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Singularities in a three-layered fluid medium

B.N. MANDAL AND R.N. CHAKRABARTI
Department of Applied Mathematics, University College of Science, 92, A.P.C. Road, Calcutta 700 009, India.

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Abstract

Velocity potentials due to the presence of different types of singularities oscillating harmonically with small amplitudes located in one of the three fluids of a three-layered fluid medium with horizontal surfaces of separation, the middle fluid being of finite depth and the other two fluids being of infinite height and depth respectively, are obtained. These are required to study internal waves at the surfaces separating the fluids. If the density of the upper fluid is made zero, known results are recovered.

Key words: Three-fluid problem, surface of separation (SS), linearised theory, Laplace's equation, SS condition, oscillating line and point singularities, potential functions.

1. Introduction

Different types of singularities that can be used in solving one-fluid problems concerning scattering or generation of surface waves of small amplitudes by obstacles present in the fluid have been surveyed in some detail initially by Thorne[1] who neglected the effect of surface tension and later by Rhodes-Robinson[2] who included it. The singularities are mainly submerged in an one-fluid medium of finite or infinite depth. The study of internal waves at the surfaces of separation of a multi-layered fluid medium necessitates the consideration of different types of singularities in the fluid. For the two-fluid case, velocity potentials describing different types of singularities were obtained by Gorgui and Kassem[3] when the upper fluid is unbounded and the lower fluid is of either finite or infinite depth, and by Kassem[4] when both the fluids are of finite depths, the surface tension effect being neglected in all the cases. The effect of surface tension is included in the problem considered independently by Rhodes-Robinson[5] and Mandal[6] when both the fluids are unbounded and later by Chakrabarti[7] when the upper fluid is unbounded and the lower fluid is of finite depth. Also Chakrabarti and Mandal[8] considered different types of singularities submerged in a two-fluid medium where the upper fluid is of finite depth with a free surface and the lower fluid is of infinite depth, the surface tension being neglected.

These two-fluid problems naturally motivate us to extend the results for a multi-layered medium. For this reason, a three-layered fluid medium is considered where the upper fluid is unbounded, the middle fluid is of finite depth and the lower fluid is of infinite depth, the two surfaces of separation being horizontal planes of infinite extent. In the present paper, we give a discussion of the basic line and point singularities oscillating with small amplitudes present in each of the three fluids. The time harmonic singularities are described by harmonic potential functions which are typical singular solutions of Laplace's equation in the neighbourhood of the singularities. Under the given boundary conditions at the two mean surfaces of separation and the radiation condition that there are only outgoing waves in the far field, unique solution will be found for each type of singularity concerned, the proofs depending upon the use of appropriate integral representations for singular harmonic functions. Detailed method of calculations for finding the different potential functions in different media is given in the case of a line singularity present in the middle fluid only. For other cases the final results are mostly stated.

2. Statement and formulation of the problem

We consider the irrotational motion of three non-viscous fluids under the action of gravity. The middle fluid is of finite depth 'h' while the upper and lower fluids are unbounded. The two mean surfaces of separation are horizontal planes of infinite extent. The motion is due to a singularity oscillating harmonically with small amplitudes in one of the three fluids. The motion in each case can be described by velocity potentials which are simple harmonic in time with period $2\pi/\sigma$ and thus it is more convenient to use complex valued potentials $\phi_j \exp(-i\sigma t)$ $(j=1,2,3)$ of which the actual velocity potentials are real parts, where the subscripts 1, 2, 3 are used for lower, middle and upper media respectively.

The origin O is taken on the mean surface of separation of the middle and lower fluids and the axis Oy pointing vertically downwards into the lower fluid is chosen in such a way that it passes through the singularity, so that the point at which the velocity potential has a singularity is taken conveniently as any one of the points $(0, \eta)$, $(0, -\eta)$, $(0, -2h+\eta)$ $(\eta>0)$ according to which the singularity is in the lower, middle or upper fluid respectively. The velocity potential then satisfy

$$\begin{aligned} \nabla^2 \phi_1 &= 0, & y &> 0 \\ \nabla^2 \phi_2 &= 0, & -h &< y < 0 \\ \nabla^2 \phi_3 &= 0, & y &< -h \end{aligned}$$

except at the point of singularity. The linearised surface of separation conditions are

$$K\phi_1 + \frac{\partial \phi_1}{\partial y} = s_1\left(K\phi_2 + \frac{\partial \phi_2}{\partial y}\right) \text{ on } y=0, \tag{2.1}$$

$$K\phi_2 + \frac{\partial \phi_2}{\partial y} = s_2\left(K\phi_3 + \frac{\partial \phi_3}{\partial y}\right) \text{ on } y=-h, \tag{2.2}$$

