Picard iterative methods for steady nonlinear ground water flow analysis

M.S. MOHAN KUMAR, K. SRIDHARAN AND N.S. LAKSHMANA RAO
Department of Civil Engineering, Indian Institute of Science, Bangalore 560 012.

Abstract

Numerical fluctuations are of importance in solving unconfined ground water flow problems as some nodes may get artificially dropped out of the computation. The Picard iterative methods associated with Successive Over Relaxation method, Line Alternating Direction Implicit method and Strongly Implicit Procedure are compared based on two steady-state test problems with one having a thin saturated zone in part of the flow domain. SIP is found to be vastly superior to the other methods. It is found that the use of larger number of iteration parameters in the SIP method is not desirable and the best choice may be four parameters. A corrective measure is proposed to avoid artificial desaturation without any significant reduction in the convergence rate.

Key words: Groundwater, steady, nonlinear, Picard methods.

1. Introduction

Because of the nonlinearity involved, unconfined flow problems are more difficult to solve than other types of ground water flow problems. Analytical solutions for simplified well field problems in unconfined aquifers have been obtained by several investigators[1-4]. In modelling an unconfined aquifer over a nonhomogeneous region, numerical methods are used. There are many methods of linearising the nonlinear unconfined flow equation. The Picard's method[5] is one of the most popular methods used in ground water flow studies[6,7].

The unconfined flow equation is nonlinear as the coefficient in the flow term, the transmissivity, depends on the unknown water level. In the Picard iterative method, the coefficient is approximated based on the water levels obtained in the previous iteration (for the first iteration, the water levels are assumed). With this approximation, a set of linear equations are obtained for the nodal water levels. The structure of the resulting matrix equation facilitates the use of iterative methods for its solution, such as Successive Over Relaxation method (SOR), Line Alternating Direction Implicit methods (LADI) and Strongly Implicit Procedure (SIP).

There are two possible approaches to the solution. The set of linear equations can be solved to the required accuracy using SOR, LADI or SIP and then the coefficients on the flow term updated using the improved water levels. The next Picard iteration again involves

solution of the linear matrix equation with the improved coefficients. Alternatively, these two steps can be combined, wherein, after each SOR/LADI/SIP iteration, the coefficients in the flow term can be updated using the latest water levels. The total computational performance is found to be better for the latter approach and hence it is adopted in the present studies.

It has been established in recent literature that the Strongly Implicit Procedure[8] is one of the best finite difference methods to solve the matrix equation resulting from ground water flow problems[7]. The Strongly Implicit Procedure is an iterative method for solving the sparse matrix difference equations arising from the application of five point formula for the two-dimensional parabolic (unsteady-state problems) and elliptic equations (steady-state problems). The method uses a set of iteration parameters cyclically to accelerate convergence. It is generally assumed that the choice of number of iteration parameters does not make a serious difference to the convergence characteristics, provided at least four parameters are used[5,7,8]. The present study shows that the choice of number of iteration parameters makes an important difference for nonlinear ground water flow problems.

A study is also made to compare SOR, LADI and SIP, which are all used with Picard approximation (referred as PSOR, PLADI and PSIP). These methods are applied on two steady-state unconfined flow problems with one having a thin saturated zone in a part of the flow domain. Steady-state flow problems are used for illustration, as they are more difficult to solve than transient flow problems[7].

While using an iterative method such as SIP the water level at a node may fluctuate from iteration to iteration and in the process may overshoot the final solution. The finite difference methods used for nonlinear problems will thus lead to saturation oscillations and these oscillations may artificially drop some genuine nodes out of the computations. Earlier several investigators[7,9] have experienced this difficulty, which becomes particularly important if there are regions of thin saturated thickness. Hence the study focusses particular attention on numerical fluctuations, that is, the fluctuations in water level at a node over a set of SIP iterations.

