omega)|[(1-w)\ (3.4 + 7.2\,w + 1.44\,w^2)]^{1/2} \tag{9a}$$

For equation (4)

$$k = \frac{1.5 + (w-1)\ (1-.05\,w)}{3\,w\,(w-1)} \tag{10}$$

Using the values given by equations (9-10) as the first approximations, the roots at any value of w may be obtained to 7-8 digit accuracy with only 3-4 iterations. The values of roots k as function of w for equations (2), (3) and (4) are given in Tables I-III. Roots at any intermediate w-values may be obtained by interpolation or, more accurately, by direct application of Newton's method using the values given by the approximate formulae. In most cases, a single iteration gives values of sufficient accuracy.

3. Calculation of *H*-function approximately

Karanjai and Sen[3] have developed some approximate forms for the H-function for isotropic scattering. Using one of their forms *viz.*

$$H(w,\mu) \simeq 1 + a\mu + b\mu^2 + c\mu^3 \tag{11}$$

which satisfies the relation

$$\frac{w}{2}\int_0^1 \frac{H(w,\mu)}{1-k\mu}\,d\mu = 1 \tag{12}$$

alongwith two other relations [vide eqns. (2.2) and (2.3) of ref. (3)]. In this equation, k is the root of eqn. (3). Using the tabulated values of w and k (Table II), approximate values

Table I

w	k	w	k	w	k
1.05	2.531782178	2.20	0.3750405709	3.35	0.2205011929
1.10	1.756651966	2.25	0.3637158485	3.40	0.2166738944
1.15	1.408309379	2.30	0.3530874800	3.45	0.2129796333
1.20	1.198265001	2.35	0.3430906482	3.50	0.2094114735
1.25	1.05359183	2.40	0.3336685825	3.55	0.2059629604
1.30	0.9460002249	2.45	0.3247713194	3.60	0.2026280789
1.35	0.8618848067	2.50	0.3163546885	3.65	0.1994012164
1.40	0.7937682966	2.55	0.3083744786	3.70	0.1962771293
1.45	0.7371493353	2.60	0.3008107456	3.75	0.1932509124
1.50	0.6891305030	2.65	0.2936172350	3.80	0.1903179721
1.55	0.6477488967	2.70	0.2867708973	3.85	0.1874740016
1.60	0.6116198272	2.75	0.2802464795	3.90	0.1847149585
1.65	0.5797339320	2.80	0.2740211780	3.95	0.1820370448
1.70	0.5513352116	2.85	0.2680743428	4.00	0.1794366882
1.75	0.5258443698	2.90	0.2623872245	4.05	0.1769105258
1.80	0.5028087985	2.95	0.2569427564	4.10	0.1744553882
1.85	0.4818688934	3.00	0.2517253666	4.15	0.1720682868
1.90	0.4627347407	3.05	0.2467208149	4.20	0.1697463993
1.95	0.4451695926	3.10	0.241916051	4.25	0.1674870594
2.00	0.4289779089	3.15	0.2372990907	4.30	0.1652877457
2.05	0.4139965401	3.20	0.2328589073	4.35	0.1631460721
2.10	0.4000881199	3.25	0.2285853365	4.40	0.1610597783
2.15	0.3871360390	3.30	0.2244689924	4.45	0.1590267219
				4.50	0.1570448707

Table II

w	k	w	k
0.00	1.0	0.55	0.9355292577
0.05	0.9999999999	0.60	0.9073323166
0.10	0.9999999958	0.65	0.872065354
0.15	0.9999967606	0.70	0.8286347986
0.20	0.9999091217	0.75	0.7755163138
0.25	0.999325673	0.80	0.7104117835
0.30	0.9974138169	0.85	0.6295014764
0.35	0.9931636244	0.90	0.5254295126
0.40	0.9856238716	0.95	0.3794852067
0.45	0.9739758941	1.00	0.00
0.50	0.957504024		

of $H(w,\mu)$ may be calculated from eqn. (2.8) of ref. 3. Comparison of the calculated values of $H(w,\mu)$ by this method with the exact values given by Chandrasekhar[2] shows good agreement. H-functions for values of w not given in ref. 1 are given in Table IV. These values are correct almost to four places of decimal.