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \text{ on } y=0, \tag{2.3}$$

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial \phi_3}{\partial y} \text{ on } y = -h, \tag{2.4}$$

where $K = \sigma^2 / g$, $s_1 = \rho_2 / \rho_1$, $s_2 = \rho_3 / \rho_2$, g being the gravity, ρ_1, ρ_2 and ρ_3 being the densities of the lower, middle and upper fluids respectively ($\rho_1 > \rho_2 > \rho_3$). Also

$$\text{grad } \phi_1 \to 0 \quad \text{as } y \to +\infty, \tag{2.5}$$

$$\text{grad } \phi_3 \to 0 \quad \text{as } y \to -\infty. \tag{2.6}$$

There is another condition to be satisfied by ϕ_j ($j = 1, 2, 3$) as $|x| \to \infty$ which is the so-called radiation condition. This states that the potential function should represent diverging waves at a large distance from the singularity.

3. Line singularity submerged in the middle fluid of finite depth

Let a line singularity be placed at the point $(0, -\eta)$ in the middle fluid. Then

$$\phi_2 \sim \log R_0' \text{ as } R_0' = \{ x^2 + (y + \eta)^2 \}^{1/2} \to 0. \tag{3.1}$$

Now ϕ_1, ϕ_2, ϕ_3 can be represented as

$$\phi_1 = \sum_1^\infty f_j \log R_j + \sum_0^\infty g_j \log R_j' + \int_0^\infty A(k) \exp(-ky) \cos kx \, dk, \tag{3.2}$$

$$\phi_2 = \sum_{-\infty}^\infty c_j \log R_j + \sum_{-\infty}^\infty d_j \log R_j' + \int_0^\infty [B(k) \cosh k(h + y) + C(k)$$

$$\sinh k(h+y)] \cos kx \, dk, \tag{3.3}$$

$$\phi_3 = \sum_0^\infty p_{-j} \log R_{-j} + \sum_0^\infty q_{-j} \log R_{-j}' + \int_0^\infty D(k) \exp(ky) \cos kx \, dk, \tag{3.4}$$

where

$$R_j^2 = x^2 + (y + 2jh - \eta)^2, \; R_j'^2 = x^2 + (y + 2jh + \eta)^2, \; j = 0, \pm 1, \pm 2 \ldots$$

Because of (3.1) we choose $d_0 = 1$. Conditions (2.5) and (2.6) are automatically satisfied. $A(k)$, $B(k)$, $C(k)$, $D(k)$ and f_j, g_j, c_j, d_j, p_j, q_j are to be so chosen that the conditions (2.1), (2.2), (2.3) and (2.4) are satisfied and the different integrals converge. The radiation condition will be dealt with in the sequel.

The following integral representations will be needed in our calculations:

$$\frac{\partial}{\partial y}(\log R_j) = \pm \int_0^\infty \exp\{ \mp k(y + 2jh - \eta) \} \cos kx \, dk, \; y \gtrless -2jh + \eta.$$

$$\frac{\partial}{\partial y}(\log R_j') = \pm\int_0^\infty \exp\{\mp k(y+2jh+\eta)\}\cos kx\,dk,\ y \gtrless -(2jh+\eta)$$

where the upper signs are for '>' cases and the lower signs are for the '<' cases. Hence

$$\text{on } y=-h,\ \frac{\partial}{\partial y}(\log R_j) = \pm\int_0^\infty \exp[\mp k\{(2j-1)h-\eta\}]\cos kx\,dk,$$

$$\text{on } y=-h,\ \frac{\partial}{\partial y}(\log R_j') = \pm\int_0^\infty \exp[\mp k\{(2j-1)h+\eta\}]\cos kx\,dk,$$

$$\text{on } y=0,\ \frac{\partial}{\partial y}(\log R_j) = \pm\int_0^\infty \exp[\pm k(2jh-\eta)]\cos kx\,dk,$$

where the upper signs are for $j = 1, 2, \ldots$ and the lower signs for $j = 0, -1, -2 \ldots$, and

$$\text{on } y=0,\ \frac{\partial}{\partial y}(\log R_j') = \pm\int_0^\infty \exp[\mp k(2jh+\eta)]\cos kx\,dk,$$

where the upper sign is for $j = 0, 1, 2, \ldots$ and the lower sign is for $j = -1, -2 \ldots$.
After using these integral representations in appropriate places the condition (2.1) gives

