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BOOK REVIEWS

A world on paper: Studies on the second scientific revolution by Enrico Bellone. The MIT Press, 28, Carleton Street, Cambridge, Massachusetts, 02142, USA, pp. 220, $9.14.

In the famous series 'Lectures on Physics' Richard Feynman writes "From a long view of the history of mankind-seen from, say, ten thousand years from now-there can be a little doubt that the most significant event of the 19th Century will be judged as Maxwell's discovery of the laws of electrodynamics". The Maxwell's equations in theoretical physics indeed laid down the scientific and technical world on paper. The second scientific revolution as compared to Galilean revolution is very appropriately entitled by Enrico Bellone as ' A World on Paper'. From the point of view of Galileo's splendid thought 'Our discourses must relate to the sensible world and not one on paper' Enrico Bellone draws attention to the increasingly important role played by mathematics in the physical theories developed after Galilean revolution. Because of this the theories gradually evolved in direction which can no longer be explained by remaining entrenched in philosophical realism. In sum, the new sciences of the second scientific revolution hold discourses about the 'sensible world' and about themselves.

Enrico Bellone places the second revolution in science between the period 1750 and 1900. This revolution saw the starting of the new theories of thermodynamics, radiation, the electromagnetic field and statistical mechanics. All these theories raised questions about the structure of matter and the very meaning of physical law, and thus influences, in different form and measure, both the Galilean-Newtonian tradition and the other sciences. Enrico Bellone in the seven chapters, in his book analyzes in a very powerful style the mental processes of the well-known physicists of this period engaged in answering these questions.

In the first essay Enrico Bellone addresses to the important question: What is the relationship between the theory and experiment? The most important example of the post-Newtonian era about the relationship between Maxwell's formulation of the electromagnetic theory of light and the observational data collected by Faraday is taken. Were Maxwell's equations just a translation of Faraday's empirical knowledge into mathematical formulas? Why such an idea is so simplistic and what are the points a historian of science should keep in view to understand the most profound structure of Maxwell's equations? This discussion is carried on to the second chapter on the reflections on the history of the physical sciences and is best understood when Professor Bellone says: 'The second scientific revolution does not consist merely of a long and complicated process of restructuring of the physical laws; it also entails an awareness of the depth of such restructuring and of its historical development. It is precisely on the basis of this awareness that we can judge the philosophical dimensions of the second scientific revolution'.

Ludwig Boltzmann played a leading role in the nineteenth-century movement toward reducing the phenomena of heat, light, electricity and magnetism to matter and motion — in other words to atomic models based on Newtonian mechanics. His own greatest contribution was to show how mechanics, which had previously been regarded as deterministic and reversible in time, could be used to describe irreversible phenomena in the real world on a statistical basis. Why then scientists like Lord Kelvin (hailed by his contemporaries as a second Newton) and P.G. Tait pronounced harsh indictment against the body of theoretical studies developed by Boltzmann in his attempt to understand and explain the laws of molecular motion? The Tait-Boltzmann controversy with an explicit role played by Kelvin in urging Tait, cover the major parts of the third, fourth and last chapters of the book. The charges against Boltzmann were attempting to substitute mathematical deduction for physical thought! As Professor Bellone writes the bone of contention is the understanding of the relationship that should exist between mathematics and physics (Reader is referred to my earlier review of the book 'Mathematics and Physics', by Yu. I. Manin, in the same columns, Feb. 1983, 64(B), pp. 63-64). This difficult relationship is discussed in an interesting manner from the point of view of developments in mathematics carried out by Lame, Poisson, Riemann, Fourier and Herschel. John Herschel was striving to introduce into England the refined mathematics of the continental school! Boltzmann's philosophical battle with the English critics forms the topic of Chapter 4. In support of his theory, Boltzmann defends himself by attacking Ostwald's superficial philosophy. In Boltzmann's opinion, as Professor Bellone points out, Ostwald's philosophy was based on a misunderstanding of Mach's thoughts. Very valuable light is thrown on this aspect in this chapter.

