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The limit-2 case of a second-order differential system

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Abstract

A technique is developed for identifying the system

$$M[\Phi] \equiv \begin{pmatrix} -\dfrac{d}{dx}\left(p\dfrac{d}{dx}\right)+q_1 & q_2 \\ q_2 & -\dfrac{d}{dx}\left(r\dfrac{d}{dx}\right)+q_3 \end{pmatrix} \Psi = \lambda\Psi,\ \lambda\epsilon c$$

to be in the limit-2 case at infinity

Key words: Limit-2 case at infinity, Hilbert space, spectral theory, Lebesgue integrable, linear manifold, bilinear form

1. Introduction

Let M denote the formally symmetric second-order vector-matrix differential expression given by

$$M[\Psi] \equiv \begin{pmatrix} -\dfrac{d}{dx}\left(p\dfrac{d}{dx}\right)+q_1 & q_2 \\ q_2 & -\dfrac{d}{dx}\left(r\dfrac{d}{dx}\right)+q_3 \end{pmatrix} \Psi \qquad (1.1)$$

Ψ being a complex-valued vector function $\Psi = \binom{f}{g}$, suitably differentiable on the interval $(0, \infty)$ and where the coefficients p, r and q_j $(j = 1,2,3)$ satisfy the following conditions:

(i) $p(x)$, $r(x)$ are real-valued and positive for all x on $(0,\infty)$ and are absolutely continuous on all compact sub-intervals of $(0,\infty)$.

(ii) q_j $(j = 1,2,3)$ are real-valued and continuous on $(0,\infty)$.

The Hilbert space H in which the spectral theory of M is developed is that of complex-valued vector-functions $\Psi = \binom{f}{g}$ such that

$$\int_0^\infty \{ |f|^2 + |g|^2 \} \, dx < \infty \tag{1.2}$$

or, equivalently, each of Re(f), Re(g), Im(f), Im(g) is square-integrable on $(0,\infty)$; we express these by writing Re(f), Re(g), Im(f), Im(g) $\in L^2(0,\infty)$. The inner product of two vectors $\Psi = \binom{f}{g}$ and $\Phi = \binom{u}{v}$ is defined by

$$(\Psi,\Phi) = \int_0^\infty (f\bar{u} + g\bar{v}) \, dx.$$

It is known [See Chakravarty [1], Sengupta [2], Naimark [3] (§. 17.5 VII) and Glazman [4] (Ch. I. §. 13] that the differential system

$$M[\Psi] = \lambda\Psi, \qquad \text{Im}\lambda \neq 0 \tag{1.3}$$

possesses at least two and at most four linearly independent solutions on $(0,\infty)$ which lie in H. $M[\cdot]$ is said to be in the limit-S case at infinity if the differential system (1.3) has exactly S number of linearly independent solutions in H. Given p, r, q_1, q_2, q_3 the number S is independent of λ, as long as im $\lambda \neq 0$. The idea of this paper is to establish a general set of sufficient conditions on the coefficients p,r,q_1,q_2,q_3 so that $M[\cdot]$ is in the limit-2 case at infinity. Several methods have been used for investigating the limit-2 case for the system (1.3) or for one similar to it. In 1954, Lidskii [5] showed that the system

$$-Y'' + QY = \lambda Y, \ \text{Im}\ \lambda \not\lessgtr 0 \tag{1.4}$$

possesses k number of linearly independent square-integrable solutions on $(0,\infty)$ provided the square hermitian matrix $Q(x)$ of order k satisfies

$$(Q(x)\, h,h) \geq -N(x) \ \|h\|^2$$

for any constant k-vector h, where the positive continuous function $N(x)$ satisfies

(i) $\int_0^\infty [N(x)]^{-1/2}\, dx$ diverges

and, either

(ii) $N(x)$ is monotone

or,

(iii) $N(x)$ is differentiable and $\limsup_{x\to\infty} |N'(x)| \ [N(x)]^{-3/2} < \infty$. (1.5)