$$\begin{aligned}
&K\Big[\sum_1^\infty f_j \log\{x^2+(2jh-\eta)^2\}^{1/2} + \sum_0^\infty g_j \log\{x^2+(2jh+\eta)^2\}^{1/2} + \\
&+\int_0^\infty A\cos kx\,dk\Big] + \int_0^\infty \Big[\sum_1^\infty f_j \exp\{-k(2jh-\eta)\} + \\
&+\sum_0^\infty \exp\{-k(2jh+\eta)\} - kA\Big]\cos kx\,dk \\
&= s_1 K\Big[\sum_1^\infty c_j \log\{x^2+(2jh-\eta)^2\}^{1/2} + \sum_0^\infty c_{-j}\log\{x^2+(2jh+\eta)^2\}^{1/2} + \\
&+\sum_0^\infty d_j \log\{x^2+(2jh+\eta)^2\}^{1/2} + \sum_1^\infty d_{-j}\log\{x^2+(2jh-\eta)^2\}^{1/2} + \\
&+\int_0^\infty (B\cosh kh + C\sin h\,kh)\cos kx\,dk\Big] + s_1\int_0^\infty \Big[\sum_1^\infty c_j \exp\{-k(2jh-\eta)\} \\
&\sum_0^\infty c_{-j}\exp\{-k(2jh+\eta)\} + \sum_0^\infty d_j \exp\{-k(2jh+\eta)\} - \sum_1^\infty d_{-j}\exp\{-k \\
&(2jh-\eta)\} + k(B\sin h\,kh + C\cosh kh\Big]\cos kx\,dk.
\end{aligned} \tag{3.5}$$

By equating the coefficients of similar logarithmic terms (3.5) gives

$$\left.\begin{array}{l} f_j = s_1 \, (c_j + d_{-j}), \, j = 1, 2, \dots \\ g_j = s \, (d_j + c_{-j}), \, j = 0, 1, 2, \dots \end{array}\right\} \tag{3.6}$$

Since $d_0 = 1$ we obtain $g_0 = s_1 \, (c_0 + 1)$.

Again, the condition (2.2) similarly gives

$$K \Big[\sum_1^\infty c_j \, \log \{ x^2 + ((2_j - 1) \, h - \eta)^2 \}^{1/2} + \sum_0^\infty c_{-j} \log \{ x^2 + ((2j+1) \, h + \eta)^2 \}^{1/2}$$

$$+ \sum_1^\infty d_j \log \{ x^2 + ((2j-1) \, h + \eta)^2 \}^{1/2} + \sum_0^\infty d_j \log \{ x^2 + ((2j+1) \, h - \eta)^2 \}^{1/2} +$$

$$+ \int_0^\infty B \cos kx \, dk \Big] + \int_0^\infty \Big[\sum_1^\infty c_j \exp \{ -k \, ((2j-1) \, h - \eta) \} -$$

$$- \sum_0^\infty c_{-j} \exp \{ -k \, ((2j+1) \, h + \eta) \} + \sum_1^\infty d_j \exp \{ -k \, ((2j-1) \, h + \eta) \}$$

$$- \sum_0^\infty d_{-j} \exp \{ -k \, ((2j+1) \, h - \eta) \} + k \, C \Big] \cos kx \, dk =$$

$$= s_2 \, K \Big[\sum_0^\infty p_{-j} \log \{ x^2 + ((2j+1) \, h + \eta)^2 \}^{1/2} + \sum_0^\infty q_{-j} \log \{ x^2 + ((2j+1) \, h - \eta^2 \}^{1/}$$

$$+ \int_0^\infty D \exp(-kh) \cos kx \, dk + s_2 \int_0^\infty \Big[- \sum_0^\infty p_{-j} \exp \{ -k \, ((2j+1) \, h + \eta) \}$$

$$- \sum_0^\infty q_{-j} \exp \{ -k \, ((2j+1) \, h - \eta) \} + k D \exp(-kh) \Big] \cos kx \, dk \tag{3.7}$$

from which we obtain similarly

$$c_{j+1} + d_{-j} = s_2 \, q_{-j} \text{ and } d_{j+1} + c_{-j} = s_2 \, p_{-j}, \, j = 0, 1, 2 \dots \tag{3.8}$$

so that $c_1 + 1 = s_2 \, q_0$ as $d_0 = 1$. Condition (2.3) gives

$$\sum_1^\infty \left(f_j - 2c_j + \frac{f_j}{s_1} \right) \exp \{ -k \, (2jh - \eta) \} + \sum_0^\infty \left(g_j - 2d_j + \frac{g_j}{s_1} \right)$$

$$\exp \{ -k \, (2jh + \eta) \} = k \, (A + B \sinh kh + C \cosh kh \,]. \tag{3.9}$$

