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On the spectral resolution of a differential operator I

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Abstract

The actual construction of the explicit form of the matrix $H(x, y, \lambda)$ generating the spectral resolution (*i.e.*, the resolution of the identity) of the matrix differential operator

$$M = \begin{pmatrix} -D^2 + p & r \\ r & -D^2 + q \end{pmatrix}$$

has been made by deriving the explicit form of the Green's matrix in the singular case $(-\infty, \infty)$.

Key words: Spectral resolution, bilinear concomitant, wronskian, Green's matrix, generalized Parseval's theorem, Cauchy's singular integral, generalized orthogonal relation, Carleman-type kernel.

1. Introduction

Consider the differential equation

$$MU = \lambda U \tag{1.1}$$

where

$$M = \begin{pmatrix} -D^2 + p(x) & r(x) \\ r(x) & -D^2 + q(x) \end{pmatrix}, \; D \equiv \mathrm{d}/\mathrm{d}x, \; U = \binom{u}{v}$$

and λ is the complex parameter, $p(x)$, $q(x)$, $r(x)$ are the real valued $C_{1-k}(a, b)$ $(k=0,1)$-class functions of x, integrable over (a, b), finite or infinite; where by $C_k(\alpha, \beta)$-class functions we mean (real or complex-valued) functions which are k times continuously differentiable with respect to x defined in (α, β), finite or infinite. The matrix differential expression is symmetric and the Hilbert space $\mathscr{H}$ in which we go in for the definition of the spectral resolution (or the resolution of the identity) of M is that of vector valued functions $f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ where $\int_{-\infty}^{\infty}(f,f)\,dt<\infty$, (.,.) denotes the usual inner product of the vectors.

Let T be a linear operator. The spectral resolution or the resolution of the identity of the operator T or the spectral family[1] (P.13) is defined as a one parameter family of projection operators E_t, $t\in[,a,b]$, where a, b are finite or infinite, where $E_{-\infty}=\lim_{t\to-\infty}E_t$, $E_\infty=\lim_{t\to\infty}E_t$, such that (i) $E_a=0$, $E_b=E$ (ii) for $a<t<b$, $E_{t-0}=E_t$, (iii) $E_\mu E_\nu=E_s$, $s=\min(\mu,\nu)$, (see Akhiezer and Glazman[2]). T is connected with E_t by means of the relation $T=\int_{-\infty}^{\infty}\lambda\, dE(\lambda)$.

The boundary conditions at a,b satisfied by a solution $U(x,\lambda)$ of (1.1) are

$$[U(x,\lambda),\phi_l]_a=0,\ [U(x,\lambda),\phi_j]_b=0,\ l=1,2;\ j=3,4 \tag{1.1a}$$

with $[\phi_1,\phi_2]_a=[\phi_3,\phi_4]_b=0$, where ϕ_i are the 'boundary condition vectors'—solutions of (1.1) which together with their first derivatives take prescribed constant values at (a or b) and $[U,V]_\alpha$, the value at $x=\alpha$ of

$$\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} + \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$$

the bilinear concomitant of the vectors

$$U=\begin{pmatrix} u_1 \\ v_1 \end{pmatrix},\quad V=\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

The boundary condition vectors at a,b are linearly independent of each other.

It is well-known[3,4] that the system (1.1) along with the boundary conditions (1 1a) leads to a self-adjoint eigenvalue problem for the finite interval (a,b). The extension problem for the singular case $[0,\infty)$ was dealt with by Chakravarty[4]; the problem for the interval $(-\infty,\infty)$ is first discussed in the following and then we obtain an explicit expression for a matrix $H(x,y,\lambda)$, λ real, which generates an expression connected in the same way with the differential operator M as the spectral resolution with the operator T. We call $H(x,y,\lambda)$ the spectral resolution or the resolution of the identity in the present discussion.

Let

$$\phi_r\equiv\phi_r(x,\lambda)=\begin{pmatrix} u_r \\ v_r \end{pmatrix}\quad r=1,2$$

be the vectors which are the solutions of (1.1) satisfying at $x=0$, the conditions

$$\left.\begin{aligned} (u_1, v_1, u_1', v_1')\,|_{x=0} &= (1,0,0,0) \\ (u_2, v_2, u_2', v_2')\,|_{x=0} &= (0,1,0,0) \end{aligned}\right\} \tag{1.2}$$

Also let the non-homogeneous system corresponding to (1.1) (which is the homogeneous system) be

$$MU - \lambda U = f(x) \tag{1.3}$$

where $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

Corresponding to the solution vectors ϕ_r, let us choose another pair of solution vectors $\theta_k \equiv \theta_k(x, \lambda) = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ of (1.1) related with ϕ_r by means of the relations

$$[\phi_r\ \theta_k] = \delta_{rk},\ [\theta_1,\ \theta_2] = 0, \quad r,k = 1,2 \tag{1.4}$$

where $[\cdot]$ represents the bilinear concomitant of the vectors concerned. Evidently, given ϕ_r, the choice of θ_k by (1.4) is not unique; in fact, three more independent relations are necessary to determine θ_k, θ_k', $k = 1,2$ completely. The vectors $\phi_1, \phi_2, \theta_1, \theta_2$ form a fundamental set, the wronskian $W = W(\phi_1, \phi_2, \theta_1, \theta_2)$ being equal to 1.

The procedure adopted for the extension to the case $(-\infty, \infty)$ is to assume the results for the interval (a,b) and then to pass on to the desired case by making $a \to -\infty$, $b \to \infty$, by considering the intervals $(0,\ b)$ and $(a,\ 0)$ separately. (For extension problem, see Chakravarty[4]).

2. The extension process

As in Chakravarty[4], there exists the symmetric matrix $(l_{rs}(\lambda))$, depending on λ, b, and the coefficients in the boundary conditions at $x = b$, where l_{rs} have an infinite number of simple poles on the real axis and for fixed b, $l_{rs} = O(1/|\nu|)$ as $\nu \to 0$, where $\nu = \text{im}\,\lambda$.