Sear's result[6] can be derived from Lidskii's result by putting $k=1$. Chakravarty[7] (Th. III) proved in a different way that the system

$$M_1[\psi] \equiv \begin{pmatrix} q_1 & -\frac{d^2}{dx^2}+q_2 \\ -\frac{d^2}{dx^2}+q_2 & q_3 \end{pmatrix} \psi = \lambda\psi \tag{1.6}$$

is in the limit-2 case at infinity if q_1, q_2, q_3 are all $0(x^2)$ as $x\to\infty$. Anderson[8] discussed the system

$$\Psi^{(2n)} + Q\Psi = \lambda\Psi \tag{1.7}$$

where Q is a $k \times k$ matrix of real measurable functions which are Lebesgue integrable on compact sub-intervals of $(0,\infty)$ and Ψ is a k-vector, and extended the results of Lidskii[5] to the case when the system (1.7) possesses the minimum number (*viz. nk*) of square-integrable solutions on $(0,\infty)$. The method applied by Anderson is similar to that applied by Hinton[9] to the corresponding scalar equation. In particular, if $n = 1$, $k = 2$ Anderson proved that [Th. 2.4], the system

$$\psi'' + \begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix} \psi = i\psi$$

is in the limit-2 case at infinity if $q_1, q_3, |q_2| \leq N(x)$ for $N(x)$ as in (1.5). Following Titchmarsh[10] (Th. 2.20) Bhagat and Guma[11] (§ 5) pointed out that the system (1.3) with $p=r=1$ is in the limit-2 case at infinity, if $q_2=0(1)$ and $q_1, q_3 \geq -N(x)$ is a positive, continuous non-decreasing function of x satisfying condition (i) of (1.5). A complete analysis of the system (1.6) has been made by Eastham[12] when $q_j's$, $j=1,2,3$ are multiples of powers of x, giving conditions under which $S=2$ or $S=3$ or 4. In this connection mention should also be made of the papers by Titchmarsh[13], Shaw and Bhagat[14], Sengupta[15,16], Eastham[17] and Everitt[18-20]

In this paper, we present a simpler method to establish that the system (1.3) is in the limit-2 case at infinity under suitable conditions imposed on the coefficients p, r, q_1, q_2, q_3 which will include the cases mentioned earlier. The method employed is based on an extension of a technique given in Levinson[21] or Coddington and Levinson[22] (Th. 2.4 Ch. 9, Sec. 2). The result obtained is given in the following theorem:

Theorem: Let $N(x)$ be a positive, absolutely continuous and non-decreasing function of x such that

(i) $\int_0^{\infty} [PN]^{-1/2}\,\mathrm{d}x$ *diverges*, $P = max\,(p, r)$ (1.8)

(ii) *lim* $sup_{x\to\infty}\; N' \sqrt{[p/N^3]}$ *and lim* $sup_{x\to\infty}\; N' \sqrt{(r/N^3)}$ *exist finitely and moreover,* (1.9)

(iii) $q_1(x) \geq -k_1\, N(x)$, $q_3(x) \geq -k_3\, N(x)$ *and* $|\,q_2(x)| \leq k_2\, N(x)$ (1.10)

(k_1, k_2, k_3 *are all finite positive constants) hold for all sufficiently large values of* x.

Then $M[\cdot]$ *is in the limit-2 case at infinity.*

The proof is given in the following section. In proving the theorem we extract a function

$$W(x) = \int_a^x [(\theta')^T R\, \theta'/N]\,\mathrm{d}x \qquad [R = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix}],$$

from the equation

$$\int_a^x (\theta^T\; M[\theta]/N)\;\mathrm{d}x = i\int_a^x (\theta^T\,\theta/N)\;\mathrm{d}x,$$

converging to a finite limit as $x\to\infty$, which later produces $(R\theta'/\sqrt{PN}) \in H$ for all $\theta \in D$ [See section 2 for definition of D]. Finally the theorem follows on utilising the last result along with (1.8) and (2.1).

