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Short Communication

A note on the separation property of symmetric fourth-order differential expressions

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Abstract

The idea of a 'separated' fourth-order differential expression has been introduced. It has been proved that if a fourth-order symmetric differential expression is separated then it satisfies the Dirichlet property and hence it is in the strong limit-2 case at infinity. The above results have been generalized to symmetric even-order differential expressions.

Key words: Strong limit-2 case at infinity, Dirichlet property, separation.

1. We consider the fourth-order differential expression

$$L[f] \equiv (rf^{(2)})^{(2)} - (pf^{(1)})^{(1)} + qf \text{ on } [0, \infty), \tag{1.1}$$

where the real-valued coefficients r, p and q satisfy the following:

(i) q is locally Lebesgue integrable on $[0, \infty)$,
(ii) $r^{(1)}$ and p are locally absolutely continuous on $[0, \infty)$ and
(iii) $r(x)$ and $p(x)$ are positive for all $x \in [0, \infty)$. (1.2)

The *linear manifold* $\Delta \subset L^2(0, \infty)$ may be defined as $f \in \Delta$ if (i) $f \in L^2(0, \infty)$ (ii) $f^{(3)}$ is locally absolutely continuous on $[0, \infty)$ and (iii) $L[f] \in L^2(0, \infty)$. (1.3)

We define the *linear manifold* $\Delta_L \subset \Delta \subset L^2(0, \infty)$ as $f \in \Delta_L$, if f satisfies (i) — (iii) of (1.3) and in addition (iv) $f(0) = f^{(1)}(0) = f^{(2)}(0) = 0$ (1.4)

Then both Δ and Δ_L are dense in $L^2(0, \infty)$ [4].

The operator $L[\cdot]$ is said to satisfy the *Dirichlet property* if

(i) $r^{1/2}f^{(2)}, p^{1/2}f^{(1)}, |q|^{1/2}f \epsilon L^2(0,\infty)$ for all $f \epsilon \Delta$ and (ii) $\int_0^\infty (rf^{(2)}\bar{g}^{(2)} + pf^{(1)}\bar{g}^{(1)} + qf\bar{g})dx$
$= [(r\bar{g}^{(2)})f - rf^{(1)}\bar{g}^{(2)} - pf\bar{g}^{(1)}](0) + \int_0^\infty fL[g]$. (1.5)

$L[\cdot]$ is said to be *separated* in Δ if $(rf^{(2)})^{(2)}, (pf^{(1)})^{(1)}$ and qf are separately in $L^2(0,\infty)$ for all $f \epsilon \Delta$. (1.6)

$L[\cdot]$ is said to be in the *strong limit-2 case* at infinity if $\lim_{X\to\infty} [pf\bar{g}^{(1)} + rf^{(1)}\bar{g}^{(2)} - f(r\bar{g}^{(2)})^{(1)}](X) = 0$. (1.7)
for all $f, g \epsilon \Delta$.

We have,

$$\int_0^X (rf^{(2)}\bar{g}^{(2)} + pf^{(1)}\bar{g}^{(1)} + qf\bar{g})\,dx = [pf\bar{g}^{(1)} + rf^{(1)}\bar{g}^{(2)} - f(r\bar{g}^{(2)})]_0^X + \int_0^X f\,L[\bar{g}]\,dx \quad (1.8)$$

for all $f, g \epsilon \Delta$.

In particular, for $f = g \epsilon \Delta$ we have, from (1.7),

$$\int_0^X \{r|f^{(2)}|^2 + p|f^{(1)}|^2 + q|f|^2\}\,dx = [pf\bar{f}^{(1)} + rf^{(1)}\bar{f}^{(2)} - f(r\bar{f}^{(2)})^{(1)}](0) + \int_0^X f\,L[f]\,dx \quad (1.9)$$

From § 3[1], if (1.9) holds for real-valued functions, then it is also true for complex-valued functions. Hence, though Δ and Δ_L are defined for complex-valued functions it is sufficient if we consider our results for real-valued functions only.

Some properties of functions, square integrable on $[0,\infty)$ are enlisted in the following lemmata.

Lemma 1. If $f \epsilon L^2(0,\infty)$ or $f \epsilon L(0,\infty)$, and $f^{(1)}$ absolutely continuous on any compact subinterval of $[0,\infty)$ then $f(x), f^{(1)}(x), f^{(2)}(x)$ tend to 0 as $X \to \infty$.

Proof: If $\lim_{x\to\infty} f(x) = \eta \neq 0$, then $|f(x) - \eta| < \epsilon$ for $x > X$
i.e., $\eta - \epsilon < f(x) < \eta + \epsilon$ for $x > X$.

As $f \epsilon L^2(0,\infty)$, for $X_1 > X, \int_X^{X_1} f^2(x)\,dx \leq \int_0^\infty f^2(x)\,dx < \infty$.

But $\int_X^{X_1} f^2(x)\,dx > \int_X^{X_1} (\eta - \epsilon)^2\,dx = (\eta-\epsilon)^2\,(X_1 - X) \to \infty$ as $X_1 \to \infty$

which is contradictory to the above result.