Also there exists a pair of vectors $\psi_r(b, x, \lambda) \equiv \psi_r = \begin{pmatrix} \psi_{r1} \\ \psi_{r2} \end{pmatrix} = l_{r1}\phi_1 + l_{r2}\phi_2 + \theta_r$, $r=1,2$, obviously solutions of the given system (1.1), such that

$$\| \psi_r(b,x,\lambda) \|_{0,b} = -1/\nu\ \text{im}\,(l_{rr}),\ r = 1,2.$$

Similarly, there exist the symmetric matrix $(L_{rs}(\lambda))$ and vectors $\chi_r(a, x, \lambda)$ which behave in $(a,0)$ in the same way as (l_{rs}) and $\psi_r(b, x, \lambda)$ respectively in (o, b). Thus

$$L_{rs} = O(1/|\nu|), \text{ as } \nu \to 0,\ \nu = \text{im}\,\lambda$$

$$\chi_r(a, x, \lambda) \equiv \chi_r = \begin{pmatrix} \chi_{r_1} \\ \chi_{r_2} \end{pmatrix} = L_{r1}\phi_1 + L_{r2}\phi_2 + \theta_r,\ r = 1,2,$$

where $\| \chi_r(a,x,\lambda) \|_{a,0} = 1/\nu \operatorname{im}(L_{rr}),\ r = 1,2.$

ψ_1, ψ_2 as also χ_1, χ_2 are linearly independent pairs. Further, ψ_j, χ_j are constructed in terms of boundary condition vectors at a, b (Chakravarty[4]) which are linearly independent of each other. It follows that ψ_j, χ_j are also linearly independent of each other. Thus the wronskian $W(a,b,\lambda)$ of $\psi_j, \chi_j, j = 1,2$ does not vanish identically.

We have
$$[\chi_1, \chi_2] = [\psi_1\ \psi_2] = 0 \tag{2.1}$$
$$[\chi_r, \psi_s] = L_{rs} - l_{rs}$$

and
$$W(a,b,\lambda) = [\chi_1, \psi_1][\chi_2, \psi_2] - [\chi_1, \psi_2][\chi_2, \psi_1]$$
$$= (L_{11} - l_{11})(L_{22} - l_{22}) - (L_{12} - l_{12})^2 \neq 0.$$

Let

$$\bar{\psi}_r(a,b,x,\lambda) \equiv \bar{\psi}_r = \begin{pmatrix} \bar{\psi}_{r1} \\ \bar{\psi}_{r2} \end{pmatrix} = \frac{[\chi_s, \psi_s]\psi_r - [\chi_s, \psi_r]\psi_s}{W(a,b,\lambda)} \tag{2.2}$$

where $s = 2$ when $r = 1$ and $s = 1$ when $r = 2$.

Put
$$\bar{\psi}(a,b,x,\lambda) = \begin{pmatrix} \bar{\psi}_{11}(a,b,x,\lambda) & \bar{\psi}_{21}(a,b,x,\lambda) \\ \bar{\psi}_{12}(a,b,x,\lambda) & \bar{\psi}_{22}(a,b,x,\lambda) \end{pmatrix}$$

$$\bar{\chi}(a,x\lambda) = \begin{pmatrix} \chi_{11}(a,x,\lambda) & \chi_{21}(a,x,\lambda) \\ \chi_{12}(a,x,\lambda) & \chi_{22}(a,x,\lambda) \end{pmatrix}$$

and construct the matrix

$$G(a,b,x,y,\lambda) = (G_{ij}(a,b,x,y,\lambda))^T = \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix}$$
$$\left.\begin{aligned} &= \bar{\psi}(a,b,x,\lambda)\ \bar{\chi}^T(a,y,\lambda),\ y \leq x \\ &= \bar{\chi}(a,x,\lambda)\ \bar{\psi}^T(a,b,y,\lambda),\ y > x \end{aligned}\right\} \tag{2.3}$$

Then $G(a,b,x,y,\lambda)$ is the Green's matrix for the system (1.1) for the interval $[a,b]$ with usual properties, as can be easily verified by using the following easily deducible identities

$$\bar{\psi}_{rr}^{(n-1)}\ \chi_{rs} - \bar{\psi}_{rs}\ \chi_{rr}^{(n-1)} + \bar{\psi}_{jr}^{(n-1)}\ \chi_{js} - \bar{\psi}_{js}\ \chi_{jr}^{(n-1)} = \delta_{rs}$$

where δ^{rs} is the Kronecker delta and $n, r, s = 1,2$; when $r=1, j=2$ and $r=2, j=1$ and $f^{(0)}=f$, $f^{(1)}=f'$.

It easily follows from (2.2) that

$$[\bar{\psi}_1, \phi_1] = \frac{l_{22} - L_{22}}{W(a,b,\lambda)}, \quad [\bar{\psi}_2, \phi_2] = \frac{l_{11} - L_{11}}{W(a,b,\lambda)}$$

$$\text{and} \quad [\bar{\psi}_1, \phi_2] = [\bar{\psi}_2, \phi_1] = -\frac{l_{12} - L_{12}}{W(a,b,\lambda)} \tag{2.4}$$

with similar results for $[\bar{\psi}_j, \theta_k]$, $j,k = 1,2$.

To determine the θ uniquely, in addition to the relations (1.4) we choose three more relations as

$$W(\phi_1, \phi_2, \theta_r, \bar{\psi}_r) = 0 \quad \text{and} \quad [\bar{\psi}_1, \theta_2] = [\bar{\psi}_2, \theta_1]$$

Hence on slight reduction, we obtain the following canonical representation for $\bar{\psi}_r$, *viz.*,

$$\bar{\psi}_1(a,b,x,\lambda) = \frac{l_{11}}{l_{11} - L_{11}} \phi_1(x,\lambda) + \frac{l_{12}}{l_{11} - L_{11}} \phi_2(x,\lambda) + \frac{1}{l_{11} - L_{11}} \theta_1(x,\lambda)$$

$$\bar{\psi}_2(a,b,x,\lambda) = \frac{l_{12}}{l_{11} - L_{11}} \phi_1(x,\lambda) + \frac{l_{22}}{l_{11} - L_{11}} \phi_2(x,\lambda) + \frac{1}{l_{11} - L_{11}} \theta_2(x,\lambda) \tag{2.5}$$

with $l_{11} - L_{11} = l_{22} - L_{22}$, $l_{12} = L_{12}$ (2.6)

By following the Chakravarty analysis[4] we obtain that

$$\lim_{\substack{b\to\infty \\ a\to-\infty}} G(a,b,x,y,\lambda) = G(x,y,\lambda) = \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix},$$

the Green's matrix for the singular case $(-\infty\ \infty)$; the Green's vectors

$$G_l(x,y,\lambda) = \begin{pmatrix} G_{l1} \\ G_{l2} \end{pmatrix} \epsilon L_2(-\infty\ \infty),\ l = 1,2,$$