2. Proof of the theorem

We introduce a linear manifold D as follows:

A vector-valued function $\Psi = \binom{f}{g}$ is in D if and only if

(i) $\Psi \in H$
(ii) f', g' are absolutely continuous on $(0, \infty)$
(iii) $M[\Psi] \in H$

For $\Psi = \binom{f}{g}$, $\Phi = \binom{u}{v} \in D$, it is known from Green's formula that

$$\int_0^x \bar{\Phi}^T\; M[\Psi]\,\mathrm{d}x - \int_0^x \Psi^T\, \overline{M[\Phi]}\;\mathrm{d}x = \{\,p\,(f\bar{u}' - f'\bar{u}) + r\,(g\bar{v}' - g'\bar{v})\,\}_0^x$$

and the bilinear form

$$[\Psi\;\Phi] = p\,(f\bar{u}' - f'\bar{u}) + r\,(g\bar{v}' - g'\bar{v}) \text{ tends to a finite limit as } x\to\infty \tag{2.1}$$

and that

$$\lim_{x\to\infty}\;[\Psi\;\Phi] = 0 \tag{2.2}$$

for all Ψ, $\Phi \in D$ if and only if M is in the limit-2 case at infinity [See Sengupta[2] Th. 6.2; Naimark[3] § 18.3 lemma].

Since the number of L^2–solutions of the system (1.3) remains uncharged as long as im $\lambda \neq 0$, we start to prove the theorem by choosing $\lambda = i$ in it.

Let $\Psi = \begin{pmatrix} f \\ g \end{pmatrix} \equiv \begin{pmatrix} f_1 + if_2 \\ g_1 + ig_2 \end{pmatrix} \epsilon\ D$ be a solution of $M[\Psi] = i\Psi$ satisfying the initial conditions

$$\left.\begin{aligned} f(a) = \alpha,\ g(a) = \beta \\ p(a) f'(a) = \gamma\ ,\ r(a) g'(a) = \delta \end{aligned}\right\} a > 0$$

$\alpha,\beta,\gamma,\delta$ are finite complex constants [For existence of the initial conditions, see Sengupta[2] Th. 3.1]. Multiply both sides of $M[\Psi] = i\Psi$ by $(\overline{\Psi}^{-T}/N)$, integrate between a and x, and then integrating the right-hand side by parts, we get

$$-\int_a^x \frac{q_1 f\bar{f} + q_2(f\bar{g} + \bar{f}g) + q_3 g\bar{g}}{N}\,dx + i\int_a^x \frac{|f|^2 + |g|^2}{N}\,dx = -\int_a^x \frac{(pf')'\bar{f} + (rg')'\bar{g}}{N}\,dx$$

$$= -\left[\frac{pf'\bar{f} + rg'\bar{g}}{N}\right]_a^x + \int_a^x \frac{pf'\bar{f}' + rg'\bar{g}'}{N}\,dx - \int_a^x \frac{(pf'\bar{f} + rg'\bar{g})N'}{N^2}\,dx$$

Taking real parts from both the sides,

$$-\int_a^x \frac{q_1|f|^2 + 2q_2(f_1g_1 + f_2g_2) + q_3|g|^2}{N}\,dx = -\frac{p(f_1f_1' + f_2f_2') + r(g_1g_1' + g_2g_2')}{N}\Big|_a^x +$$

$$+\int_a^x \frac{p|f'|^2 + r|g'|^2}{N}\,dx - \int_a^x \frac{p(f_1f_1' + f_2f_2') + r(g_1g_1' + g_2g_2')}{N^2}\,N'\,dx$$

then by condition (1.10) l.h.s. satisfies the inequality

$$-\int_a^x \frac{q_1|f|^2 + 2q_2(f_1g_1 + f_2g_2) + q_3|g|^2}{N}\,dx \leq -\int_a^x \frac{q_1|f|^2 + q_3|g|^2}{N}\,dx +$$

$$+2\int_a^x \frac{|q_2|\,|f_1g_1 + f_2g_2|}{N}\,dx < k_1\int_a^x |f|^2 dx + k_3\int_a^x |g|^2 dx + 2k_2\int_a^x |f_1g_1 + f_2g_2|\,dx$$