The following chapter discusses the development of the theory of electricity and magnetism by going back to Coulomb and Ampere. This again is to understand the clash between Kelvin's and Boltzmann's view of mathematics.

The seventh chapter makes a very interesting reading. Professor Bellone takes the reader back a little further in the past to find a very different physics: a physics that seeks qualitative answers by manipulating hypotheses and inferences over an ocean of observations and empirical data. Professor Bellone observes that only few years separate Nollet from Cavendish, Aepinus and Coulomb; yet in those years the empirical ocean is radically reduced, subdivided and reshaped by means of systematic deductions that the new theorists borrow from mathematics and from the theory of motion many regard as nothing more than a branch of analysis. Enrico Bellone reexamines the Nollet's empirical ocean and discusses how Coulomb and Ampere built their electric and magnetic theories on that of scientists like Nollet.

The book ends with another look at Boltzmann-Kelvin problem.

All readers interested in the development of science, historically or philosophically, will find the book very interesting and valuable. The English translation by Mirella and Riccardo Giacconi is appreciable.

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CHANCHAL UBEROI

Nonlinear stochastic problems edited by R.S. Bucy and J.M.F. Moura. NATO ASI Series, D. Reidel, Dordrecht, 1983, pp. 623, D. Fl. 183.

This volume is a collection of papers presented at the NATO Advanced Study Institute on nonlinear stochastic problems held in Algarve, Portugal, in May 1982. The goal of this Institute, to quote the editors, was to present a 'blend of theoretical issues, algorithmic implementation aspects and application examples' in topics such as identification, signal processing, control and nonlinear filtering. Accordingly, one finds a total of forty-one different papers assembled under the headings 'spectral estimation', 'identification', 'system theory', 'adaptive/stochastic control', 'optimal control', 'nonlinear filtering', 'stochastic processes' and 'applications'. Under the first two headings, one finds an assortment of topics in time series analysis, ranging from radar signal processing to parameter estimation algorithms. Simulation studies or case studies are often presented in support of the theoretical work. The next three closely related topics cover contributions in stochastic realization theory, adaptive control algorithms and stochastic control. There is also a solo tutorial paper on calculus on manifolds with a view to applications in control and identification. Under 'nonlinear filtering', one has approximation techniques, finite dimensional filters, filtering for Hilbert space-valued signals, etc. 'Stochastic processes', with few exceptions, contains studies in stochastic calculus and differential equations. 'Applications' covers varied topics like data compression, radar tracking, population dynamics, ship positioning, etc. The list of contributors includes some well-known names like C.I. Byrnes, L.A. Shepp, R.S. Bucy, S.K. Mitter, etc.

I would like to make the following comments concerning this book: On the whole, there seems to be a lack of a clear-cut editorial policy. The choice of topics is a bit too wide and far flung to make a coherent presentation. By spreading itself too thin, the book loses its utility as a source book in any one topic. One feels the omission of good survey/tutorial papers which should have accompanied and preceded the special topics. This is particularly true because most papers here are more of short notes, which either present a small ramification of some on-going program or give a very, very brief summary of results that appeared elsewhere and thus tend to be of rather limited interest. The quality and nature of the contents are highly nonuniform. Moreover, the classification seems somewhat arbitrary. It is instructive to compare this volume with another in the same series, *viz Stochastic systems: The mathematics of filtering and identification and applications* edited by M. Hazewinkel and J.C. Willems. The latter is a far better organized and better presented proceedings, contains more significant papers in terms of both pedagogical and research value, and therefore makes an invaluable reference for anyone working in the field. In comparison, the volume under review is just another of the numerous conference proceedings being churned out every year, ill-conceived and ill-presented, and relatively insignificant.

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Optimal control of partial differential equations edited by K.-H. Hoffmann and W. Krabs. Birkhauser Verlag, Basel, Switzerland, 1984, pp. 261, S. Fr. 78.