$\psi_r(b,x,\lambda)$ tends to $\psi_r(x,\lambda)$, as $b\to\infty$, where $\psi_r \epsilon L_2[0,\infty)$

and

$$\psi_r = m_{r1}\,\phi_1 + m_{r2}\,\phi_2 + \theta_r \tag{2.7}$$

$\chi_r(a,x,\lambda)$ tends to $\chi_r(x,\lambda)$, as $a \to -\infty$, where $\chi_r \epsilon L_2(-\infty, 0]$

and

$$\chi_r = M_{r1}\,\phi_1 + M_{r2}\,\phi_2 + \theta_r \tag{2.8}$$

$l_{rs}(\lambda) \to m_{rs}(\lambda)$, $m_{rs} = m_{sr}$, as $b \to \infty$

$L_{rs}(\lambda) \to M_{rs}(\lambda)$, $M_{rs} = M_{sr}$, as $a \to -\infty$.

$$\| \psi_r(x,\lambda) \|_{0,\infty} = -1/\nu \text{ im } \{ m_{rr}(\lambda) \}$$

and $$\| \chi_r(x,\lambda) \|_{-\infty,0} = 1/\nu \text{ im } \{ M_{rr}(\lambda) \} \tag{A}$$

Thus from (2.5) and (2.6), since ϕ_1, ϕ_2 are linearly independent, it follows by making $a \to -\infty$ and $b \to \infty$,

$$\bar{\psi}_1(x,\lambda) = \frac{m_{11}}{m_{11} - M_{11}}\,\phi_1(x,\lambda) + \frac{m_{12}}{m_{11} - M_{11}}\,\phi_2(x,\lambda) + \frac{1}{m_{11} - M_{11}}\,\theta_1(x,\lambda)$$

$$\bar{\psi}_2(x,\lambda) = \frac{m_{12}}{m_{11} - M_{11}}\,\phi_1(x,\lambda) + \frac{m_{22}}{m_{11} - M_{11}}\,\phi_2(x,\lambda) + \frac{1}{m_{11} - M_{11}}\,\theta_2(x,\lambda) \tag{2.9}$$

$M_{11} - m_{11} = M_{22} - m_{22}$, $M_{12} = m_{12}$ and

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \bar{\psi}_r(a,b,x,\lambda) = \bar{\psi}_r(x,\lambda).$$

If $\bar{\Psi}(x,\lambda)$ is the $\bar{\psi}(a,b,x,\lambda)$, as $a \to -\infty$, $b \to \infty$, and $\bar{\chi}(x,\lambda)$, the $\bar{\chi}(x,\lambda)$, as $a \to -\infty$, with

$$\bar{\psi}_r = \begin{pmatrix} \bar{\psi}_{r1}(x,\lambda) \\ \bar{\psi}_{r2}(x,\lambda) \end{pmatrix} \quad \chi_r = \begin{pmatrix} \chi_{r1}(x,\lambda) \\ \chi_{r2}(x,\lambda) \end{pmatrix}$$

it follows from (2.3) that the Green's matrix in the singular case $(-\infty, \infty)$ has the representation

$$G(x,y,\lambda) = \bar{\psi}(x,\lambda)\ \bar{\chi}^T(y,\lambda),\ Y \leq x \\ = \chi(x,\lambda)\ \bar{\psi}^T(y,\lambda),\quad y > x \tag{2.10}$$

$G(x,y,\lambda)$ is not necessarily unique. For uniqueness of $G(x,y,\lambda)$ we require a number of stringent conditions on p,q,r (See Chakravarty[4], where the problem is discussed for the interval $[0,\infty)$).

Finally, as in Chakravarty[4], if $f(x) \in L_2(-\infty,\infty)$ be an arbitrary vector, the vector

$$\Phi(x,\lambda) \equiv \Phi(x,\lambda,f) = \int_{-\infty}^{\infty} G(x,y,\lambda) f(y)\,dy \tag{2.11}$$

satisfies the non-homogeneous system (1.3) and $\Phi(x,\lambda,f) \in L_2(-\infty,\infty)$.

3. Derivation of the generalized Parseval theorem for the system (1.1) in the singular case $(-\infty\ \infty)$

In (2.10) we substitute the explicit expressions for $\psi_{rs}(x,\lambda)$, $\chi_{rs}(x,\lambda)$ as obtained in (2.9) and (2.8). Then, since

$$\mathrm{im}\left[\frac{M_{rr}}{M_{rr}-m_{rr}}\right] = \mathrm{im}\left[\frac{m_{rr}}{M_{rr}-m_{rr}}\right],\quad \frac{m_{12}\,M_{11}+m_{22}\,M_{12}}{M_{11}-m_{11}} = \frac{m_{11}\,M_{12}+m_{12}\,M_{22}}{M_{11}-m_{11}}$$

etc., and $\phi_r(x,\lambda)\ \theta_r(x,\lambda)$ take real values for real λ, it follows after some reductions that for $y \leq x$,

$$\lim_{\nu\to 0} \mathrm{im}\, G_{11}(x,y,\lambda) = (u_1\ u_2)(d\,\xi_{ij})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (u_1\ u_2)(d\,\eta_{ij})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \\ + (x_1\ x_2)(d\,\eta_{ij})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (x_1\ x_2)\,d\,\zeta_{11}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where u_r, x_r are the elements of $\phi_r = \begin{pmatrix} u_r \\ v_r \end{pmatrix}$, $\theta_r = \begin{pmatrix} x_r \\ y_r \end{pmatrix}$

respectively, $\lambda = \mu + i\nu$ and $\xi_{ij} \equiv \xi_{ij}(\mu)$, $\eta_{ij} \equiv \eta_{ij}(\mu)$, $\zeta_{11} = \zeta_{11}(\mu)$;

$$\xi_{ij} = \xi_{ji},\quad \eta_{ij} = \eta_{ji},\quad i,j = 1,2,$$

are defined by

$$\xi_{rr}(\mu) = \lim_{\nu\to 0} \int_0^{\mu} -\operatorname{im} \left[\frac{m_{rr}(u+i\nu)\, M_{rr}(u+i\nu) + m_{rs}^2(u+i\nu)}{M_{11}(u+i\nu) - m_{11}(u+i\nu)} \right] \mathrm{d}u$$

where $r = 1,\ s = 2$ and $r = 2,\ s = 1$;

$$\xi_{12}(\mu) = \lim_{\nu\to 0} \int_0^{\mu} -\operatorname{im} \left[\frac{m_{12} M_{11} + m_{22} M_{12}}{M_{11} - m_{11}} \right] \mathrm{d}u \tag{3.1}$$

$$\eta_{rs}(\mu) = \lim_{\nu\to 0} \int_0^{\mu} -\operatorname{im} \left[\frac{m_{rs}}{M_{11} - m_{11}} \right] \mathrm{d}u,\ r,s = 1,2,$$

and $$\zeta_{11}(\mu) = \lim_{\nu\to 0} \int_0^{\mu} -\operatorname{im} \left[\frac{1}{M_{11} - m_{11}} \right] \mathrm{d}u$$