Hence there exists a constant K such that

$$K > -\frac{p(f_1f_1' + f_2f_2') + r(g_1g_1' + g_2g_2')}{N} + \int_a^x \frac{p|f'|^2 + r|g'|^2}{N}\,dx -$$

$$-\int_a^x \frac{p(f_1f_1' + f_2f_2') + r(g_1g_1' + g_2g_2')}{N^2}\,N'dx,\ (\forall x) \tag{2.3}$$

Now it is to be proved that if the solution $\Psi \in D$ then the integral

$$\int_a^\infty \frac{p|f'|^2 + r|g'|^2}{N} \, dx$$

converges. For, suppose conversely that this integral diverges, then the function

$$W(x) = \int_a^x \frac{p|f'|^2 + r|g'|^2}{N} \, dx$$

is positive, monotonically increasing and tends to $+\infty$ as $x \to \infty$. Using condition (1.9) and then the Cauchy-Schwartz inequality results in

$$\left| \int_a^x \frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N^2} N' \, dx \right|$$

$$< \int_a^x [\, | \sqrt{p/N^3}\, N' \sqrt{P/N}(f_1 f_1' + f_2 f_2') | + | \sqrt{r/N^3}\, N' \sqrt{r/N}(g_1 g_1' + g_2 g_2') | \,] dx$$

$$< K_1 \int_a^x [\, \sqrt{P/N}\, | f_1 f_1' + f_2 f_2' | + \sqrt{r/N}\, | g_1 g_1' + g_2 g_2' | \,] \, dx$$

$$\leq K_1 \int_a^x \{ (f_1^2 + f_2^2) + (g_1^2 + g_2^2) \}^{1/2} \left\{ \frac{p(f_1'^2 + f_2'^2) + r(g_1'^2 + g_2'^2)}{N} \right\}^{1/2} dx$$

$$< K_1 \left\{ \int_a^x (|f|^2 + |g|^2) \, dx \right\}^{1/2} \left\{ \int_a^x \frac{p|f'|^2 + r|g'|^2}{N} \, dx \right\}^{1/2}$$

$$< K_2 \sqrt{W(x)}$$

Applying these results in (2.3), we find that

$$K > W(x) - \frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N} - K_2 \sqrt{W(x)}, \; (\forall x)$$

Since $W(x) \to \infty$, as $x \to \infty$, the last inequality can hold only if

$$\frac{p(f_1 f_1' + f_2 f_2') + r(g_1 g_1' + g_2 g_2')}{N} > 1/2\, W(x)$$

for all sufficiently large x. As p, r and N are positive it appears from the above inequality that at least one of the pairs $f_1, f_1'; f_2, f_2'; g_1, g_1'; g_2, g_2'$ is of the same sign for large x. In this situation at least one of the four integrals

$$\int^{\infty} f_1^2\, dx,\ \int^{\infty} f_2^2\, dx,\ \int^{\infty} g_1^2\, dx,\ \int^{\infty} g_2^2\, dx$$

fails to exist and this contradicts the fact that $\Psi \in D$. Thus, $W(x)$ remains finite for $\Psi \in D$ and that

$$\int_a^{\infty} \frac{p|f'|^2 + r|g'|^2}{N}\, dx < \infty,$$

it then follows

$$\sqrt{p/N}\,|f'|,\quad \sqrt{r/N}\,|g'| \quad \in \quad L^2(0,\infty)$$

and consequently

$$\frac{p\,|f'|}{\sqrt{PN}} \leq \frac{p\,|f'|}{\sqrt{pN}} = \sqrt{p/N}\,|f'| \in L^2(0,\infty), \tag{2.4a}$$

$[P = \max(p, r)]$ and like wise

$$\frac{r\,|g'|}{\sqrt{PN}} \in L^2(0,\infty) \tag{2.4b}$$

for all $\Psi \in D$.