2. Governing equation and finite difference approximation

The governing equation of flow in a two-dimensional unconfined aquifer is given by

$$\frac{\partial}{\partial x}\left[K_x(h-B_1)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y(h-B_1)\frac{\partial h}{\partial y}\right] = S_y\frac{\partial h}{\partial t} + Q \tag{1}$$

where h is the ground water level, B_1 is the bed rock level, K_x and K_y are hydraulic conductivities in x and y directions, S_y is the specific yield, Q is the net withdrawal rate per unit area, x, y are cartesian coordinates and t is time. Applying the block centered finite difference scheme to eqn. (1) yields (fig. 1)

$$(B_{i,j} + B'_{i,j})\,h_{i,j-1} + (D_{i,j} + D'_{i,j})\,h_{i-1,j} + (E_{i,j} + E'_{i,j})\,h_{i,j}$$

$$+ (F_{i,j} + F'_{i,j})\,h_{i+1,j} + (H_{i,j} + H'_{i,j})\,h_{i,j+1} = \frac{S_{y_{i,j}}\,h_{0_{i,j}}}{\Delta t} - \frac{Q_{i,j}}{\Delta x_i\,\Delta y_j} \tag{2}$$
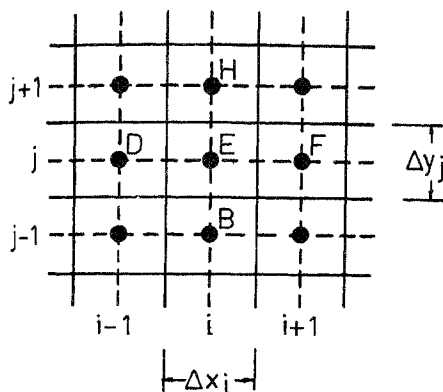
Fig. 1. Finite difference grid.

where $q_{i,j}$ is the net withdrawal rate at node (i,j) for the entire block. The coefficients in eqn. (2) are given by

$$B_{i,j} = -\frac{(K_{y_{i,j-1}} h_{i,j-1} + K_{y_{i,j}} h_{i,j} + K_{y_{i,j-1}} h_{i,j} + K_{y_{i,j}} h_{i,j-1})}{2 \Delta y_j (\Delta y_j + \Delta y_{j-1})}$$

$$D_{i,j} = -\frac{(K_{x_{i-1,j}} h_{i-1,j} + K_{x_{i,j}} h_{i,j} + K_{x_{i-1,j}} h_{i,j} + K_{x_{i,j}} h_{i-1,j})}{2 \Delta x_i (\Delta x_i + \Delta x_{i-1})}$$

$$F_{i,j} = -\frac{(K_{x_{i+1,j}} h_{i+1,j} + K_{x_{i,j}} h_{i,j} + K_{x_{i+1,j}} h_{i,j} + K_{x_{i,j}} h_{i+1,j})}{2 \Delta x_i (\Delta x_i + \Delta x_{i+1})}$$

$$H_{i,j} = -\frac{(K_{y_{i,j+1}} h_{i,j+1} + K_{y_{i,j}} h_{i,j} + K_{y_{i,j+1}} h_{i,j} + K_{y_{i,j}} h_{i,j+1})}{2 \Delta y_j (\Delta y_j + \Delta y_{j+1})}$$

$$B'_{i,j} = \frac{(K_{y_{i,j}} B_{i,j} + K_{y_{i,j-1}} B_{i,j-1})}{\Delta y_j (\Delta y_j + \Delta y_{j-1})}$$

$$D'_{i,j} = \frac{(K_{x_{i,j}} B_{i,j} + K_{x_{i-1,j}} B_{i-1,j})}{\Delta x_i (\Delta x_i + \Delta x_{i-1})}$$

$$F'_{i,j} = \frac{(K_{x_{i,j}} B_{i,j} + K_{x_{i+1,j}} B_{i+1,j})}{\Delta x_i (\Delta x_i + \Delta x_{i+1})}$$

$$H'_{i,j} = \frac{(K_{y_{i,j}} B_{l_{i,j}} + K_{y_{i,j+1}} B_{l_{i,j+1}})}{\Delta y_i (\Delta y_j + \Delta y_{j+1})}$$

$$E_{i,j} = -(B_{i,j} + D_{i,j} + F_{i,j} + H_{i,j})$$

$$E'_{i,j} = B'_{i,j} + D'_{i,j} + F'_{i,j} + H'_{i,j} + \frac{S_{y_{i,j}}}{\Delta t} \tag{3}$$

In an implicit procedure, eqn. (2) is written at all the nodes of the aquifer. They will give rise to a set of equations which can be solved using any of the iterative methods such as ADI, if the coefficients are known. However, the coefficients in eqn. (2) are functions of the unknown water table head. In the Picard method, the coefficients are taken at the latest available iteration.