For convergence of the integrals in (3.2), (3.3) and (3.4), the expression in the left side of (3.9) must vanish for $k = 0$ so that

$$\sum_1^\infty (f_j - 2c_j + \frac{f_j}{s_1}) + \sum_0^\infty (g_j - 2d_j + \frac{g_j}{s_1}) = 0.$$

This is satisfied by choosing

$$f_j = \frac{2s_1}{1+s_1} c_j \qquad j=1,2,\ldots.$$

and (3.10)

$$g_j = \frac{2s_1}{1+s_1} d_j \quad j=0,1,2,\ldots.$$

Finally the condition (2.4) gives

$$\sum_1^\infty [c_{j+1} - \frac{f_j}{s_1} + c_j + \frac{1}{s_2}(c_{j+1} + \frac{f_j}{s_1} - c_j)] \exp[-k\{(2j+1)h - \eta\}] +$$

$$+ \sum_0^\infty [-\frac{g_j}{s_1} + d_j + d_{j+1} + \frac{1}{s_2}(d_{j+1} + \frac{g_j}{s_1} - g_j)] \exp[-k\{(2j+1)h + \eta\}]$$

$$+ \{c_1 - 1 + \frac{1}{s_2}(c_1 + 1)\} \exp\{-k(h-\eta)\} + k\{D \exp(-kh) - C\}. \tag{3.11}$$

The left side of (3.11) must vanish for $k = 0$ from the convergence consideration so that

$$\sum_1^\infty [(1 + \frac{1}{s_2}) c_{j+1} + (1 - \frac{1}{s_2}) c_j + (\frac{1}{s_2} - 1)\frac{f_j}{s_1}] + \sum_0^\infty [(1 + \frac{1}{s_2}) d_{j+1} +$$

$$(1 - \frac{1}{s_2}) d_j + (\frac{1}{s_2} - 1)\frac{g_j}{s_1}] + [(\frac{1}{s_2} + 1). c_1 + (\frac{1}{s_2} - 1)] = 0.$$

This is satisfied by choosing

$$c_{j+1} - \gamma c_j + \frac{\gamma}{s_1} f_j = 0 \qquad j=1,2,\ldots.$$

$$d_{j+1} - \gamma d_j + \frac{\gamma}{s_1} g_j = 0 \qquad j=0,1,2,\ldots. \tag{3.12}$$

$c_1 = -\gamma$ where $\gamma = (1 - s_2) / (1 + s_2)$.

Then from (3.6), (3.8), (3.10) and (3.12), we can obtain

$$
\begin{aligned}
f_j &= \frac{2s_1}{1+s_1}(-1)^j \gamma^j \mu^{j-1} && j=1,2,3,\dots \\
g_j &= \frac{2s_1}{1+s_1}(-1)^j (\mu\gamma)^j && j=0,1,2,\dots \\
c_j &= (-1)^j \gamma^j \mu^{j-1} && j=1,2,3,\dots \\
d_j &= (-1)^j (\mu\gamma)^j && j=0,1,2,\dots \\
c_{-j} &= (-1)^j \gamma^j \mu^{j+1} && j=0,1,2,\dots \\
d_{-j} &= (-1)^j (\mu\gamma)^j && j=0\ 1,2,3,\dots \\
p_{-j} &= \frac{2}{1+s_2}(-1)^j \gamma^j \mu^{j+1} && j=0,1,2,\dots \\
q_{-j} &= \frac{2}{1+s_2}(-1)^j (\mu\gamma)^j && j=0,1,2,\dots
\end{aligned}
\tag{3.13}
$$

where $\mu = (1-s_1)/(1+s_1)$.

Using these in (3.5), we can obtain

$$(k-K)A + s_1(K\cosh kh + k\sinh kh)B + s_1(K\sinh kh + k\cosh kh)C$$

$$= \frac{2s_1\mu[\exp(-k\eta) - \gamma\exp\{k(\eta-2h)\}]}{1+\mu\gamma\exp(-2kh)} = E(k), \text{say.} \tag{3.14}$$