The book under review is a volume in the Birkhauser series on numerical mathematics and comprises fifteen papers presented at a conference on optimal control of p.d.e.s held at Oberwolfach in December 1982. The papers are devoted to control of parabolic and hyperbolic p.d.e.s., parameter estimation in parabolic and elliptic p.d.e.s, and optimal design. In the first category, some of the issues discussed are: numerical schemes for control problems (based on Galerkin-type approximations by finite dimensional problems or discretization methods), control of free boundary problems, approximation of certain variational inequalities, controllability in boundary control, etc. Of the two papers on parameter identification, the first considers a numerical procedure for parameter identification in a parabolic problem, formulated as a function space optimization problem, while the second deals with the specific situation of estimating the permeability coefficient in the porous flow potential equation based on measured values of the potential. One paper on optimal design deals with optimal shape design when the state equation is replaced by a state inequality Another paper studies the stability of solutions to a p.d.e. with respect to variations in the domain.

The papers in this collection present important, purely theoretical, contributions both to the field as well as numerical schemes. The latter are provided with rigorous theoretical justification (convergence analysis, etc.) and in addition, are backed by concrete numerical studies. Most of the problems studied here have a strong motivation from engineering problems. These include vibration problems, fluid flow, optimization of radiation patterns, problems arising in metallurgy (such as steel casting), etc.

On the whole, this book will serve as a valuable reference both for a theoretically-inclined engineer interested in mathematical formulation and analysis of engineering problems, and an applied mathematician interested in numerical analysis or optimization problems involving p.d.e.s. Since this field is going through a phase of rapid development at present, the appearance of this book in the market is particularly well-timed, as it can serve as a pointer to the various issues that emerge in this process.

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Numerical treatment of eigenvalue problems, Vol. 3 (International Series of Numerical Mathematics, No. 69) edited by J. Albrecht, L. Collatz and W. Velte, Birkhauser Verlag, Basel, Switzerland, 1984, pp. 214, S. Fr. 54.

The determination of the eigen elements of a linear system is of fundamental importance in several branches of mathematics, physics and engineering. Since these are often impossible to compute exactly, numerical approximations are called for. The current volume is a collection of papers presented on this subject at a workshop held at Oberwolfach in June 1983.

About a third of the papers presented deal with the computation of sharp upper and lower bounds of eigenvalues of differential equations occurring in theoretical physics. Schrodinger operators govern the energy of atoms and their eigenvalues are related to the atomic energy levels. Hence the importance of computing sharp bounds to these is evident. Upper bounds are obtained by Hartree-Fock methods. Fox's method and another new method using the effective field method are presented for computing lower bounds. Estimates by using difference methods and inclusion by quotient theorems are also presented.

An interesting generalization of Collatz's inclusion principle is discussed.

Finite element methods can be used effectively to compute eigenvalues and eigenvectors of systems which occur in engineering, especially structural problems. This is again amply demonstrated by the examples presented in this volume. An interesting example of the nodal condensation technique, together with a new error estimate shows the efficiency of this method.

The next generalizations from a linear eigenvalue problem is the non-linear eigenvalue problem and problems of bifurcation. A new method to compute solutions near a bifurcation point is described. This provides for efficient branch switching without actually computing the bifurcation point itself.

One paper is devoted to existence theory of non-linear equations based on the Galerkin method. While the questions raised by its author are interesting and results promising, the notations are very confusing and so the paper is not very clear. References have been once again made to unpublished manuscripts.

About a third of the papers in this volume are in German. This seems to be a common feature in several publications these days, where publishers bring out conference proceedings and seminar work-outs as books. While it is normal for a conference to have more than one official language, I feel that a book must rise above the level of a journal (where a certain amount of inhomogeneity can be allowed). It would be preferable, when conference proceedings are published as books, that all the articles be in the same language and thus be accessible to a larger number of potential users.

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Numerical methods of approximation theory (Vol. 7) edited by L. Collatz, G. Meinardus, and H. Werner (International Series of Numerical Mathematics, No. 67). Birkhauser Verlag, Basel, Switzerland, 1984, pp. 148, S. Fr. 44.