3. Picard iterative methods

When eqn. (2) is written at all the nodes of the aquifer, a set of simultaneous equations for unknown water table heads is obtained. This set can be solved using an iterative method with the Picard approximation for the coefficients. In this study, SOR, LADI and SIP methods[5] have been considered. The LADI method is similar to LSOR except that iterations are done both row and columnwise alternately. Thus unlike in the classical ADI[5], a single value of relaxation parameter ω is used and the updated value of water level is used for previous row (or column). SOR and LADI are selected because of less numerical fluctuations and SIP is selected because of its rapid rate of convergence. LSOR is not considered because it is good only for anisotropic problems, and ADI is not considered because of severe numerical fluctuations over iterations. The only change in the Picard method with respect to the methods used for linear problems lies in incorporating the coefficients B, D, E, F and H associated with the latest available $h_{i,j}$ values. For example, in a Picard SIP iteration for obtaining $h_{i,j}^{(n+1)}$, $h_{i,j}^{(n)}$ values will be used for calculation of B, D, E, F and H in eqn. (3), where n refers to the iteration number. As stated earlier, this approach must be distinguished from completing the solution of the linear equations and then updating the coefficients.

The SOR and LADI methods are used with different choices of relaxation parameter ω in the range 1 to 2. For the SIP method, computations are done with different choices of the number of iteration parameters, P. Once the number of iteration parameters is chosen, the actual parameter values, ω_1, ω_2, ... ω_p, are obtained based on the hydraulic conductivity values and choice of Δx_i and Δy_j, as recommended by Stone[8]. The value of ω_{min} will be greater than zero and the value of ω_{max} will be less than one.

4. Test problems

Two test problems have been chosen to study the relative merits of different Picard iterative methods. The first test problem has a single pumping well in a rectangular, isotropic flow domain (fig. 2). The river acts as a recharge source to the aquifer through leakage, which facilitates a steady-state solution. The bed level is taken to be homogeneous with the river water level 12m above. The hydraulic conductivity is taken to be homogeneous and equal to

65.84 m/day and the pumping rate is 2447 m³/day. The leakage coefficient for the river bed is 0.108/day. This test problem is a slight modification of the problem given by Lin[10].

The second test problem has two pumping wells in a rectangular isotropic flow domain (fig. 3). Pumping well A is pumped at a rate of 223.8 m³/day, while pumping well B is pumped at a rate of 1296 m³/day. The bed level is varied from 10 to 0 m as shown in fig. 3. The hydraulic conductivities at the nodes are varied randomly by generation of uniformly distributed random numbers. Conductivities are varied from 0 to 80 m/day. A Dirichlet boundary is imposed at the centre of the flow domain to obtain a steady-state solution. This effectively makes Test problem 2 as two separate problems, with pumping in a thin and thick saturated zone. Thus this problem clearly gives a contrast of the effect of numerical fluctuations in thin and thick saturated zones.

The solutions for steady-state head distribution are also shown in figs. 2 and 3.
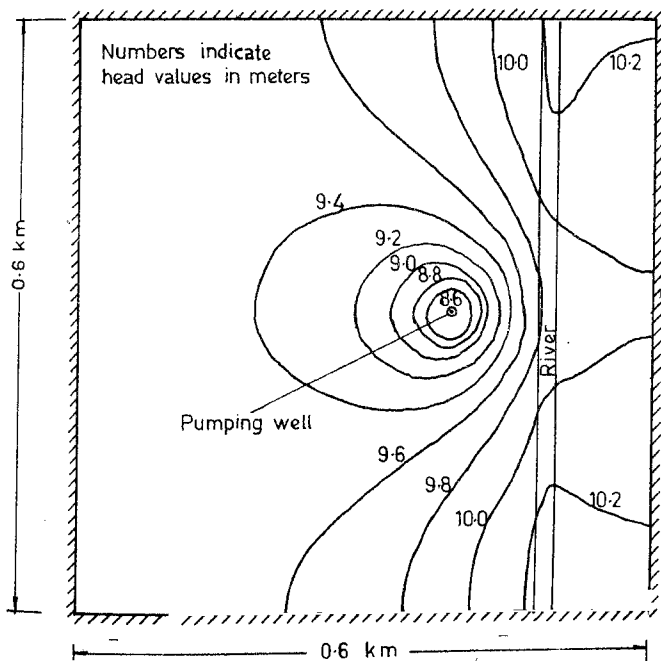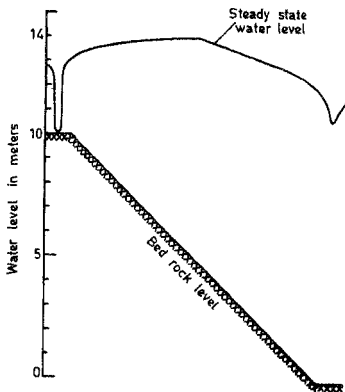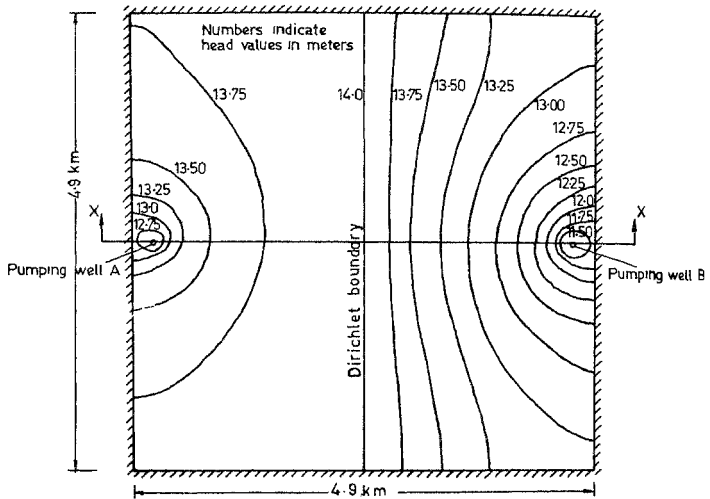