(3.7) gives

$$KB + KC - s_2(k+K)\exp(-kh)D$$

$$= \frac{2\gamma\exp(-kh)\{\mu\exp(-k\eta) + \exp(k\eta)\}}{1+\mu\gamma\exp(-2kh)} = F(k), \text{say} \tag{3.15}$$

and from (3.9) and (3.11) we can obtain

$$A + B \sinh kh + C \cosh kh = 0, \tag{3.16}$$

$$D \exp(-kh) - C = 0; \tag{3.17}$$

solving for A, B, and D from (3.14), (3.15), (3.16) and (3.17), we can obtain

$$A = -\frac{F}{K} \sinh kh + \left[\frac{1}{K}\{k - s_2(k+K)\} \sinh kh - \cosh kh\right] \frac{W}{\Delta},$$

$$B = \frac{F}{K} + \frac{1}{K}\{s_2(k+K) - k\} \frac{W}{\Delta}, \tag{3.18}$$

$$C = \frac{W}{\Delta},$$

$$D = \frac{W}{\Delta} \exp(kh),$$

where $W(k) = E - \dfrac{F}{K}\{(K - k + s_1 K)\sinh kh + s_1 K \cosh kh\}$ (3.19)

and $\Delta(k) = \left\{\dfrac{1}{K}(K - k + s_1 k)(s_2 k + s_2 K - k) + s_1 K\right\} \sinh kh +$

$$+ \{s_1 s_2 (k+K) + K - k\} \cosh kh. \tag{3.20}$$

Now $\Delta(k)$ has three zeros at $k = K_1, k_0, -k_0'$, say, all on the real axis and complex zeros at $k = k_n$, say, $(n \geq 1)$, where $k_n = \alpha_n + i\beta_n$, say. It may be noted that when $s_2 = 0$, K_1 becomes K. Thus $A(k)$, $B(k)$, $C(k)$ and $D(k)$ have simple poles at $k = K_1$ and $k = k_0$ on the positive real axis. In the line integrals from 0 to ∞ we make indentations below these poles which account for the behaviour of the potential functions at infinity particularly as $|x| \to \infty$. This will be evident later.

Thus using the above results, we can obtain

$$\phi_1 = \frac{2s_1}{1+s_1} \sum_{1}^{\infty} (-1)^j (\mu\gamma)^j \log R_j + \frac{2s_1}{1+s_1} \sum_{0}^{\infty} (-1)^j (\mu\gamma)^j \log R_j'$$

$$-\int_0^\infty \frac{F}{K} \sinh kh \exp(-ky) \cos kx \, dk +$$

$$+\int_0^\infty \frac{1}{K} \left[\{ k - s_2 (K+k) \} \sinh kh - \cosh kh \right] \frac{W}{\Delta} \exp(-ky) \cos kx \, dk, \tag{3.21}$$

$$\phi_2 = \sum_1^\infty (-1)^j \gamma^j \mu^{j-1} \log R_j + \sum_0^\infty (-1)^j \gamma^j \mu^{j+1} \log R_{-j} + \sum_0^\infty (-1)^j \cdot (\mu\gamma)^j$$

$$\log R_j' + \sum_1^\infty (-1)^j (\mu\gamma)^j \log R_{-j}' + \int_0^\infty \frac{F}{K} \cosh k(h+y) \cos kx \, dk +$$

$$+\int_0^\infty \left[\frac{1}{K} \{ s_2 (k+K) - k \} \cosh k(h+y) + \sinh k(h+y) \right] \frac{W}{\Delta} \cos kx \, dk, \tag{3.22}$$

$$\phi_3 = \frac{2}{1+s_2} \left[\sum_0^\infty (-1)^j \gamma^j \mu^{j+1} \log R_{-j} + \sum_0^\infty (-1)^j (\mu\gamma)^j \log R_{-j}' \right] +$$

$$+\int_0^\infty \frac{W}{\Delta} \exp(k(h+y)) \cos kx \, dk. \tag{3.23}$$

Putting $s_2 = 0$ we find that the expressions for ϕ_1 and ϕ_2 agree with the corresponding results in the case of a two-fluid medium with upper fluid of finite depth and the lower fluid of infinite depth obtained by Chakrabarti and Mandal[8], and further letting $h \to \infty$ (the case of a two-fluid medium when both the fluids are unbounded) the results given by Gorgui and Kassem[3] are recovered. Also, if we put $\rho_1 = \rho_2 = \rho_3$, then the three-layered medium reduces to a single fluid medium of infinite extent, and in that case $s_1 = 1, s_2 = 1$ so that $\mu = 0, \gamma = 0$. Then it is easily seen that (3.21), (3.22) and (3.23) readily give $\phi_1 = \phi_2 = \phi_3 = \log R_0'$ which is in fact the potential function in an infinite fluid due to a line singularity of logarithmic type at $(0, -\eta)$.