This volume consists of papers presented at a Workshop on Numerical Methods of Approximation Theory, held at Oberwolfach in March 1983. A third of the papers presented are in German.

The topics covered include Chebychev approximation, Padé approximation and Interpolation. One of the papers compares the error due to real and complex rational Chebychev approximations of (analytic or continuous) functions in complex domains symmetric about the real axis. Operator Padê approximations are investigated in another paper and these are shown to be efficient tools in accelerating the convergence when solving systems of non-linear equations and for the approximation of multivariate functions. Another paper presents several algorithms to compute co-efficients in a recursive scheme to generate polynomials and solve Padê-like approximation problems. Relationships to digital filters are examined.

Some of the papers border on Numerical Analysis. Volterra integral equations of the second kind are approximated and new step-size strategies are presented. These improve on the classical strategies based on the approximation of ordinary differential equations. Another paper describes strategies to use when assembling the stiffness matrices when using finite element methods, by taking into account the physics of the problem concerned so that the computer time is reduced and accuracy increased.

One important method in solving non-linear problems is that of successive approximations. The book contains a good survey on the convergence of successive approximations for non-expansive mappings. It would have been nice if some open problems were also discussed.

One paper which is completely different from the rest in this volume is related to the mathematical formulation of a real situation. In this case a problem of navigating a ship in a fog is considered. The paper is writeen in rather vague terms but probably its inclusion is justified as it will stimulate thought and research more than the presentation of a problem already solved completely.

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Tata Lectures on Theta II (PM 43) by David Mumford. Birkhauser Verlag, CH-4010, Basel, Switzerland, 1984, pp. 272, S. Fr. 54.

This second volume of 'Tata lectures on theta' presents Chapter III devoted to 'Jacobian theta functions and differential equations'. One ought to be greatly indebted to the author for this brilliant and extremely lucid account of Jacobian varieties of hyperelliptic curves, (the associated) hyperelliptic theta functions and differential equations satisfied by these functions.

The classical theta function $\vartheta(z, \tau)$ holomorphic in the complex variables z and τ (with $Im\ \tau > 0$) and its generalization $\vartheta = \vartheta(\underline{z}, \Omega)$ for $\underline{z} \in \mathbb{C}^g$ and *a* (g,g) complex symmetric matrix Ω 'with positive-definite imaginary part' (*i.e.* Ω belonging to the Siegel upper half-plane H_g of degree g) were analysed exhaustively in the first volume; the generalized theta functions arise in a natural fashion when one starts from a compact Riemann surface C of genus $g(\geq 1)$ and looks for meromorphic functions on C. For $g \geq 4$, the set of all $(g,g,)$ matrices Ω which occur as period matrices for Riemann surfaces C (of genus g) depends on a smaller number of (independent) parameters than the complex dimension $g(g+1)/2$ of H_g; consequently, such Ω and the associated complex tori X_Ω (known also as the Jacobian variety of C) are distinguished naturally by rather special and subtle properties. A remarkable function-theoretic (special) property is that ϑ's (coming from such Ω) give rise to solutions of many well-known non-linear partial differential equations (p.d.e) familiar in 'Applied Mathematics'; indeed, such ϑ yield solutions of non-linear p.d.e.'s of low degree such as the Korteweg-de Vries (K.-d.V.) equation and the Sine-Gordan equation in the case of Ω coming from hyperelliptic C and further, the Kadomstev-Petviashvili (K.-P.) equation for Ω arising from general Riemann surfaces C. A principal objective of this volume is precisely to elucidate these phenomena.

After a self-contained review of the concepts from algebraic geometry needed in the sequel, the basic projective model of hyperelliptic Jacobians and its use by J. Moser for solving the Neumann system of ordinary differential equations is nicely presented. This algebraic construction of Jacobians is then linked with their analytic construction (given in the first volume), by invoking theta functions as 'coordinates'.