ξ_{rs}, η_{rs} and ζ_{11} are non-decreasing functions of μ (Proof given in § 4) with similar expressions for the remaining $G_{ij}(x, y, \lambda)$ for $y \leq x$. Hence for $y \leq x$, we have

$$\lim_{\nu\to 0} \operatorname{im} G(x,y,\lambda) = \phi(x,\mu)\, \mathrm{d}\xi\, \phi^T(y,\mu) + \phi(x,\mu]\, \mathrm{d}\eta\theta^T(y,\mu)$$

$$+ \theta(x,\mu)\, \mathrm{d}\eta\, \phi^T(y,\mu) + \theta(x,\mu)\, \mathrm{d}\zeta\, \theta^T(y,\mu) \tag{3.2}$$

where $\mathrm{d}\xi(\mu) = (\mathrm{d}\xi_{ij}(\mu))$, $\mathrm{d}\eta(\mu) = (\mathrm{d}\eta_{ij}(\mu))$, $\mathrm{d}\zeta(\mu) = \mathrm{d}\zeta_{11}(\mu)\, I$,

I, unit 2×2 matrix, and $\phi(x,\mu) = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$ and $\theta(x,\mu) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$

The superscript T denotes the transpose of a matrix.

A similar result holds when $y > x$.

As in Chakravarty[4], the vector $\Phi(x, \lambda, f)$ satisfies the relations

$$\| \Phi(x,\lambda,f) \|_{-\infty,\infty} \leq \nu^{-2} \|f\|_{-\infty,\infty},$$

$$\lambda\, \Phi(x,\lambda,f) = [f(x) + \Phi(x,y,\tilde{f})], \tag{3.2a}$$

where $\lambda = \mu + i\nu$, $\tilde{f}(x) \equiv Mf$, $f \in L_2(-\infty,\infty)$.

Also, by utilizing the formula of type

$$(\xi-x)^2\,\Phi(x,\lambda) = \int_x^{\xi} (\xi-y)^2 (y-x)\{M(y,\lambda)\,\Phi(y,\lambda) - F(y)\}\,dy -$$

$$-\int_x^{\xi} (6y - 2x - 4\xi)\,\Phi(y,\lambda)\,dy$$

where $M(y,\lambda) = \begin{pmatrix} p-\lambda & r \\ r & q-\lambda \end{pmatrix}$ and $F(y) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

we obtain (see Chakravarty[4], p. 411)

$\Phi(x,\lambda) = O(|\lambda|^{-1/4}\,|\nu|^{-1})$, for $f(x) \in L_2(-\infty,\infty)$, x fixed and $\nu \neq 0$ (compare Titchmarsh[5], p. 34).

Hence

$$f(x) = \lim_{\substack{R\to\infty \\ \nu\to 0}} \operatorname{im}\left[1/\pi \int_{-R+i\nu}^{R+i\nu} \Phi(x,\lambda)\,d\lambda\right]$$

$$= \lim_{\substack{R\to\infty \\ \nu\to 0}} \operatorname{im} 1/\pi \int_{-R+i\nu}^{R+i\nu} \left[\int_{-\infty}^{\infty} G(x,y,\lambda)\,f(y)\,dy\right] d\lambda$$

(see Titchmarsh[5], pp. 39-40).

Thus,

$$\pi f(x) = \int_{-\infty}^{\infty} \Phi(x,\mu)\,d\xi(\mu) \int_{-\infty}^{\infty} \phi^T(y,\mu) f(y)\,dy + \int_{-\infty}^{\infty} \phi(x,\mu)\,d\eta(\mu) \int_{-\infty}^{\infty} \theta^T(y,\mu) f(y)\,dy +$$

$$+ \int_{-\infty}^{\infty} \theta(x,\mu)\,d\eta(\mu) \int_{-\infty}^{\infty} \phi^T(y,\mu) f(y)\,dy + \int_{-\infty}^{\infty} \theta(x,\mu)\,d\zeta(\mu) \int_{-\infty}^{\infty} \theta^T(y,\mu) f(y)\,dy \quad (3.3)$$

Since for square matrices A, B, C, of the same order

$(ABC)^T = C^T B^T A^T$, $(A^T)^T = A$ and $A^T = A$ when the matrix is symmetric, the above expansion formula leads formally to the following theorem

Theorem: For two vectors $f(x)$, $g(x) \in L_2(-\infty,\infty)$

$$\int_{-\infty}^{\infty} (f^T(x), g(x))\,dx = I/\pi\left[\int_{-\infty}^{\infty} E_1^T(\lambda)\,d\xi(\lambda)\,F_1(\lambda) + \int_{-\infty}^{\infty} E_2^T(\lambda)\,d\eta(\lambda)\,F_1(\lambda) +\right.$$

$$\left. + \int_{-\infty}^{\infty} E_1^T(\lambda)\,d\eta(\lambda)\,F_2(\lambda) + \int_{-\infty}^{\infty} E_2^T(\lambda)\,d\zeta(\lambda)\,F_2(\lambda)\right] \quad (3.4)$$

where (.,.) is the usual inner product of two vectors:

$$E_1(\lambda) = \begin{pmatrix} E_{11} \\ E_{12} \end{pmatrix} = \int_{-\infty}^{\infty} \Phi^T(x,\lambda) f(x)\,dx; \quad E_2(\lambda) = \begin{pmatrix} E_{21} \\ E_{22} \end{pmatrix} = \int_{-\infty}^{\infty} \theta^T(x,\lambda) f(x)\,dx;$$

$$F_1(\lambda) = \begin{pmatrix} F_{11} \\ F_{12} \end{pmatrix} = \int_{-\infty}^{\infty} \Phi^T(x,\lambda)\, g(x)\,dx; \; F_2(\lambda) = \begin{pmatrix} F_{21} \\ F_{22} \end{pmatrix} = \int_{-\infty}^{\infty} \theta^T(x,\lambda)\, g(x)\,dx;$$

and the elements of each of the matrices, ξ, η, ζ are non-decreasing functions of the real variable λ.

