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Short Communication

Derivation of the solution of a special singular integral equation

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Abstract

A direct method involving minimum usage of complex analysis is presented for solving a special singular integral equation which is a generalization of one occurring in wave-guide theory.

Key words: Singular integral equation, wave-guide theory, Hilbert problem.

1. Introduction

The integral equation

$$a\,T\phi + T(b\phi) = f(x) \tag{1}$$

with $a(x) = (1-x^2)^{1/2}$, $b(x) = c(1-x^2)^{1/2}$ (c = a constant),

$$T\phi = \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)\,\mathrm{d}y}{y-x} \tag{2}$$

and $f(x)$ is differentiable in $-1<x<1$, arises in wave-guide theory and the theory of dislocations, as has been shown by Lewin[1] and Williams[2].

Systems of singular integral equations of the type

$$\left.\begin{aligned} &a_1(x)f_1(x) + \frac{1}{\pi}\int_{\alpha}^{\infty}\frac{b_1(t)f_2(t)\,\mathrm{d}t}{t-x} = g_1(x)\\ \text{and}\quad &\\ &a_2(x)f_2(x) + \frac{1}{\pi}\int_{\alpha}^{\infty}\frac{b_2(t)f_1(t)\,\mathrm{d}t}{t-x} = g_2(x)\end{aligned}\right\}\quad (0<\alpha<x<\infty), \tag{3}$$

can also be shown to be equivalent to the integral equation

$$A\ T_\alpha \phi + T_\alpha (B\phi) = G_2(\lambda) - \frac{A_2(\lambda)}{B_1(\lambda)} T_\alpha^{-1}(G_1) - C_1 A(\lambda) \tag{4}$$

by utilising the following transformation of the variables:

$$\left.\begin{aligned} & t = \frac{\xi + \alpha}{\alpha - \xi} + \alpha, \quad x = \frac{\lambda + \alpha}{\alpha - \lambda} + \alpha \\ & g_i(x) = (\lambda - \alpha)\ G_i(\lambda),\ a_i(x) = A_i(\lambda),\ b_i(t) = B_i(\xi) \\ & f_i(t) = F_i(\xi)(\xi - \alpha), \qquad (i = 1,2), \end{aligned}\right\} \tag{5}$$

where

$$\left.\begin{aligned} & \phi(\lambda) = A_1(\lambda)\ F_1(\lambda)(\lambda^2 - \alpha^2)^{1/2}, \\ & T_\alpha \phi = \frac{1}{\pi} \int_{-\alpha}^{\alpha} \frac{\phi(\xi)\, d\xi}{\xi - \lambda} \\ & T_\alpha^{-1} g = C_1 (\lambda^2 - \alpha^2)^{-1/2} - (\lambda^2 - \alpha^2)^{-1/2}\ T_\alpha [(\lambda^2 - \alpha^2)^{1/2} g], \\ & A(\lambda) = \frac{A_2}{B_1}(\lambda^2 - \alpha^2)^{-1/2}, \quad B(\lambda) = \frac{B_2}{A_1}(\lambda^2 - \alpha^2)^{-1/2}, \end{aligned}\right\} \tag{6}$$

with C_1 as an arbitrary constant.

In an earlier paper[3], a method was developed by using the inverse operator T^{-1} [$= T_1^{-1}$, cf. equations (6)] of the operator T in equation (2), to reduce the integral equation (1) into two independent Carleman-type singular integral equations, in the case when $ab = c(1 - x^2)$. As a further application of similar ideas together with some basic results of functions of a complex variable, the case of the integral equation (1), when $a = b$, is considered in the present work and it is shown, as in the earlier paper[3], that an elaborate analysis as given by Peters[4], for more general problems, can be avoided in the special case of equation (1). Though the results obtained in the present paper agree with known results, the analysis and the derivation are different.

2. The analysis

By a procedure similar to the one described in the earlier paper[3] and by setting

$$f(x) = T[(1 - x^2)^{1/2} g], \tag{7}$$

equation (1) can be cast, by using the inverse operator T^{-1}, into the form

$$L^2\phi - \frac{ab}{1-x^2}\phi = -ag\,(1-x^2)^{-1/2} \tag{8}$$

where the operator L is defined as

$$L\phi = a\,(1-x^2)^{-1/2}\quad T\phi \tag{9}$$

In the case under consideration, *i.e.*, when $a = b$, equation (8) can be identified with the following system of Carleman type integral equations:

$$\psi - \mu\, T\phi = 0,$$

and (10)

$$\mu\phi - T\psi = g,$$

with $\mu = a/(1-x^2)^{1/2}$

Using the standard Riemann-Hilbert arguments[5], the system of equations (10) can be reduced to a system of Hilbert problems for the determination of the unknown functions $\Phi^{\pm}(x)$ and $\Psi^{\pm}(x)$ denoting, in usual notations, the limiting values on the cut (-1, 1) of the sectionally analytic functions $\Phi(z)$ and $\Psi(z)$ as defined by ($z = x+iy$):

$$\Phi(z) = 1/\pi\int_{-1}^{1}\frac{\phi(t)\,dt}{t-z},\quad \Psi(z) = 1/\pi\int_{-1}^{1}\frac{\psi(t)\,dt}{t-z} \tag{11}$$