Next, a fundamental 'vanishing property' for theta characteristics is established; combined with Riemann's theta formula (expounded in the first volume), this is shown to lead to a generalized Frobenius' theta formula (which in the hyperelliptic case, goes back essentially to Frobenius). As applications, the connection between the algebraic and the analytic theories of the Jacobian is further 'strengthened' and explicit solutions for Neumann's dynamical system are provided *via* theta functions; a formula of Thomae relating the cross-ratios of branch points to 'theta constants' is also derived. As a natural application of this theory (and as an important consequence of the 'vanishing property' above), the author takes us through his novel characterisation of hyperelliptic period matrices among all Ω.

The following two sections discuss the McKean-van Moerbeke theory describing the differential identities satisfied by theta functions for hyperelliptic Ω, the link to the K.-d.V. equation and the Iits-Matveev formula for a solution thereof and finally the question of integrating the K.-d.V. dynamical system through hyperelliptic Jacobians.

The second part of Chapter III is concerned with the study of general Jacobian theta functions (*i.e.* those for which Ω is the period matrix of an arbitrary Riemann surface). All such ϑ are shown to satisfy Fay's 'trisecant identity'; formulae for solutions (in terms of theta functions) for the K.-P. equation in general and the K.-d.V. and Sine-Gordan equations in the case of hyperelliptic Ω are derived and further, the connection with the 'soliton' solutions for the K.-d.V. equation is given. The pictorial simulation of a solution of the K.-d.V. equation by the genus 2 ϑ-function as well as of the 'soliton waves' presented opposite to the inside title page is really impressive, like the many other beautiful illustrations in the book.

The third part of Chapter III deals with the resolution of general algebraic equations with the help of hyperelliptic theta functions. After the impossibility of solving, by radicals, the general polynomial equation of degree ≥ 5 was established by Abel (and Galois), it was shown by Hermite and Kronecker that polynomial equations of degree 5 could be solved explicitly by using elliptic modular functions and elliptic integrals. Developing ideas of Jordan and using Thomae's formula above, Umemura derived a simple expression involving

Siegel modular functions and hyperelliptic integrals for writing down the roots of an arbitrary polynomial equation; an exposition of this result by Umemura himself is presented in this concluding section.

One of the several striking unresolved problems in this frontier area is the 'Schottky problem' of characterising the period matrices of Riemann surfaces among all Ω; even as the book was going to press, 'substantial progress' appears to have been made by several distinguished mathematicians towards the resolution of such frontline problems.

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Theory of functions on complex manifolds by G.M. Henkin and J. Leiterer. Birkhauser Verlag, Basel, Switzerland, 1984, pp. 226, S. Fr. 68.

A large number of problems in the theory of analytic functions of (one or more) complex variables quickly reduce to the solution of the partial differential equation $\bar{\partial} u = f$ where f is a so-called (O,q)-form, $q \geq 1$, on an open set D in $\mathbb{C}^n$ (or more generally a complex n-manifold). For $q = 1$, and $D \subset \mathbb{C}^n, f = \Sigma f_i d\bar{Z}_i$, and the problem is to find a function u on D such that $\partial u / \partial \bar{z}_i = f_i$ for all i. As usual, the integrability conditions $\partial f_i / \partial \bar{Z}_j = \partial f_j / \partial \bar{Z}_i$ $1 \leq i, j \leq n$ are necessary for the local solvability of the equation (for $q > 1$, the analogous condition is written as $\bar{\partial} f = o$; u will be a $(o,q\text{-}1)$-form).

Till the 60's, the theory of functions was concerned with the characterisation of open sets $D = \mathbb{C}^n$ (or complex manifolds) where these conditions are also sufficient: the domains of holomorphy and more generally, Stein manifolds. The methods used for patching up local solutions to global ones were abstract (sheaf-theory); in particular there was no way of relating global growth properties of a 'good' solution u to those of f.

In the last two decades, two new methods have been developed to find explicit solutions u which reflect the global growth properties of f. One method, developed by J.J. Kohn, L. Hörmander and others, is derived from the L^2 methods of the theory of partial differential equations; the method has led to important results on the boundary behaviour of analytic mappings.