The rigorous derivation of (3.3) that is the expansion formula and (3.4), that is the generalized Parseval relation, follow in exactly the same manner as Titchmarsh[5] (Chapters II-III), the only difference lies in proving the non-decreasing characters of each of the elements of the matrices ξ, η, and ζ.

4. On the matrices ξ, η, ζ

Let $\lambda_{n,a,b}$ $\psi_n(a,b,x) = \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}$ be the eigenvalues and eigenvectors for the interval (a,b).

Then,

$$\int_a^b \psi_n^T(a,b,y)\, G_r(a,b,x,y,\lambda)\,dy = \frac{\psi_{rn}(a,b,x)}{\lambda - \lambda_{n,a,b}}$$

where $G_r(\ldots)$ are the Green's vectors (*i.e.*, the column vectors of the Green's matrix $G(a,b,x,y,\lambda)$ with elements $G_{ij}(\cdot)$).

Differentiating with respect to x we have

$$\int_a^b \psi_n^T(a,b,y)\, G_r'(a,b,x,y,\lambda)\,dy = \frac{\psi_{rn}'(a,b,x)}{\lambda - \lambda_{n,a,b}}$$

If $K(\ldots..)$ denotes the various constants depending on the arguments shown, then

Lemma 1. $$\sum_{n=-\infty}^{\infty} \frac{\psi_{rn}^2(a,b,x)}{1 + \lambda_{n,a,b}^2} < K(x)$$

i.e., the left hand side is bounded independently of a, b.

Lemma 2. If $\lambda = \mu + i\nu, \quad 0 < \nu \leq 1$

$$\int_{\alpha}^{\beta} | \operatorname{im} G(x,y,\lambda) | \, d\mu < K(x,y,\alpha,\beta),$$

i.e., the left hand side is bounded independently of ν.

Lemma 3. If $x \neq y$, $0 < \nu \leq 1$,

$$\int_{\alpha}^{\beta} | G(x,y,\lambda) | \, d\mu < K(x,y,\alpha,\beta) \, \nu^{-1/2}$$

Lemma 4. If $0 < \nu \leq 1$

$$\| \int_{\alpha}^{\beta} \operatorname{im} G_r(y,x,\lambda) \, d\mu \|_{-\infty,\infty} < K(x,\alpha,\beta)$$

Lemma 5. If $0 < \nu \leq 1$, and α,β and x fixed,

$$\int_{\alpha}^{\beta} d\mu \, \| G_r(y,x,\lambda) \|_{-\infty,\infty} < K(x,\alpha,\beta) \, \nu^{-1}$$

Lemma 6. For $0 < \nu \leq 1$ and α, β, x fixed

$$\int_{\alpha}^{\beta} d\mu \, \| G_r'(y,x,\lambda) \|_{-\infty,\infty} < K(x,\alpha,\beta) \, \nu^{-1}$$

The lemmas follow in the same way as Titchmarsh[6] (pp. 28-40) and Titchmarsh[5] (p. 57) (also see Tiwary[7], pp. 45-48 and p. 108 for $G_r' \in L_2(-\infty,\infty)$). If $x=0$, by virtue of the initial conditions (1.2) and (1.4), the Green's matrix (2.10) takes the simpler form

$$G(0,y,\lambda) = \frac{1}{m_{11} - M_{11}} (m_{ij}(\lambda)) \, \bar{\chi}^T(y,\lambda), \; y \leq 0$$

$$= \frac{1}{m_{11} - M_{11}} (M_{ij}(\lambda)) \, \psi^T(y,\lambda), \; y > 0$$

where $\psi = \begin{pmatrix} \psi_{11} & \psi_{21} \\ \psi_{12} & \psi_{22} \end{pmatrix}$ and $\psi_r = \begin{pmatrix} \psi_{r1} \\ \psi_{r2} \end{pmatrix}$

Consider first the case $y \leq 0$. Then utilizing the inequality $|a|^2 \leq 2(|a+b|^2 + |b|^2)$, a, b complex, and the lemma 5, it follows that

$$\int_{\mu_1}^{\mu_2} \frac{|m_{rr}|^2}{|M_{11} - m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_r|^2 dy\right) d\mu - 2 \int_{\mu_1}^{\mu_2} \frac{|m_{rs}|^2}{|M_{11} - m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_s|^2 dy\right) d\mu = O(1/\nu)$$

and

$$\int_{\mu_1}^{\mu_2} \frac{|m_{rs}|^2}{|M_{11} - m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_s|^2 dy\right) d\mu - 2 \int_{\mu_1}^{\mu_2} \frac{|m_{rr}|^2}{|M_{11} - m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_r|^2 dy\right) d\mu = O(1/\nu)$$

Thus

$$\int_{\mu_1}^{\mu_2} \frac{|m_{rr}|^2}{|M_{11}-m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_r|^2 \mathrm{d}y\right)\mathrm{d}\mu; \quad \int_{\mu_1}^{\mu_2} \frac{|m_{rs}|^2}{|M_{11}-m_{11}|^2} \left(\int_{-\infty}^{0} |\chi_s|^2 \mathrm{d}y\right)\mathrm{d}\mu \tag{4.1}$$

are each $O(1/\nu)$, where $r \neq s = 1,2$, and χ_r are the column vectors of $\chi(x, \lambda)$.

In exactly similar manner, by considering the case $y > 0$,

$$\int_{\mu_1}^{\mu_2} \frac{|M_{rr}|^2}{|M_{11}-m_{11}|^2} \left(\int_{0}^{\infty} |\psi_r|^2 \mathrm{d}y\right)\mathrm{d}\mu; \int_{\mu_1}^{\mu_2} \frac{|M_{rs}|^2}{|M_{11}-m_{11}|^2} \left(\int_{0}^{\infty} |\psi_s|^2 \mathrm{d}y\right) \mathrm{d}\mu \tag{4.2}$$

are each $O(1/\nu)$ where $r \neq s = 1,2$ and Ψ_r are the column vectors of $\Psi(x, \lambda)$.