Now to investigate the behaviour of ϕ_1, ϕ_2 and ϕ_3 for large $|x|$ we note that we have to consider only the behaviour of the last integral in each expression. We put $2\cos kx = \exp(ik|x|) + \exp(-ik|x|)$ in these integrals so that

$$\int_0^\infty 1/K \{ k - s_2 (k+K) \} \sinh kh - \cosh kh] \exp(-ky) \frac{W}{\Delta} \cos kx \, dk$$

$$= \int_0^\infty I e^{ik|x|} dk + \int_0^\infty I e^{-ik|x|} dk, \text{ say.} \tag{3.24}$$

For the first integral of (3.24) we consider in the complex k-plane a contour in the first quadrant bounded by a portion of the real axis of large length X_1 with indentations below the poles at $k = K_1, = k_0$, a circular arc Γ of radius X_1 with centre at the origin and the line joining the origin with point $X_1 e^{i\alpha}$ where $0 < \alpha < \pi/2$. Now the integrals along the arc Γ and this line become exponentially small for large $|x|$. The contribution from the poles $\alpha_m + i\beta_m$, say, in the first quadrant which lie inside the contour has also a factor $\exp(-\beta_m|x|)$ which becomes exponentially small for large $|x|$. The line may cross some complex zeros of $\Delta(k)$ in the first quadrant. To account for this, if it crosses a zero of $\Delta(k)$ we indent the line about it so that it lies outside the region bounded by these contours, and the contribution for this indentation will also contain a factor which becomes exponentially small for large $|x|$. Thus for considering the behaviour as $|x| \to \infty$, we only need to consider the behaviour of the integral arising from the residues at $k = K_1$ and $k = k_0$. Hence making $X_1 \to \infty$ we find that, as $|x| \to \infty$,

$$\int_0^\infty I \exp(ik|x|)\, dk \to 2\pi i \{ \text{sum of the residues of}$$

$$I \exp(ik|x|) \text{ at } k = K_1 \text{ and } k = k_0 \}.$$

For the second integral of (3.24) we consider in the complex k-plane a contour in the fourth quadrant bounded by the real axis from 0 to X_1 with indentations below the poles at $k = K_1$ and $k = k_0$, a circular arc Γ' of radius X_1 with centre at the origin and the line joining the origin with the point $X_1 \exp(-i\alpha)$ where $0 < \alpha < \pi/2$. Since now the singularities on the positive real axis are taken to be outside this contour, following a similar argument as above we obtain as $|x| \to \infty$.

$$\int_0^\infty I \exp(-ik|x|)\, \mathrm{d}k \to 0. \text{ Hence we find that as } |x| \to \infty.$$

$$\phi_1 \to \pi i \Big[\frac{1}{K} \{ K_1 - s_2(K_1 + K) \} \sinh K_1 h - \cosh K_1 h \Big] \Big(\frac{W}{\Delta'} \Big)_{k = K_1}$$

$$\times \exp\{ K_1 (i|x| - y) \} + \pi i \Big[\frac{1}{K} \{ k_0 - s_2(k_0 + k) \}$$

$$\sinh k_0 h - \cosh k_0 h \Big] \Big(\frac{W}{\Delta'} \Big)_{k = k_0} \exp\{ k_0 (i|x| - y) \}$$

where $\Delta' = \mathrm{d}\Delta/dk$.

Similarly, we can obtain as $|x| \to \infty$,

$$\phi_2 \to \pi i \Big[\frac{1}{K} \{ s_2(K_1 + K) - K_1 \} \cosh K_1 (h+y) + \sinh K_1 (h+y) \Big] \times$$

$$\times \Big(\frac{W}{\Delta'} \Big)_{k=K_1} \exp(i K_1 |x|) + \pi i \Big[\frac{1}{K} \{ s_2(k_0 + K) - k_0 \} \cosh k_0 (h+y) +$$

$$+ \sinh k_0 (h+y) \Big] \Big(\frac{W}{\Delta'} \Big)_{k = k_0} \exp(ik_0 |x|),$$

$$\phi_3 \to \pi i \left(\frac{W}{\Delta'} \right)_{k=K_1} \exp \{ K_1 (h+y+i|x|) + \pi i \left(\frac{W}{\Delta'} \right)_{k=k_0}$$

$$\exp \{ k_0 \ (h+y+i|x|) \}.$$

Thus ϕ_1, ϕ_2, ϕ_3 satisfy the radiation condition as $|x| \to \infty$. Putting $s_2 = 0$, the far field behaviour of ϕ_1 and ϕ_2 agrees with the results obtained earlier by Chakrabarti and Mandal[8]