The second method, which is expounded in this book, is very specifically adapted to the $\bar{\partial}$-equation, and consists in generalising the one-variable Cauchy-Green formula to several variables. Specifically, let D be a bounded open set in $\mathbb{C}^n$. Then a Leray map $w(Z, \zeta) = (w_1(Z, \zeta) \ldots\ldots w_n(Z, \zeta) \ldots\ldots w_n(Z, \zeta))$ for D is a C^1 map $D \times U\, \mathbb{C}^n$, U a neighbourhood of the boundary ∂D of D such that $\Sigma\, w_i(Z, \zeta) \cdot (\zeta_i - Z_i)$ is never zero on $D \times \partial D$. If the Leray map $w(Z, \zeta)$ can be chosen to be holomorphic in the Z-variables, then for any continuous (o,q) form f on $\bar{D}$ such that $\bar{\partial} f = O$ in D, one can explicitly write down a $(0,q\text{-}1)$ form u in D (in terms of $w(Z, \zeta)$ and f) which satisfies $\bar{\partial} u = f$ (Leray-Koppelmann formula). The global behaviour (or the behaviour near ∂D) of this u can therefore be studied in detail. (In all of the above, ∂D is supposed to be at least piecewise C^1.)

Chapter I of this book surveys the elementary notions and general results in the theory of analytic functions (one or more variables) and concludes with a general Leray-Koppelman formula. The basic problem of constructing a Leray map $w(Z,\zeta)$ depending holomorphically on Z is solved for a bounded $D \subset \mathbb{C}^n$ with C^2 strictly pseudo-convex boundary in Chapter II. From this (as is well-known), the basic general (qualitative) results on Stein manifolds follow quickly. Thus the first two chapters also serve as a quick, elementary and self-contained course on several-variable theory.

Chapters III and IV are much more technical. Chapter III solves the equation $\bar{\partial} u = f$ with uniform estimates on strictly pseudo-convex bounded domains with non-smooth boundary. Chapter IV is concerned with representations by integral formulas for solutions u of $\bar{\partial} u = f$ on strictly pseudo-convex domains in Stein manifolds. To illustrate the nature of the results obtainable by the methods of this book, we quote (in part) the final theorem of the book: let X be a Stein manifold, D a strictly pseudo-convex bounded open set, and Y a closed complex submanifold of some neighbourhood of $\bar{D}$; then every bounded holomorphic function on $Y \cap D$ extends to a bounded holomorphic function on D.

The senior author of the book, G.M. Henkin, is a major contributor to the subject. The book is clearly written. By the very nature of the methods employed, analytic functions appear to be nowhere in the scene on most pages of the book. However, at the end of each chapter, there are two very valuable sections titled 'Notes' and 'Exercises, remarks and problems' respectively, in which the results obtained in the chapter in question are placed in their proper perspective, compared to related results obtained by other methods, etc; the exercises are to be regarded as essential for a proper understanding of the subject. The open problems listed in the book greatly enhance the value of the book. The book is strongly recommended to mathematicians interested in Complex Analysis.

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Number theory, an approach to history from Hamurapi to Legendre by Andre Weil. Birkhauser Verlag, CH-4010, Basel, Switzerland, 1984, pp. 375, S. Fr. 64.

A beautiful and breathtaking account of the evolution of Number Theory over 'thirtysix centuries of arithmetical work' from one who has been bestriding the mathematical world like a Colossus, this book presents the illustrious author's 'detailed study and exposition of the achievements of four famous mathematicians — Fermat, Euler, Lagrange and Legendre', the 'founders of modern number theory'. True to the dictum 'Nihil tetigit quod non ornavit', the author's analysis of the successes (and failures) of the various mathematicians concerned is indeed superb, truly revealing and quite authentic; we feel we can do no better than quote profusely, in the sequel, from this very interesting book of inestimable value alike for lovers of mathematics or its history.