After a little readjustment we express the system of Hilbert problems to be solved as

$$\underset{\sim}{P}^{+} + \underset{\approx}{J}\,\underset{\sim}{P}^{-} = \underset{\sim}{S}(x), \tag{12}$$

where

$$\underset{\approx}{J} = \begin{pmatrix} 0, & -i/\mu \\ -i/\mu, & 0 \end{pmatrix}\ \underset{\sim}{S}(x) = \left(\frac{ig}{\mu}, -g\right)^T, \text{ and } \underset{\sim}{P}^{\pm} = (\Phi^{\pm}, \Psi^{\pm})^T, \tag{13}$$

We note that if in particular $\mu = 1$ (Lewin[1]), the system (12) can be decoupled easily and solutions obtained therefrom solve the known particular case ($ab = 1-x^2$) of equation (1).

In the case when $\mu \neq 1$, we use the idea that the function $\lambda(z) = (z^2-1)^{1/2}$, with the positive branch in mind, represents a sectionally analytic function in the complex z-plane cut along the real axis from $x=-1$ to $x=+1$ and has the limiting values $\lambda^{\pm}(x) = \pm i(1-x^2)^{1/2}$

as z tends to $x \in (-1,1)$, from above and from below respectively. We then define $X(z) = \lambda(z)\Psi(z)$ which helps us to recast the system to Hilbert problems (12) into a new system as given by

$$P'^{+} + J' P'^{-} = S'(x), \tag{14}$$

where

$$P'^{\pm} = (\Phi^{\pm}, X^{\pm})^T, \; S' = \left(\frac{ig}{\mu} \; \frac{-iga}{\mu}\right)^T$$

and (15)

$$J' = \begin{pmatrix} 0, & 1/a \\ a, & 0 \end{pmatrix}$$

The system of Hilbert problems (14) can be easily decoupled by using the factorization

$$\frac{1}{a(x)} = \hat{r}(x) = r^{+}(x)\, r^{-}(x), \quad (-1 < x < 1) \tag{16}$$

where, by means of standard arguments[5], we have that

$$r^{\pm}(x) = \hat{r}^{1/2}(x) \exp\left[\mp i/2 \,(1-x^2)^{1/2} \cdot T\{(1-t^2)^{-1/2} \log \hat{r}(t)\}\right] \tag{17}$$

with

$$r(z) = \exp\left[-\frac{(z^2-1)^{1/2}}{2\pi} \int_{-1}^{1} \frac{\log \hat{r}(t)\, dt}{(1-t^2)^{1/2}\,(t-z)}\right]$$

We finally find that the decoupled system, as obtained from (14) by using (17), is given by

$$\left(\frac{\Phi}{r} \pm rX\right)^{+} \pm \left(\frac{\Phi}{r} \pm rX\right)^{-} = \frac{ig}{\mu}\left(\frac{1}{r^{+}} \mp \frac{1}{r^{-}}\right) \tag{18}$$

The solution of (18) gives rise to the result

$$\Phi(z) = \frac{r(z)}{4\pi}\left[\int_{-1}^{1} \frac{g(t)}{t-z}\left\{ (r^{+}(t)+r^{-}(t))\,(1-t^2)^{1/2} + i\left(\frac{1}{z^2-1}\right)^{1/2}(1-t^2)\,(r^{-}(t)-r^{+}(t))\right\} dt\right] + c' r(z)\,(z^2-1)^{-1/2} \tag{19}$$

with c' as an arbitrary constant.

The solution of the integral equation (1) with $a = b$ can ultimately be derived by using the Plemelj formula

$$\phi(x) = 1/2i\,[\Phi^{+}(x) - \Phi^{-}(x)]. \tag{20}$$

The particular well-known case, when $a=b=a$ constant, simplifies the functions $r^{\pm}(t)\,[r^{+} = r^{-} = a^{-1/2}]$ and we obtain from equations (19) and (20) that

$$\phi(x) = T^{-1}\,(f/2a). \tag{21}$$

by using equation (7), and this is in agreement with the well-known result[5].

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