4. Line singularity submerged in lower fluid

Let there be a logarithmic type singularity at the point $(0, \eta)$, then $\phi_1 \to \log R_0$ as $R_0 \to 0$. (4.1)

Proceeding similarly as in § 3, we can obtain

$$\phi_1 = \log R_0 - \mu \log R_0 + \frac{4 s_1}{(1+s_1)^2} \sum_0^\infty (-1)^j \gamma^j \mu^{j-1} \log R$$

$$- \int_0^\infty \frac{F_1}{K} \ \sinh kh \exp(-ky) \cos kx \, dk +$$

$$+ \int_0^\infty \left[\frac{1}{K} \ \{ k - s_2 (k+K) \} \ \sinh kh - \cosh kh \right] \frac{W_1}{\Delta} \exp(-ky) \cos kx \, dk,$$

$$\phi_2 = \frac{2}{1+s_1} \left[\log R_0 + \sum_1^\infty (-1)^j \mu^j \gamma^j \log R_{-j} + \sum_1^\infty (-1)^j \mu^{j-1} \gamma^j \log R'_{-j} \right]$$

$$+ \int_0^\infty \frac{F_1}{K} \ \cosh k\,(h+y) \cos kx \, dk$$

$$+ \int_0^\infty \left[\frac{1}{K} \ \{ s_2 (k+K) - k \} \cosh k(h+y) + \sinh k\,(h+y) \right] \frac{W_1}{\Delta} \ \cos kx \, dk,$$

$$\phi_3 = \frac{4}{(1+s_1)(1+s_2)} \sum_0^\infty (-1)^j \gamma^j \mu^j \log R_{-j}$$

$$+ \int_0^\infty \frac{W_1}{\Delta} \ \exp\{ k(h+y) \} \cos kx \, dk \tag{4.4}$$

where Δ is given by (3.20), and

$$W_1 = E_1 - \frac{F_1}{K} \{ (K-k+s_1 k) \sinh kh + s_1 \ K \cosh kh \},$$

$$E_1 = -2\,\mu \exp(-k\eta)\ [\ 1 + \frac{2s_1\ \gamma \exp(-2kh)}{(1+s_1)(1+\mu\gamma\exp(-2kh))}\ ],$$

$$F_1 = \frac{4\gamma \exp\{-k(h+\eta)\}}{(1+s_1)[1+\mu\gamma\exp(-2kh)]}\ . \tag{4.5}$$

If we now put $\rho_1=\rho_2=\rho_3$ so that $\mu=\gamma=0$ in (4.3), (4.4), (4.5) then we obtain $\phi_1=\phi_2=\phi_3=\log R_0$ which is the potential function for a line source at $(0,\eta)$ in an infinite fluid,

As $|x| \to \infty$ we can show that

$$\phi_1 \sim \pi i\ [\frac{1}{K}\ \{K_1 - s_2(K_1+k)\}\sinh K_1 h - \cosh K_1 h\ ]\ (\frac{W_1}{\Delta'})_{k=K_1} \times$$

$$\times \exp K_1(i\,|x| - y) +$$

$$+\ \pi i\ [\ \frac{1}{K}\{k_0 - s_2(k_0+K)\}\sinh k_0 h - \cosh k_0 h\ ]\ (\frac{W_1}{\Delta'})_{k=k_0} \times$$

$$\times \exp\{k_0(i|x|-y\},$$

$$\phi_2 \sim \pi i\ [\ \frac{1}{K}\ \{s_2(K_1+k) - K_1\}\cosh K_1(h+y) + \sinh K_1(h+y)\ ]\ (\frac{W_1}{\Delta'})_{k=K_1} \times$$

$$\times \exp(iK_1\,|x|\,) +$$

$$+\ \pi i\ [\ \frac{1}{K}\{s_2(k_0+K) - k_0\}\cosh k_0(h+y) + \sinh k_0(h+y)\ ]\ (\frac{W_1}{\Delta'})_{k=k_0} \times$$

$$\times \exp(ik_0|x|),$$

$$\phi_3 \sim \pi i\ (\frac{W_1}{\Delta'})_{k=K_1} \exp\{\ K(h+y+i\,|x|\,)\} + \pi i\ (\frac{W_1}{\Delta'})_{k=k_0} \exp\{\ k_0(h+y+i\,|x|)$$

where $\Delta' = d\Delta/dk$, Δ being given by (3.20).