According to Chapter I entitled 'Protohistory', modern number theory 'seems to have been twice-born', first somewhere 'between 1621 and 1636 A.D.' when Fermat began to study carefully (the Greek text of) Bachet's edition of Diophantus and then again in 1729 A.D. when Goldbach, in a 'fateful postscript' brought to Euler's attention a statement of Fermat on the primality of integers of the form $1 + 2^r$ ($r = 1,2,4,8,16,...$) and Euler subsequently reported that he had 'just been reading Fermat' and was 'greatly impressed by Fermat's assertion that every integer is a sum of four squares (and also of 3 triangular numbers of 5 pentagonal numbers, etc.)'. Number theory which continued thereafter to be fondly nursed by Euler followed by Lagrange and Legendre, reached full maturity under the patronage of Gauss who inaugurated a new era therein with his profound work. Chapter I mentions a few highlights from the scanty remains from ancient (Greek Mesopotamian) or medieval (Western/Oriental) mathematics and in particular, of number-theoretic work before the 17th century. Basic multiplicative properties of positive integers are dealt with quite well in Euclid's books; a general assertion about the unique factorization of integers into primes seems, however, to be missing therein, although one can find the famous proof for the infinitude of primes, which could have been due to Euclid or discovered earlier. Solution (in integers) of indeterminate equations of the first degree (or of linear congruences) 'must have occurred quite early in various cultures, either as puzzles' (or 'as calendar problems'); the 'Euclidean algorithm' for finding the greatest common divisor of two integers gives 'essentially the general method of solution'. 'If we leave China aside, the first explicit description of the general solution occurs' in *Aryabhatiya* (5th-6th century); 'in later Sanskrit texts, this became known as the 'kuttaka (= pulverizer)' method (possibly of Greek origin), 'recalling to our mind Fermat's 'infinite descent'...'. It is hard to assert, 'either that they (Pythagoreans) rediscovered it (Pythogorean triples) or .... that it came to them .... from Mesopotamia'. The problem of finding three squares in arithmetic progression with the (common) difference 5 'became the object of the Liber Quadratorum' of Leonardo Pisano (Fibonacci) who also considered the question of the area of pythagorean triangle never being a square; the latter 'was to be one of Fermat's major discoveries' based on 'descent'. Problems like finding integers x,y satisfying the equation $x^2 - Ny^2 = \pm m$ for given positive integers N,m must have occurred early in Greek mathematics, possibly towards getting good rational approximations to $\sqrt{N}$ (for non-square N)-cf. Archimedes' approximations for $\sqrt{3}$ or his 'problem bovinum'. Questions asking for rational numbers x,y satisfying 'Pell's equation': $x^2 - Ny^2 = 1$ occur in Diophantus. However, using the 'bhavana' rules (laws of composition for the binary form $X^2 - NY^2$), Brahmagupta (7th century) could find integers x,y to solve 'Pell's equation' for a number of N, although still far short of the general solution; such a solution is to be found in the work of Bhaskara (12th century) and of ('an otherwise unknown') Jayadeva (11th century) and is based on the 'chakravala (= cyclic process)' of undetermined origin. Although 'for the Indians, the effectiveness of the 'chakravala' could be no more than an experimental fact', to have 'developed' it and 'applied it successfully to such difficult numerical cases as $N = 61$ or $N = 67$ had been no mean achievement'. However, 'it was Fermat who was the first to perceive the need for a general proof and Lagrange the first to publish one'.