FIG. 2. Details of Test problem 1.

FIG. 3. Details of Test-problem 2.

5. Discussion of results

5.1. Numerical fluctuations

Numerical fluctuations during the early iterations may lead to the water levels at certain nodes, to drop below the bed rock level thereby eliminating certain genuine nodes, creating spurious desaturation. Figures 4 and 5 present the water table head at pumping well with iterations for PSIP method for Test problems 1 and 2. The fluctuations are less for the SOR and LADI methods. An interesting trend may be observed from the results presented for the PSIP method (figs. 4 and 5), with numerical fluctuations showing a clear tendency to increase with increase in the number of parameters. Because of the numerical fluctuations, the head at the Pumping well A, goes below bed level for P ≥ 8 in the PSIP method for Test problem 2 (fig. 5).
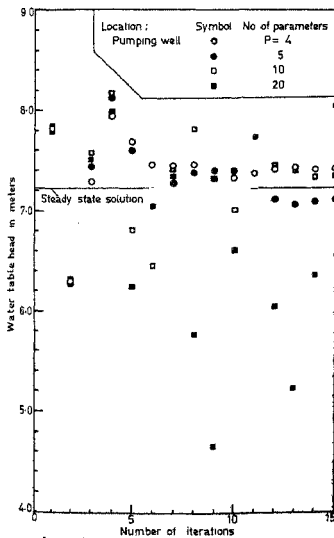

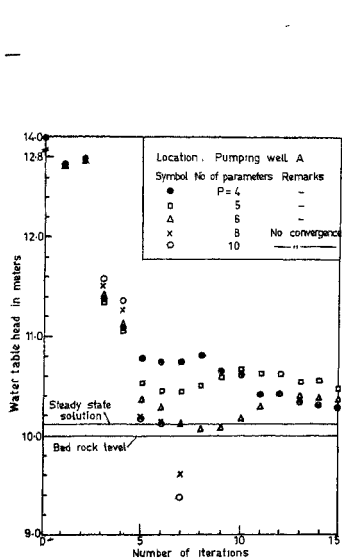FIG. 4. Numerical fluctuations in PSIP method — Test problem 1.

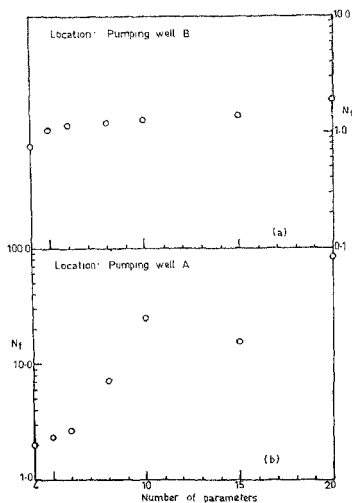FIG. 5. Numerical fluctuations in PSIP method — Test problem 2.

FIG. 6.   Effect of PSIP parameters on numerical fluc-
tuations — Test problem 2.