Table III

w	k	w	k	w	k
1.05	9.83869266	2.05	0.376004299	3.05	0.169949321
1.10	4.84368698	2.10	0.356148362	3.10	0.165075774
1.15	3.18168691	2.15	0.338106669	3.15	0.160447343
1.20	2.35272464	2.20	0.321647874	3.20	0.156046632
1.25	1.85682828	2.25	0.306577972	3.25	0.151867827
1.30	1.52735575	2.30	0.292733088	3.30	0.147866522
1.35	1.29290012	2.35	0.279975875	3.35	0.144059564
1.40	1.1177662	2.40	0.268181109	3.40	0.140429925
1.45	0.98212965	2.45	0.257252206	3.45	0.136951581
1.50	0.87410078	2.50	0.247098427	3.50	0.133629415
1.55	0.78611728	2.55	0.237642634	3.55	0.130449123
1.60	0.7131407	2.60	0.228817466	3.60	0.127402137
1.65	0.65168534	2.65	0.220563844	3.65	0.12448055
1.70	0.59926346	2.70	0.212829746	3.70	0.12167706
1.75	0.55405237	2.75	0.205569193	3.75	0.118984911
1.80	0.51468642	2.80	0.198741396	3.80	0.116397842
1.85	0.48012245	2.85	0.192310054	3.85	0.113910044
1.90	0.45500417	2.90	0.186242758	3.90	0.111516124
1.95	0.42233023	2.95	0.180510482	3.95	0.109211062
2.00	0.39795254	3.00	0.17508716	4.00	0.106990183

Acknowledgement

The author thanks the University of North Bengal for financial assistance.

References

1. WILLIAMS, M.M.R. *Math. Proc. Camb. Phil. Soc.*, 1978, **84**, 549.
2. CHANDRASEKHAR, S. *Radiative Transfer*, Dover, 1960.
3. KARANJAI, S. AND SEN, M. *Astrophys. Sp. Sci.*, 1971, **13**, 267.

Appendix I

Consider equation (3) *viz.*

$$1 - \frac{w}{2k} \ln \frac{1+k}{1-k} = 0, \quad 0 \le k \le 1$$

Expanding in power series, the above equation becomes

$$1 - \frac{w}{k} \left(k + \frac{k^3}{3} + \frac{k^5}{5} + \frac{k^7}{7} + \ldots \right) = 0$$

Table IV

	0.225	0.275	0.325	0.375	0.425
0.1	1.0290315	1.0361292	1.0435109	1.0512071	1.0592541
0.2	1.0442234	1.0553004	1.0669361	1.0791955	1.0921572
0.3	1.0549788	1.0689727	1.0837715	1.0994754	1.1162056
0.4	1.0632161	1.0794988	1.0968050	1.1152685	1.1350520
0.5	1.0698042	1.0879513	1.1073160	1.1280637	1.1503969
0.6	1.0752271	1.0949315	1.1160261	1.1387063	1.1632117
0.7	1.0797857	1.1008148	1.1233887	1.1477304	1.1741142
0.8	1.0836806	1.1058529	1.1297089	1.1554973	1.1835248
0.9	1.0870523	1.1102228	1.1352023	1.1622633	1.1917432
1.0	1.0900030	1.1140538	1.1400265	1.1682170	1.1989903
	0.475	0.525	0.575	0.625	0.675
0.1	1.0676960	1.0766876	1.0859971	1.0960129	1.1067527
0.2	1.1059170	1.1205929	1.1363356	1.1533402	1.1718677
0.3	1.1341106	1.1533762	1.1742389	1.1970080	1.2221010
0.4	1.1563565	1.1794340	1.2046080	1.2323042	1.2631008
0.5	1.1745664	1.2008839	1.2297731	1.2617596	1.2975879
0.6	1.1898406	1.2189716	1.2510952	1.2868644	1.3271771
0.7	1.2028834	1.2344751	1.2694533	1.3085989	1.3529344
0.8	1.2141769	1.2479458	1.2854752	1.3276230	1.3756106
0.9	1.2240664	1.2597774	1.2995907	1.3444697	1.3957583
1.0	1.2328080	1.2702635	1.3121382	1.3594892	1.4137978
	0.725	0.775	0.825	0.875	0.925
0.1	1.1183796	1.1311320	1.1453855	1.1618028	1.1818032
0.2	1.1922829	1.2151219	1.2412851	1.2721319	1.3110653
0.3	1.2501043	1.2818872	1.3188392	1.3634468	1.4211145
0.4	1.2978163	1.3376733	1.3846417	1.4422781	1.5183882
0.5	1.3383117	1.3855164	1.4417775	1.5117863	1.6059422
0.6	1.3733210	1.4272472	1.4921492	1.5738950	1.6856238
0.7	1.4039970	1.4640963	1.5370476	1.6299201	1.7586996
0.8	1.4311590	1.4969469	1.5774091	1.6808267	1.8261064
0.9	1.4554152	1.5264614	1.6139435	1.7273554	1.8885718
1.0	1.4772317	1.5531520	1.6472059	1.7700926	1.9466808