By using the relation (A) of § 2, it follows from (4.1) and (4.2) that

$$\int_{\mu_1}^{\mu_2} \frac{1}{|M_{11}-m_{11}|^2} [\,|m_{rr}|^2 \operatorname{im} M_{rr} - |M_{rr}|^2 \operatorname{im} m_{rr}]\mathrm{d}\mu = O(1) \tag{4.3}$$

and

$$\int_{\mu_1}^{\mu_2} \frac{|m_{rs}|^2}{|M_{11}-m_{11}|^2} (\operatorname{im} M_{rr} - \operatorname{im} m_{rr})\, \mathrm{d}\mu = O(1) \tag{4.4}$$

(4.3) and (4.4) are equivalent to

$$\int_{\mu_1}^{\mu_2} \operatorname{im} \left[\frac{m_{rr} M_{rr}}{M_{11}-m_{11}} \right] \mathrm{d}\mu = O(1) \tag{4.5}$$

and

$$\int_{\mu_1}^{\mu_2} |m_{rs}|^2 \operatorname{im} \left(\frac{1}{M_{11}-m_{11}}\right) \mathrm{d}\mu = O(1) \tag{4.6}$$

Again, from (2.10), by differentiating with respect to x, and then putting $x = 0$, we have on utilizing the initial conditions (1.2) and (1.4)

$$G'(0,y,\lambda) = \frac{1}{m_{11} - M_{11}} \bar{\chi}^T(y,\lambda), y \leq 0$$

$$= \frac{1}{m_{11} - M_{11}} \psi^T(y,\lambda), y > 0$$

$\bar{\chi}$ and ψ being defined as before.

Hence, by making use of lemma 6 and the relation (A) of § 2 we have

$$\int_{\mu_1}^{\mu_2} \frac{d\mu}{| M_{11} - m_{11} |^2} (\text{im}\, M_{rr} - im\, m_{rr}) = O(1)$$

which is equivalent to

$$\int_{\mu_1}^{\mu_2} \text{im} \left(\frac{1}{M_{11} - m_{11}}\right) d\mu = O(1) \tag{4.7}$$

By using the Titchmarsh inequality[5] (p. 57) *viz.*,

$[\text{im}\,(a/a-b)]^2 \leq \text{im}\,(1/a-b)\ \text{im}\,(ab/a-b)$, a,b complex, $a \neq b$,

we obtain from (4.5) and (4.7)

$$\int_{\mu_1}^{\mu_2} \text{im} \left[\frac{m_{rr}}{M_{11} - m_{11}} \right] d\mu = O(1) \tag{4.8}$$

The analysis adopted above remains true if M_{kj}, m_{kj} are replaced by iM_{kj} and im_{kj} respectively. Hence as in (4.6) we obtain

$$\int_{\mu_1}^{\mu_2} |m_{rs}|^2 \text{re} \left(\frac{1}{M_{11} - m_{11}}\right) d\mu = O(1) \tag{4.9}$$

From (4.6) and (4.9), we have

$\int_{\mu_1}^{\mu_2} \text{re}\, m_{rs}^2\, \text{im} \left(\frac{1}{M_{11} - m_{11}}\right) d\mu$, $\int_{\mu_1}^{\mu_2} \text{im}\, m_{rs}^2\, \text{re} \left(\frac{1}{M_{11} - m_{11}}\right) d\mu$ are each $O(1)$.

Hence

$$\int_{\mu_1}^{\mu_2} \text{im} \left(\frac{m_{rs}^2}{M_{11} - m_{11}} \right) d\mu = O(1) \tag{4.10}$$

It is easy to verify the identity

$(\text{im}\, ab)^2 = \text{im}\, b.\, \text{im}\,(a^2 b) + (\text{im}\, a)^2\, |b|^2$, for complex numbers a,b. From this, if im b, im $(a^2 b)$ are of the same sign, we have by the obvious inequality $a^2+b^2 \leq (a+b)^2, a,b \geq 0$,

$$|\text{im}\, ab| \leq |\text{im}\, b.\, \text{im}\,(a^2 b)| + |\text{im}\, a|\, |b| \tag{4.11}$$

If im b, im $(a^2 b)$ be of different sign, we have

$$|\text{im}\, ab| \leq |\text{im a}|\, |b| \tag{4.12}$$

In (4.11), put $a = m_{rs}$, $b = \dfrac{1}{M_{11} - m_{11}}$ so as to obtain by the Schwarz inequality

$$\int_{\mu_1}^{\mu_2} \left|\text{im}\, \frac{m_{rs}}{M_{11} - m_{11}}\right| d\mu \leq \left(\int_{\mu_1}^{\mu_2} \text{im}\left(\frac{1}{M_{11} - m_{11}}\right) d\mu\right)^{1/2} \left(\int_{\mu_1}^{\mu_2} \text{im}\left(\frac{m_{rs}^2}{M_{11} - m_{11}}\right) d\mu\right)^{1/2} +$$

$$+ \int_{\mu_1}^{\mu_2} \text{im}\, m_{rs}\, \frac{1}{|M_{11} - m_{11}|}\, d\mu$$

The first term on the right is $O(1)$, by (4.10) and (4.6), the second term is also $O(1)$, since $M_{11} \neq m_{11}$ implies $|M_{11} - m_{11}|\, \delta > 0$

$$\text{and} \int_{\mu_1}^{\mu_2} \text{im}\, m_{rs}\, d\mu = O(1) \text{ (compare Tiwary[7])} \tag{4.13}$$

$$\text{Consequently,} \quad \int_{\mu_1}^{\mu_2} \text{im}\left(\frac{m_{rs}}{M_{11} - m_{11}}\right) d\mu = O(1) \tag{4.14}$$

The result also holds, if $\text{im}\left(\dfrac{1}{M_{11} - m_{11}}\right)$ and $\text{im}\left(\dfrac{m_{rs}^2}{M_{11} - m_{11}}\right)$ differ in sign: to prove this case we use (4.12).

$$\text{Again,} \quad \int_{\mu_1}^{\mu_2} |m_{rs}|^2\, d\mu < \infty,\ r,s = 1,2 \tag{4.15}$$

(compare Titchmarsh[5], p. 43).