The effect of parameters on numerical fluctuation may be clearly seen by considering the variable N_f given by

$$(N_f)_{i,j} = \underset{n}{\text{Max}} \; \left[h_{i,j}^{(n)} \right] \; \underset{-n}{\text{Min}} \; \left[h_{i,j}^{(n)} \right] n = 1, 2,... P \tag{4}$$

N_f thus indicates the magnitude of numerical fluctuation during the first cycle of iteration. The variations of N_f with P at Pumping wells B and A for the PSIP method are shown in fig. 6 respectively. The strong effect of number of iteration parameters P on fluctuations in zones of thin saturated thickness may be seen from fig. 6.

Thus the results on numerical fluctuation suggest that it may be advantageous to choose a smaller number of iteration parameters in the PSIP method.

5.2. Sensitivity studies of relaxation or iteration parameters

This section presents the results of the effect of variation in iteration or relaxation parameters on the rate of convergence, for the two test problems. For convergence check, the maximum normalised residue has been used. The residue is calculated using eqn. (2) by taking all terms to one side. This residue is multiplied by the area represented by the node at

which the maximum residue occurs and this is normalised with respect to the total pumping in the system. The PSOR method is found to be very slow even for optimal value for both the test problems. Hence detailed results are not presented for this method.

Test problem 1

The rate of convergence in the PSIP method is not influenced much by the variation in the number of iteration parameters. It is found that 4 and 5 parameters give a marginally better convergence rate than a larger number of parameters. Unlike PSIP, PLADI shows a clear dependence on the value of the relaxation parameters. For optimal ($\omega_{opt} = 1.95$), a maximum normalised residue (R_{max}) of 10^{-6} is reached in 160 iterations, while for $\omega = 1.7$, in the same number of iterations a value of $R_{max} = 10^{-3}$ only is achieved.

Test problem 2

Unlike the conclusions of Stone[8], for nonlinear problems, SIP shows a clear dependence on the number of parameters if there is a zone of thin saturated thickness as in Test problem 2. For purposes of the present study, a method is treated to be a failure if during the computation, the water level at any node goes below the bed level (for the steady-state all heads are above the bed level vide fig. 3). It was actually found that for some cases in the PSIP method (not always), where the water level goes marginally below the bed level at Pumping well A, convergence was obtained if the computations were continued. However, such cases are still treated as failures in the present study, as the recovery of water level above the bed again depends on the extent of depletion below the bed level and also as continuation of computations lack physical validity once the water level goes below the bed level, unless the relevant node is dropped out of the computations.

In the PSIP method, the number of iteration parameters has been varied from 4 to 20 and the method fails to converge for $P > 8$, with the water level at the Pumping well A going below the bed level (fig. 5). The reason for this may be seen from fig. 6, which indicates that the magnitude of the numerical fluctuation increases with the increase in the number of iteration parameters.

Several techniques may be used to prevent artificial desaturation. Computations are made for $P = 10$ using alternative methods to ensure covergence. Trescott and Larson[7] suggested the use of an underrelaxation parameter, where the SIP corrections in each iteration are multiplied by a factor $\beta' < 1.0$. Use of a safe underrelaxation parameter $\beta' = 0.3$ for the case of $P = 10$ leads to convergence, but the rate of convergence becomes slow. A residue of 10^{-6} is reached in 192 iterations for $\beta' = 0.3$ and $P = 10$ against 84 iterations for $\beta' = 1.0$ and $P = 4$.