Hence $\lim_{x\to\infty} f(x) = 0$. A similar proof holds if $f \epsilon L(0,\infty)$.

Also $\int_0^\infty f^{(1)}(t)\,dt = \lim_{x\to\infty} [f(x) - f(0)] = -f(0) => f^{(1)} \epsilon L(0,\infty) => \lim_{x\to\infty} f^{(1)}(x) = 0$

Lemma 2. If $f \epsilon L^2(0,\infty)$, and $f, f^{(1)}, f^{(2)}$ are continuous on $[0,\infty)$, then for large $X, f(x)f^{(1)}(x) < 0$ and $f^{(1)}(x)f^{(2)}(x) < 0$.

Proof: If $ff^{(1)}(x) > 0$, then f and $f^{(1)}$ are of the same sign, say $f(x) > 0$ and $f^{(1)}(x) > 0$; then $f(x) > 0$ and f is increasing.

But $f \in L^2(0,\infty) \Rightarrow \lim_{x\to\infty} f(x) = 0$ by lemma 1. This leads to a contradiction.

By a similar argument, we can prove that $f^{(1)}(x)f^{(2)}(x) < 0$ for large x.

Lemma 3. For any $f \in \Delta_L$ and q satisfying (1.2) if $qf \in L^2(0,\infty)$, then $|q|^{1/2} f \in L^2(0,\infty)$.

The proof is exactly similar to that of lemma 1, § 3[2] and hence is omitted here.

In this paper we prove the following theorem.

Theorem. If $L[\cdot]$ is separated in $L^2(0,\infty)$, then $L[\cdot]$ satisfies the Dirichlet property and hence $L[\cdot]$ is in the strong limit-2 case at infinity.

Proof: We assume that $L[\cdot]$ is separated *i.e.,* $L[\cdot]$ satisfies (1.6).

Since Δ_L is dense in Δ, it is sufficient if we prove the theorem for $f \in \Delta_L$. So we consider any $f \in \Delta_L$.

From (1.6) we know that $qf \in L^2(0,\infty)$. Hence from lemma 3, we have $|q|^{1/2} f \in L^2(0,\infty)$ $\forall$ $f \in \Delta_L$. (1.11)

$$\text{Now} \int_0^X pf^{(1)} g^{(1)}\,dx = p(X)f(X)g^{(1)}(X) + \int_0^X f[-(pg^{(1)})^{(1)} + qg]\,dx - \int_0^X qfg\,dx \qquad (1.12)$$

for all $f, g \in \Delta_L$ as $f(0) = g^{(1)}(0) = 0$.

In particular for $f = g$

$$\int_0^X pf^{(1)2}\,dx = (pff^{(1)})(X) + \int_0^{X_1} f[-(pf^{(1)})^{(1)} + qf]\,dx - \int_0^X qf^2 dx \qquad (1.13)$$

Since, by the definition of separation, $(pf^{(1)})^{(1)}$ and qf are in $L^2(0,\infty)$ for all $f \in \Delta_L$ and by lemma 3, $|q|^{1/2} f \in L^2(0,\infty)$, each of the integrals on the right hand side of (1.13) tends to a finite limit as $X \to \infty$. Since $p(x) > 0$ by (1.2) (iii), the left-hand side of (1.13) is positive and tend to $+\infty$ if and only if $(pff^{(1)})(X) \to +\infty$ as $X \to \infty$. But this is not possible, as $p(x) > 0$ and $f \in L^2(0,\infty)$ implies f and $f^{(1)}$ cannot have the same sign. Hence $pff^{(1)}(X) \to$ a finite limit as $X \to \infty$. So $p^{1/2} f^{(1)} \in L^2(0,\infty)$ for all $f \in \Delta_L$. (1.14)

Hence by (1.11), (1.12) and (1.14), $p(X)f(X)g^{(1)}(X)$ tends to a finite limit as $X \to \infty$. (1.15)

In fact, it can be shown that this limit of (1.15) is zero [with the help of a lemma analogous to a lemma in § 2 [3]].

Hence

$$\lim p(X)f(X)g^{(1)}(X) = 0 \text{ for all } f, g \in \Delta_L. \qquad (1.16)$$

Now

$$\int_0^X rf^{(2)2}\,dx = rf^{(1)}f^{(2)}(X) - (rf^{(2)})^{(1)} f(X) + \int_0^X (rf^{(2)})^{(2)} f\,dx,\ f \in \Delta_L \qquad (1.17)$$

As $L[\cdot]$ is separated, $(rf^{(2)})^{(2)} \epsilon L^2(0,\infty)$ and $f \epsilon L^2(0,\infty)$ and so

$$\int_0^X f(rf^{(2)})^{(2)} < \infty \text{ as } X \to \infty.$$

By (1.2) (iii), $r(x) > 0$ and hence

$$rf^{(1)}f^{(2)}(X) - (rf^{(2)})^{(1)}f(X) \to +\infty \text{ if } \int_0^X \{rf^{(2)}\}^2 \to \infty \text{ as } X \to \infty.$$

Now $rf^{(1)}f^{(2)}(X) - (rf^{(2)})^{(1)}f(X) \to +\infty$ would mean at least one of the terms will tend to $+\infty$. We shall show that this leads to a contradication.