Chapter II begins with a brief sketch of Fermat's life and gives a detailed account of his (mathematical correspondence and) work on perfect numbers, (the little) 'Fermat's theorem', representation of integers as sums of two squares (of integers), 'descent' applied to solve some diophantine equations, 'Pell's equation' and 'Ascent' for equations of genus 1. Fermat had suggested the solution of 'Pell's equation' as a 'challenge problem to the English and all others'; his method of solution 'is equivalent (in substance) to the Indian 'chakravala' method as well as the modern treatments based on continued fractions'. With regard to the (famous) 'incautious words' in his statement of 'Fermat's last theorem' (as now 'vulgarly called'), 'what he had in mind on that day can never be known'. 'On the other hand, what we possess of his methods for dealing with curves of genus 1 'is' still the foundation for the modern theory of such curves'; 'it falls into two parts',-'the first one, directly inspired by Diophantus' which may be 'termed a method of ascent' and the second one, 'the descent which is .. Fermat's own'. 'Fermat has cast his shadow well into the present century and perhaps the next one'. Diophantus, Viete and Fermat offered, according to Leibniz, the only existing models for algebraic geometry on which the study and classification of algebraic differentials and integrals depended; no wonder then that Leibniz exhorted 'Analysts' 'to attend more diligently than heretofore to the advancement of Diophantine algebra and reap a richer harvest' or in 'Euler's oft-repeated admonition to fellow-analysts that number theory is even from their point of view, no waste of time'. Entirely appropriate excerpts with sparkling, incisive and authoritative comments by the author form thus a veritable delight for the reader, not to mention the great value of the several appendices reconstructing results of these mathematicians or presenting them in a modern setting.

The torch held aloft earlier by Fermat 'remained extinguished until 1730' when 'Euler picked it up, kindled it anew and kept it burning brightly for the next half-century'. A sizeable component of his monumental work, Euler's 'work in number theory alone would have earned him a distinguished place in the history of mathematics, had he never done anything else'. Indeed, 'no mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did, for the best part of the eighteenth century'. Despite losing his eyesight, 'he seems to have carried in his head the whole of the mathematics of his day'. Chapter III (incidentally the longest) contains a lucid survey of Euler's (number-theoretic) investigations: generalization of (the little) Fermat's theorem, representation of integers as sums of two squares or in the form $ax^2 + by^2$ (in which connection, 'Euler must have been aware of 'Brahmagupta's identity'), quadratic, cubic and biquadratic residues modulo primes (with the law of quadratic reciprocity lying 'hidden in his guesses, as perhaps the most valuable nugget of his treasure-trove' which 'he was not to recognise as such until quite late in his life'), representation of integers as sums of four squares, criteria for numbers to be prime ('numeri *idonei* (= suitable)' for testing the primality of large numbers), diophantine equations, elliptic integrals and the addition theorem. ('momentous discovery' of the) 'Euler product' for (Riemann's) zeta function and the evaluation of zeta (and some L-) series at integral arguments ('marking the birth of analytic number theory'), Partitio numerorum and modular functions. Three appendices follow, one on the quadratic reciprocity law describing facts about 'prime divisors of the form $X^2 + NY^2$' for comparison with Euler's empirical results and two very useful appendices on 'an elementary proof for sums of squares' and 'the addition theorem for elliptic curves'. Together with the three important appendices to Chapter IV on the results of Legendre and of Lagrange, they enhance the value of this fascinating book.

Chapter IV covers the work of Lagrange ('be premier des géomètres' to whom Euler 'clearly felt ready to pass the title on', by 1775) and also that of Legendre (who had been an 'eager *student*' of the work of Euler and Lagrange). The best work of Lagrange was largely inspired by that of Euler "which, already as a young man, he 'knew by heart, down to the minutest details". A brief description is given of the work of Lagrange on arithmetical topics, especially solutions of 'Pell's equation', continued fractions and applications to the solution of diophantine equations, proof for Fermat's statement on sums of four squares and diophantine equations arising from the work of Fermat and Euler and finally of his great work on 'reduction' of binary quadratic forms. This is followed by a nice sketch of the work of Legendre and in particular, his important criterion for the solvability of ternary quadratic diagonal equations. The principle of 'descent' was applied to the solutions of Fermat's equation $x^5 + y^5 = z^5$ by Dirichlet and Legendre in 1825. Legendre ('then well over 70 years old') was 'guided' by Dirichlet 'almost to the top' of this 'modest Everest' and even to 'get there first'. The concluding sentence of Chapter IV: 'As to Dirichlet, he was soon to take his flight and soar to heights undreamt of by Legendre's is truly soulstirring. Would we have the good fortune to be escorted in a future publication of the author's to these 'heights undreamt of' and higher still?

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