Utilizing the results obtained we now show that $\lim_{x\to\infty} [\Psi\,\Phi] = 0$ for any solution $\Psi, \Phi, \in D$. From (2.1) we find

$$\int_a^x \frac{|[\Psi\Phi]|}{\sqrt{PN}}\, dx = \int_a^x \frac{|pf\bar{u}' - pf'\bar{u} + rg\bar{v}' - rg'\bar{v}|}{\sqrt{PN}}\, dx$$

$$\leq \int_a^x \frac{p|f|\,|\bar{u}'| + p|f'|\,|\bar{u}| + r|g|\,|\bar{v}'| + r|g'|\,|\bar{v}|}{\sqrt{PN}}\, dx$$

The integral on the right side converges as x tending to infinity following the result (2.4) for $\Psi, \Phi \in D$ and consequently

$$\int_a^{\infty} \frac{|[\Psi\Phi]|}{\sqrt{PN}}\, dx$$

converges. Now, if $\lim_{x\to\infty} [\Psi\Phi]=k$, a finite limit ($\neq 0$), we can find an $X_k \in (0,\infty)$, depending on k such that

$$1/2\,|k| < |[\Psi\Phi]| < 3/2\,|k|$$

hold for all $x > X_k$. Then

$$\int_a^X \frac{|[\Psi\Phi]|}{\sqrt{PN}}\,dx = \left(\int_a^{X_k} + \int_{X_k}^X\right) \frac{|[\psi\phi]|}{\sqrt{PN}}\,dx \geq I_1 + 1/2\,|k| \int_{X_k}^X dx/\sqrt{PN} \to \infty,$$

as $X \to \infty$ contradictory to (2.5) and the desired result $\lim_{x\to\infty} [\Psi\Phi] = 0$ is achieved, which ensures the system $M[\cdot]$ to be in the limit-2 case at infinity.

Remark 1. Some discrepancy is found in between the papers of Gadamsi-Mahto[23] and Eastham-Gould[24]. In Theorem 3[23] the authors tried to apply the techniques of Titchmarsh[13] and Everitt[20] in proving the system (1.3) to be in the limit-2 case at infinity; the conditions taken there were

$$\text{(i) } 0 < p, r \leq k\,x^{\beta} \quad \text{(ii) } q_1, q_3 \geq -k_1 x^{\alpha},\ q_2 \geq -k_2 x^{\gamma}$$

with $\alpha+\beta \leq 2$, $\alpha \geq 0$, $\beta-\alpha \leq 2\gamma \leq \alpha$ and k, k_1, k_2 all positive finite constants; where as in Theorem 1 (ii), §. 3[24] it was proved that the system (1.3), (with $p=r=1$) is in the limit-3 case at infinity provided

$$q_1 = a|q_2|,\ q_3 = b|q_2|,\ a \geq 0,\ b \geq 0,\ ab < 1$$

where q_2 be no where zero in some interval $[X,\infty)$, $X \geq 0$ and $q_2^{-1/4}(q_2^{-1/4})'' \in L^2(X,\infty)$.

As for an example, if we take $p = r = 1$ and $q_1 = q_3 = 1/2\,x^3$, $q_2 = x^3$, $x \in (0,\infty)$, then following Eastham-Gould[24] the system (1.3) turns out to be in the limit-3 case at infinity though the coefficients p, r, q_1, q_2, q_3 satisfy the conditions of Gadamsi-Mahto[23].

Remark 2. It appears from the previous results (except Gadamsi-Mahto[23]) especially Anderson[8], Th.2.4 and the theorem of the present paper that for a system of the type (1.3), belonging to the limit-2 case at infinity, q_2 should satisfy

$$|q_2(x)| \leq K\,N(x)$$

along with other restrictions on q_1 and q_3.