Suppose $-(rf^{(2)})^{(1)}f(X) \to +\infty$ *i.e.*, $(rf^{(2)})^{(1)}f(X) \to -\infty$ as $X \to \infty$. Then $f(X)$ and $(rf^{(2)})^{(1)}(X)$ must be of different sign *i.e.*, if $f(X) > 0$, then $(rf^{(2)})^{(1)}(X) < 0$. By lemma 1, $f \epsilon L^2(0,\infty)$ implies $f(X) \to 0$ as $X \to \infty$. Hence it must be that $(rf^{(2)})^{(1)}(X) \to -\infty$ as $X \to \infty$. But this implies $rf^{(2)}(X) \to -\infty$ as $X \to \infty$. Again, as $r(x) > 0$, $f^{(2)}(X)$ must be < 0 which, by lemma 2, is impossible, since $f(X) > 0$. The case $f(X) < 0$ may be similarly discussed.

If $rf^{(1)}f^{(2)}(X) \to +\infty$, $r(X)$ being positive, it follows that $f^{(1)}, f^{(2)}$ are of the same sign. But by lemma 2 this is not possible. Hence $rf^{(1)}f^{(2)}(X) + \infty$ as $X \to +\infty$.

Therefore, not only $rf^{(1)}f^{(2)}(X) - (rf^{(2)})^{(1)}f(X) \to$ a finite limit as $X \to \infty$, but both $rf^{(1)}f^{(2)}(X)$ and $(rf^{(2)})^{(1)}f(X)$ must tend to finite limits as $X \to \infty$. It then follows from (1.17) that $r^{1/2}f^{(2)} \epsilon L^2(0,\infty)$ for all $f \epsilon \Delta_L$. (1.18)

In fact, we shall show that $\lim_{X\to\infty} rf^{(1)}f^{(2)}(X) = \lim_{X\to\infty}(rf^{(2)})^{(1)}f(X) = 0.$ (1.19)

If possible, suppose $\lim_{X\to\infty}(rf^{(2)})^{(1)}(X) = +\infty$. Then $\lim_{X\to\infty} rf^{(2)}(X) = +\infty$ *i.e.*, given $M > 0$, there exists $X > 0$, such that $r(x)f^{(2)}(x) > M$ for $x > X$. So, for $x > X$, (note that $r(x) > 0$ $\forall x \epsilon [0,\infty)$), $\int_X^x f^{(2)}(x) > \int_X^x M/r(t)\, dt$

This implies that

$f^{(1)}(X) \to +\infty$ as $X \to \infty$, which is contradictory to lemma 1.

A similar contradiction can be obtained for the case $\lim_{X\to\infty}(rf^{(2)})^{(1)}(X) = -\infty$. Hence, $(rf^{(2)})^{(1)}(X)$ tends to a finite limit as $X \to +\infty$. As $f \epsilon L^2(0,\infty)$, it therefore follows that $\lim_{X\to\infty}(rf^{(2)})^{(1)}(x)f(X) = 0$.

Now suppose, $\lim_{X\to\infty} rf^{(2)}f^{(1)}(X) =$ finite $= C$ (say) and $C \neq 0$.

Then $|rf^{(1)}f^{(2)}(x)| > C_1 > 0$ for large X.

This implies, for $X > X_0 \int_{X_0}^X rf^{(2)^2}\,dx > C_1[\log f^{(1)}(X_0) - \log f^{(1)}(X)] \to +\infty$ as $X \to +\infty$, by lemma 1. On the other hand, by (1.18), $\lim_{X\to\infty}\int_{X_0}^X rf^{(2)^2} < \infty$.

This is a contradiction and hence $\lim_{X\to\infty} rf^{(2)}f^{(1)}(X) = 0$, and (1.19) follows. In particular

$$\lim_{X\to\infty}\{rf^{(2)}g^{(1)}(X) - (rf^{(2)})^{(1)}g(X)\} = 0 \text{ for all } f,g \,\epsilon \Delta_L. \tag{1.20}$$

Combining (1.11), (1.14), (1.16), (1.18), and (1.19), it may be see that $L[\cdot]$ satisfies the Dirichlet conditions and is in the strong limit-2 case at infinity.

2. The above theorem can be generalized to the case of linear ordinary differential expression $M[\cdot]$ of order $2n$ ($n=1,2,3,$), given by, $M[f] = (-1)^n (pf^{(n)})^{(n)} + qf$ on $[0,\infty)$, (2.1)

where the coefficients p and q are real valued on $[0,\infty)$ and satisfy the following

(i) $p^{(n)}$ is continuous on $[0,\infty)$ and $p(x) > 0$ $(0 \le x < \infty)$
(ii) q is locally Lebesgue integrable on $[0,\infty)$ (2.2)

In fact, it can be shown that if $M[\cdot]$ is separated in $L^2(0,\infty)$, then $p^{1/2}f^{(n)}$ and $|q|^{1/2} f \in L^2(0,\infty)$ and $M[\cdot]$ is in the strong limit - n case at infinity.

Proof is similar to that of the theorem in § 1.

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