or $$1 - w\left(1+\frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \ldots\right) = 0 \tag{13}$$

where $x = k^2$

Rearrangement of equation (3) to the form

$$k = \frac{1 - e^{-2k/w}}{1 + e^{-2k/w}}. \tag{14}$$

shows that the two extreme roots of equation (3) are $k=0$ at $w=1$ and $k=1$ at $w=0$. Using $y=1-w$ as the variable, this becomes $k=0$ at $y=0$ and $k=1$ at $y=1$.

Equation (13) may now be written as

$$\frac{1}{1-y}-(1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+ ....) = 0$$

or

$$(y+y^2+y^3+...) - (\frac{x}{3}+\frac{x^2}{5}+\frac{x^5}{7}+ ....) = 0$$

Expanding in terms of a power series in y about $y=0$ (where $x=k^2=0$) we have

$$x = a_1y + a_2y^2 + a_3y^3 + ... \tag{15}$$

while the co-efficients may be related to the derivatives, *i.e.*,

$$a_n = \frac{1}{n!} \ \frac{d^n x}{dy^n}$$

In view of the Taylor's theorem, it is more advantageous to evaluate the values of the co-efficients for approximation to x by using a limited number of terms, *e.g.*, three to four.

Let us now consider a four-term expansion for x, *e.g.*,

$$x = a_1y + a_2y^2 + a_3y^3 + a_4y^4 \tag{16}$$

Substituting x from equation (16) in equation (13) we have

$$(I - a_1/3)\,y + (1 - a^2/3 - a_1^2/5)\,y^2 + ... = 0 \tag{17}$$

for $y << 1$, all terms of order higher than y^2 may be neglected in equation (17) and equating the co-efficients of each power of y on both sides, we have $a_1 = 3$; $a_2 = -2.4$

At $y=1$, x reaches the maximum value of 1, which gives

$$x\,(\text{at } y=1) = a_1 + a_2 + a_3 + a_4 = 1 \tag{18}$$

$$\left(\frac{dx}{dy}\right)_{y=1} = a_1 + 2a_2 + 3a_3 + 4a_4 = 0 \tag{19}$$

wherefrom we get $a_3 = 0.2$; $a_4 = 0.6$

Equation (16) then becomes

$$x = 3y - 2.4y^2 - 0.2y^3 + 0.6y^4$$
$$= (1-w)\ (1 + w + 1.6\ w^2 - 0.6\ w^3)$$

Hence $\quad k = [\ (1-w)\ (1 + w + 1.6\ w^2 - 0.6\ w^3)\ ]^{1/2}$ (20)

Use of the expression (14) gives k correct within $\pm$ 0.0008 in the entire range and may now be easily corrected to 8 digits within 2-3 iterations using Newton Raphson's method.

Appendix II

Let us consider the equation

$$1 - wk\ ln\ \frac{1+k}{k} = 0, w > 1$$

which may be rearranged as

$$1 + 1/k = e^{1/wk} \tag{21}$$

Power series expansion followed by slight rearrangement gives

$$1 - \frac{1}{w} + \frac{1}{2\ w^2k} + \frac{1}{6\ w^3k^2} + \frac{1}{24\ w^4k^3} + \ldots = 0$$

In the range $wk \gg 1$, one may neglect higher order terms, so that

$$1 - w + \frac{1}{2\ w^2\ k} + \frac{1}{6\ w^3\ k^2} \simeq 0$$

or

$$6w^2\ (w-1)\ k^2 - 3\ wk - 1 = 0$$

which gives the positive root as

$$k = \frac{3\ w \pm [9\ w^2 + 24\ w^2\ (w-1)]^{1/2}}{12\ w^2\ (w-1)}$$

$$= \frac{1 \pm [1 + (8/3)(w - 1)]^{1/2}}{4w(w - 1)} \tag{22}$$

Using binomial expansion and neglecting terms containing $(w - 1)^2$ and higher order we have

$$k = \frac{1.5 + (w - 1)}{3w(w - 1)} \tag{23}$$

For $w < 1.5$, this gives good first approximation to k. In order to correct this expression for higher values of w, additional terms in equation (22) were used with empirical co-efficients to give k correct to 3 digits at $w = 5$. With this the final expression for k becomes

$$k = \frac{1.5 + (w - 1)(1 - 0.5w)}{3w(w - 1)} \tag{24}$$