Then by the relation

$$4\, \frac{m_{rr}\, m_{rs}}{M_{rr} - m_{rr}} = \left(\frac{m_{rr} + m_{rs}}{(M_{rr} - m_{rr})^{1/2}}\right)^2 - \left(\frac{m_{rr} - m_{rs}}{(M_{rr} - m_{rr})^{1/2}}\right)^2$$

and the inequality.

$$\text{im}\,(a \pm b)^2 \leq 2\,(\,|a|^2 + |b|^2) \text{ and } |M_{11} - m_{11}| = \delta > 0$$

if follows that

$$\int_{\mu_1}^{\mu_2} \text{im}\,(\frac{m_{rr}\, m_{rs}}{M_{rr} - m_{rr}})\ d\mu = O(1) \tag{4.16}$$

Since, $\dfrac{m_{12}\, M_{11}}{m_{11} - M_{11}} = \dfrac{m_{11}\, m_{12}}{m_{11} - M_{11}} - m_{12}$

and $\dfrac{M_{12}\, m_{22}}{m_{11} - M_{11}} = \dfrac{m_{12}\, m_{22}}{m_{11} - M_{11}}$, (for $m_{12} = M_{12}$) it follows from the results obtained before that

$$\int_{\mu_1}^{\mu_2} \text{im}(\frac{m_{12}\, M_{11} + M_{12}\, m_{22}}{M_{11} - m_{11}})\ d\mu = O(1) \tag{4.17}$$

Hence (vide Titchmarsh[5], p. 43, lemma (3.3)) we can establish that the elements ξ_{rs}, η_{rs}, ζ_{11} of the matrices ξ, η, ζ respectively defined by (3.1) are non-decreasing functions of λ (λ real).

A rigorous derivation of the expansion formula (3.3) and the Parseval formula (3.4) can now be obtained by closely following Titchmarsh[5] (Chapter III).

5. The spectral resolution and the generalized orthogonal relation

Let the matrix $H(x, y, \lambda) = (H_{rs}(x, y, \lambda))$, ($\lambda$ real) be defined by

$$\begin{aligned} H(x,y,\lambda) &= \lim_{\nu \to 0} \int_0^{\lambda} \text{im } G(x,y,\sigma+i\nu)\ d\sigma \quad, \quad \lambda > 0 \\ &= -\lim_{\nu \to 0} \int_{\lambda}^{0} \text{im } G(x,y,\sigma+i\nu)\ d\sigma\ , \quad \lambda < 0 \\ &= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad , \quad \lambda = 0 \end{aligned} \tag{5.1}$$

(compare Titchmarsh[6], p. 41, Tiwary[7], p. 49). Then the properties like existence of the limits, bounded variation character of H_{rs}, etc., follow from Titchmarsh and are incorporated in Tiwary's thesis[7] (§§ 2.13-2.14).

Also in the interval $(-\infty, \infty)$

$$\| H_r\,(y, x, \lambda)\|_{-\infty,\infty} < k(x, \lambda) \tag{5.2}$$

where H_r is the rth column vector of the matrix H, and $k(x,\lambda)$ is a constant depending on the arguements shown. By making use of the relation (3.2) the matrix $H(x,y,\lambda)$ has the explicit representation

$$\begin{aligned} H(x,y,\lambda) &= \int_0^{\lambda} [\,\phi(x,\lambda)\,d\xi(\lambda)\,\phi^T(y,\lambda) + \phi(x,\lambda)\,d\eta(\lambda)\,\theta^T(y,\lambda) + \\ &\quad + \theta(x,\lambda)\,d\eta(\lambda)\,\phi^T(y,\lambda) + \theta(x,\lambda)\,d\zeta(\lambda)\,\theta^T(y,\lambda)\,]\,; \quad \lambda > 0 \\ &= -\int_{\lambda}^{0} [\,\phi(x,\lambda)\,d\xi(\lambda)\,\phi^T(y,\lambda) + \phi(x,\lambda)\,d\eta(\lambda)\,\theta^T(y,\lambda) + \\ &\quad + \theta(x,\lambda)\,d\eta(\lambda)\,\phi^T(y,\lambda) + \theta(x,\lambda)\,d\zeta(\lambda)\,\theta^T(y,\lambda)\,]\,; \quad \lambda < 0 \\ &= 0 \qquad\qquad ; \quad \lambda = 0 \end{aligned} \tag{5.3}$$

where the matrices ϕ, θ, ξ, η, ζ are defined as before.

Then the expansion formula (3.3) takes the form

$$f(x) = 1/\pi \lim_{T\to 0} \int_{-\infty}^{\infty} [\,H(x,t,T) - H(x,t,-T)\,]\,f(t)\,dt \tag{5.4}$$

By Green's theorem (see, for example, Chakravarty[3], p. 139), it follows that for non-real

$$\lambda = \mu + i\nu, \qquad \lambda' = \mu' + i\nu', \qquad \lambda \neq \lambda',$$

$$(\lambda - \lambda') \int_a^b G(a,b,t,x,\lambda)\,G(a,b,y,t,\lambda')\,dt = G(a,b,y,x,\lambda) - G(a,b,y,x,\lambda')$$

It is easy to verify that $G(a,b,t,x,\lambda)$ converges in mean square to $G(t,x,\lambda)$; therefore by the familiar extension procedure (vide Titchmarsh[6], p. 58 and Chakarvarty[4]) we have

$$\int_{-\infty}^{\infty} G(t,x,\lambda)\,G(y,t,\lambda')\,dt = \frac{G(y,x,\lambda) - G(y,x,\lambda')}{\lambda - \lambda'} \tag{5.5}$$

Hence (vide Titchmarsh[6], p. 59) we obtain after integration with respect to μ between the limits $(0,\nu)$ and making $\nu \to 0$

$$\int_{-\infty}^{\infty} H(t,x,\nu)\,G(y,t,\lambda')\,dt = \frac{H(y,x,\nu)}{\nu - \lambda'} + \int_0^{\nu} \frac{H(y,x,\mu)}{(\mu - \lambda')^2}\,d\mu \tag{5.6}$$

Equate the imaginary parts of both sides of (5.6), integrate with respect to μ' between the limits $(0,u)$ and proceed as in Titchmarsh[6] (p. 60) by using the theory of the Cauchy singular integral. Then after some reduction we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} H(t,x,v)\, H(y,t,u)\, \mathrm{d}t &= \pi\, H(y,x,u) - \pi/2\, H(y,x,0+0) \; ; \; 0<u<v \\ &= \pi/2\, [\, H(y,x,u) + H(y,x,u-0) - H(y,x,0+0)\,] \; ; \; 0<u=v \\ &= 0 \qquad ; \; 0=u\le v \\ &= -\pi/2\, H(y,x,0+0) \qquad ; \; u<0<v\end{aligned} \tag{5.7}$$

Similarly, for the case $v \le 0$; $u < v < 0$; $u = v < 0$.