A slight variation in the use of the underrelaxation factor accelerates the rate of convergence very significantly. It has been observed that, while using SIP, numerical fluctuations are significant only in the first few iterations. In fact the maximum fluctuations may be expected in the first cycle of computations for a given set of parameters. In view of this, a modification is adopted wherein $\beta' = 0.3$ is used till the maximum normalised residue is less than 0.01 and the number of iterations is less than twice the number of parameters (that is completion of one cycle of iterations). When both these conditions are satisfied, underrelaxation factor β' is changed to 1.0. Thus this method dampens the numerical fluctuations which occur initially, without sacrificing the rate of convergence significantly.
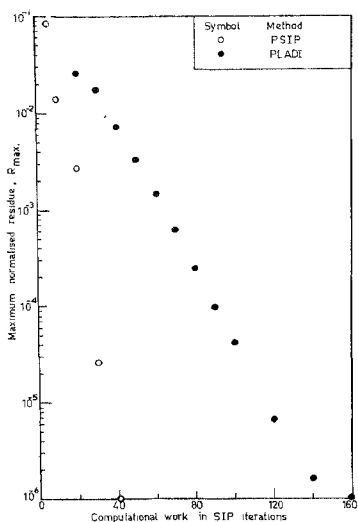
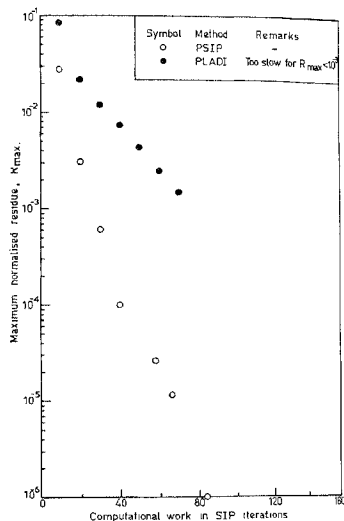FIG. 7. Comparison of convergence rate — Test problem 1.

FIG. 8. Comparison of convergence rate — Test problem 2.

For those cases for which water level does not go below bed level, all the cases studied for the PSIP method show convergence at approximately the same rate, except when a constant underrelaxation parameter $\beta' = 0.3$ is used.

5.3 Comparison of rate of convergence

To compare the rate of convergence of PSIP and PLADI (PSOR is too slow), the best cases among those tested have been selected and the maximum normalised residue is plotted against the number of iterations. The work load involved per iteration for each method has been calculated based on the number of multiplication and division operations and it has been found that both these methods take almost the same amount of work per iteration.

Figures 7 and 8 present a comparison of the rate of convergence for Test problems 1 and 2. It is seen that the PSIP method is considerably superior to PLADI method (and also PSOR method for which results are not presented). It appears that the superiority of the SIP method over the other methods (at least for predominantly isotropic problems) may be more pronounced in nonlinear problems than in linear problems.

6. Conclusions

A comparative study of three Picard iterative methods has been made for nonlinear ground water flow problems. Two steady-state test problems have been considered with one of them being designed as a critical problem with a zone of thin saturated thickness. Aspects studied include sensitivity to relaxation or iteration parameters, rate of convergence and numerical fluctuations. The PSIP method is found to be the best. It is found that the use of a larger number of iteration parameters in the PSIP method is not desirable and the best choice may be four parameters. Corrective measures to avoid artificial desaturation have been proposed. This method may be used if the PSIP method with four parameters fails to converge.

References

1. BOULTON, N.S.   Unsteady radial flow to a pumped well allowing for delayed yield from storage, Int. Ass. Sci. Hydrol , Gen. Assembly, 1954, **2**, 472.

2. BOULTON, N S.   Analysis of data from non-equilibrium pumping tests allowing for delayed yield from storage, *Proc. Instn., Civ. Engr.,* London, 1963, **26**, 469-482.

3. BOULTON N S.   The influence of delayed drainage on data from pumping tests in unconfined aquifers, *J. Hydrol.,* 1973, **19**, 157-169.

4. NEUMANN, S.P   Theory of flow in unconfined aquifers considering delayed response of the water table, *Wat. Resources Res.,* 1972, **8**(4), 1031-1045.

5. REMSON, I., HORBERGER, G.M. AND MOLZ, F.J.   *Numerical methods in subsurface hydrology,* Wiley-Interscience, New York, 1971.

6. BREDEHOEFFT, J D. AND PINDER, G.F.   Digital analysis of areal flow in multi-aquifer groundwater systems: A quasi three-dimensional model, *Wat. Resources Res.,* 1970, **6**(3), 883-888.

7. TRESCOTT, P C. AND LARSON, S.P.   Comparison of iterative methods for solving two dimensional groundwater flow equations, *Wat. Resources, Res.,* 1977, **13**(1), 125-136.

8. STONE, H.L.   Iterative solution of implicit approximations of multidimensional partial differential equations, *SIAM J. Numer. Anal.,* 1968, **5**(3), 530-558

9. KONIKOW, L.F.   Hydrogeological maps of the alluvial aquifer in and adjacent to Rocky mountain arsenal, Colorado, U.S. Geological Survey, Open file report, Reston, Va., 1975.

10. LIN, C.L.   Digital simulation of the Boussinesq equation for a water table aquifer, *Wat. Resources Res.,* 1972, **8**(3), 691-698.