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References

1. CHAKRAVARTY, N K. Some problems in eigenfunction expansions IV, *Indian J. Pure Appl. Math.*, 1970, **1**, 347-353.
2. SENGUPTA, P.K On a self-adjoint extention of a type of a matrix differential operator, *J. Indian Inst. Sci.*, 1978, **60(B)**, 225-234.
3. NAIMARK, M A. *Linear differential operators*, Part II, Harrap, London, 1968.
4. GLAZMAN, I.M. *Direct methods for the qualitative spectral analysis of singular differential operators* (Eng. trans.), Israel Programme for Sci. Trans, Jerusalem, 1965.
5. LIDSKII, V.B. On the number of quadratically integrable solutions of the system of differential equations — $Y'' + P(t)\ Y = \lambda\ Y$, *Doklad Akad. Nauk SSSR*, 1954, **95**, 217-220 (Russian).
6. SEARS, D.B. Note on the uniqueness of the Green's functions associated with certain differential equations, *Can. J. Math.*, 1950, **2**, 314-325.
7. CHAKRAVARTY, N.K. Some problems in eigenfunction expansion III, *Q.J. Math. Oxford (2)*, 1968, **19**, 397-415.
8. ANDERSON, R.L. Limit-point and limit-circle criteria for a class of singular symmetric differential operator, *Can. J. Math.*, 1976, **28**, 905-914.
9. HINTON, D. Limit point criteria for differential equations, *Can J. Math.*, 1972, **24**, 293-305.
10. TITCHMARSH, E.C. *Eigenfunction expansions associated with second-order differential equations*, Part I, 2nd edition, Oxford University Press, London, 1962.
11. BHAGAT, B. AND GUMA, S. Deficiency indices of second-order matrix differential operator, *Indian J. Pure Appl. Math*, 1982, **13**, 433-439.
12. EASTHAM, M.S.P. The deficiency index of a second-order differential system, *J. Lond. Math, Soc. (2)*, 1981, **16**, 311-320.
13. TITCHMARSH, E.C. On the uniqueness of the Green's function associated with a second-order differential equation, *Can. J. Math.*, 1950, **1**, 191-198.
14. SHAW, S. AND BHAGAT, B. On a second-order matrix differential operator, *Proc. Indian. Acad. Sci.*, 1974, **79A**, 213-222.
15. SENGUPTA, P.K. Integrable-square solutions of certain vector matrix differential equation, *J. Indian Inst. Sci.*, 1978, **60(B)**, 235-245.
16. SENGUPTA, P.K. L^2-classification of vector-matrix differential equation, *J. Pure Math.* 1981, **1**, 67-72.
17. EASTHAM, M.S.P. The limit-2 case of fourth-order differential equations, *Q. J. Math. Oxford(2)*, 1971, **22**, 131-134.
18. EVERITT, W.N. Singular differential equations II; Some self-adjoint even order case, *Q. J. Math. Oxford(2)*, 1967, **18**, 13-22.
19. EVERITT, W.N. Some positive definite differential operators, *J. Lond. Math Soc. (2)*, 1968, **43**, 465-473.
20. EVERITT, W.N. On the limit-point classification of fourth-order differential operators, *J. Lond. Math. Soc. (2)*, 1969, **44**, 273-281.
21. LEVINSON, N. Criteria for the limit-point case for 2nd-order linear differential operators, *Chasopis Pchst Math. Fys.*, 1949, **74**, 17-20.
22. CODDINGTON, E.A. AND LEVINSON, N. *Theory of ordinary differential equations*, McGraw-Hill, N.Y., 1955.

23. GADAMSI, A.M AND MAHTO, K.R. On the limit-2 case of second-order matrix differential equations, *Indian J Pure Appl. Math.* 1978, **9**, 653-660

24 EASTHAM, M.S.P. AND GOULD, K.J. Square-integrable solutions of a matrix differential expression, *J Math. Anal. Appl.* 1983, **91**, 424-433.