Let $\Lambda = (\alpha,\beta)$ and $H(t,x,\Lambda) = H(t,x,\beta) - H(t,x,\alpha)$.

Then, if $\Lambda' = (\alpha',\beta')$ such that $\Lambda \cap \Lambda' = (\alpha,\beta')$

$$\int_{-\infty}^{\infty} H(t,x,\Lambda)\, H(y,t,\Lambda')\, \mathrm{d}t = \pi\, H(y,x,\Lambda \cap \Lambda') \tag{5.8}$$

In particular,

$$\int_{-\infty}^{\infty} H(t,x,\Lambda)\, H(y,t,\Lambda]\, \mathrm{d}t = \pi\, H(y,x,\Lambda) \tag{5.9}$$

The relation (5.8) is the generalized orthogonal relation for $H(t,x,\Lambda)$.

The differential operation M defines on $C_2(-\infty,\infty)$ a symmetric operator on $L_2(-\infty,\infty)$, called the minimal unclosed differential operator. The closure T_1 of this is the minimal differential operator defined by M. Let T be the operator 'generated' by M, so that T is any self-adjoint extention of T_1 (see Glazman[8], pp. 27-28).

Put $K(x,y,\lambda) = H(x,y,\lambda-0) - H(x,y,-\infty)$, (when λ is real).

Then, $K(x,y,\lambda)$ is symmetric in the sense that $K(x,y,\lambda) = K^T(y,x,\lambda)$, H being so. Moreover, since as a function of y and for almost all x (as well as for almost all y when considered as a function of x) $H \in L_2(-\infty,\infty)$, $K(x,y,\lambda)$ does so. The (matrix) kernel $K(x,y,\lambda)$ is thus of the Carleman type.

The operator $E(\lambda)$: $f(x) \to 1/\pi \int_{-\infty}^{\infty} K(x,t,\lambda)\, f(t)\, \mathrm{d}t$

$$\text{i.e. } E(\lambda) f(x) = 1/\pi \int_{-\infty}^{\infty} K(x,t,\lambda)\, f(t)\, dt, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{5.9a}$$

is therefore a linear symmetric operator in the Hilbert space $\mathscr{H}$ (see Stone[9] pp. 101, 398).

From definition, $E(-\infty) = 0$ and from the expansion formula (5 4) $E(\infty) = 1$.
Also (see Titchmarsh[6], p.52) we have for $f, g \in L_2(-\infty, \infty)$

$$\int_{-\infty}^{\infty} (E(\lambda) f, g)\, dx = \int_{-\infty}^{\infty} (E(\lambda) g, f)\, dx$$

showing that $E(\lambda)$ is self-adjoint.

$$\text{Again, } E(\mu)\, E(\lambda)\, f = 1/\pi^2 \int_{-\infty}^{\infty} K(x, t, \mu)\, dt \int_{-\infty}^{\infty} K(t, y, \lambda)\, f(y)\, dy$$

$$= 1/\pi^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K(x, t, \mu)\, K(t, y, \lambda)\, dt \right) f(y)\, dy$$

$$= 1/\pi \int_{-\infty}^{\infty} K(x, y, \Lambda \cap \Lambda')\, f(y)\, dy, \text{ by (5.8)}$$

where

$$\Lambda : (\mu - 0, -\infty), \qquad \Lambda' : (\lambda - 0, -\infty)$$

Thus, $E(\mu)\, E(\lambda) = E(\lambda)$ for $\lambda \leq \mu$

Also evidently $E(\lambda - 0) = E(\lambda)$.

$E(\lambda)$ is thus a projection operator and is, in particular, a resolution of the identity of the operator T.

Put $F(x, \lambda, f) = 1/\pi \int_{-\infty}^{\infty} H(x, y, \lambda)\, f(y)\, dy$, for $f \in L_2(-\infty, \infty)$, so that

$E(\lambda)\, f = F(x, \lambda, f) - F(x, -\infty, f)$.

Then following Titchmarsh[6] (p. 55)

$$E(\mu)\, \tilde{f} = \lim_{\nu \to 0} 1/\pi \int_0^{\mu} \operatorname{im} \{ \Phi(x, \sigma + i\nu, \tilde{f}) - \Phi(x, -\infty + i\nu, \tilde{f}) \}\, d\sigma$$

where $\tilde{f} = Mf \in L_2(-\infty, \infty)$.

In the relation (3.2a) of § 3, we replace $\lambda\ (\equiv \sigma + i\nu)$ by $\lambda'\ (\equiv \sigma' + i\nu')$ subtract the new result from (3.2a), equate imaginary parts from both sides of the result so obtained and finally make $\nu, \nu' \to 0, \sigma \to \infty, \sigma' \to -\infty$.

Then closely following the analysis of Titchmarsh[6] (p. 55), we obtain,

$$\int_{-\infty}^{\infty} (\tilde{f}, g)\, dx = \int_{-\infty}^{\infty} \lambda \left\{ d \int_{-\infty}^{\infty} (E(\lambda) f, g)\, dx \right\}, \qquad \lambda, \text{ real} \tag{5.10}$$

The equation (5.10) is expressed as

$$T = \int_{-\infty}^{\infty} \lambda \, \mathrm{d} E(\lambda) \tag{5.11}$$

where T is the self-adjoint operator generated by the differential operation M.

The results obtained above can now be summarized in the form of the following theorem.

Theorem: To every self-adjoint boundary value problem involving the system (1.1), (1.1a) *over the interval* $(-\infty, \infty)$, *there exists a matrix* $H(x, y, \lambda)$ *explicitly defined by* (5.3) *which satisfies the generalized orthogonal relation* (5.8). $H(x, y, \lambda)$ *generates the operator* $E(\lambda)$ *given by* (5.9a) *which is associated with the self-adjoint operator* T *generated by* M *by means of the relation* (5.11). $E(\lambda)$ *is the spectral resolution or the resolution of the identity of the operator* T.

The matrix $H(x, y, \lambda)$ given by (5.3) is therefore the spectral resolution (or the resolution of the identity) of the differential operation M in (1.1).

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