5. Line singularity submerged in upper fluid

In this case, the singularity is situated at the point $(0,-2h+\eta)$, say, so that

$$\phi_3 \sim \log R_1 \text{ as } R_1 \to 0. \tag{5.1}$$

Proceeding as in § 3, it can be shown that

$$\phi_1 = \frac{4\, s_1 s_2}{(1+s_1)(1+s_2)} \sum_1^{\infty} (-1)^{j-1} (\mu\gamma)^{j-1} \log R_j - \int_0^{\infty} \frac{F_2}{K} \sinh kh$$

$$\times \exp(-ky) \cos kx\, dk + \oint_0^{\infty} \left[\frac{1}{K} \{ k - s_2 (k+K) \} \right.$$

$$\left. \sinh kh - \cosh kh \right] \frac{W_2}{\Delta} \exp(-ky) \cos kx\, dk, \tag{5.2}$$

$$\phi_2 = \frac{2\, s_2}{1+s_2} \sum_1^{\infty} (-1)^{j-1} (\mu\gamma)^{j-1} \log R_j + \frac{2\, s_2}{1+s_2} \sum_1^{\infty} (-1)^{j-1} \mu^j \gamma^{j-1} \log R'_{-j} +$$

$$+ \oint_0^{\infty} \frac{F_2}{K} \cosh k(h+y) \cos kx\, dk +$$

$$+ \oint_0^{\infty} \left[\frac{1}{K} \{ s_2 (k+K) - k \} \cosh k(h+y) + \sinh k(h+y) \right] \frac{W_2}{\Delta} \cos kx\, dk, \tag{5.3}$$

$$\phi_3 = \log R_1 + \gamma \log R'_0 + \frac{4\, s_2}{(1+s_2)^2} \sum_1^{\infty} (-1)^{j+1} \mu^j \gamma^{j-1} \log R'_{-j} +$$

$$+ \oint_0^{\infty} \frac{W_2}{\Delta} \exp\{ k(h+y) \} \cos kx\, dk$$

where

$$W_2 = E_2 - \frac{F_2}{K} \{ (K-k) \sinh kh + s_1 (K \cosh kh + k \sinh kh) \},$$

$$E_2 = \frac{4 s_1 s_2\, \mu \exp\{ k(\eta - 2h) \}}{(1+s_2)\{ 1 + \mu\gamma \exp(-2kh) \}},$$

$$F_2 = \frac{2\, s_2}{1+s_2} \exp\{ -k(h-\eta) \} \left[s_2 + \frac{\mu\gamma \exp(-2kh) - 1}{1 + \mu\gamma \exp\{-2kh\}} \right].$$

As earlier, by putting $\rho_1 = \rho_2 = \rho_3$, it is easily verified that $\phi_1 = \phi_2 = \phi_3 = \log R_1$ which is the potential function in an infinite fluid due to a line singularity at the point $(0, -2h + \eta)$.

The behaviour of ϕ_1, ϕ_2, and ϕ_3 as $|x| \to \infty$ can be shown as the outgoing waves

$$\phi_1 \sim \pi i \left[\frac{1}{K} \{ K_1 - s_2(K_1+K) \} \sinh K_1 h - \cosh K_1 h \right] \exp(-K_1 y) \times$$

$$\left(\frac{W_2}{\Delta'} \right)_{k=K_1} \exp(iK_1 |x|)$$

$$+ \pi i \left[\frac{1}{K} \{ k_0 - s_2(k_0+K) \} \sinh k_0 h - \cosh k_0 h \right] \exp(-k_0 y) \times$$

$$\times \left(\frac{W_2}{\Delta'} \right)_{k=k_0} \exp(ik_0 |x|),$$

$$\phi_2 \sim \pi i \left[\frac{s_2(K_1+K)-K_1}{K} \cosh K_1(h+y) + \sinh K_1(h+y) \right] \left(\frac{W_2}{\Delta'} \right)_{k=K_1} \times$$

$$\times \exp(iK_1 |x|) +$$

$$+ \pi i \left[\frac{s_2(k_0+K)-k_0}{K} \cosh k_0(h+y) + \sinh k_0(h+y) \right] \left(\frac{W_2}{\Delta'} \right)_{k=k_0}$$

$$\exp(ik_0 |x|),$$

$$\phi_3 \sim \pi i \left(\frac{W_2}{\Delta'} \right)_{k=K_1} \exp\{K_1(h+y+i|x|)\} + \pi i \left(\frac{W_2}{\Delta'} \right)_{k=k_0}$$

$$\exp\{k_0(h+y+i|x|)\}.$$

6. Multipoles submerged in the middle fluid

We consider only point singularities for which the y-axis is an axis of symmetry, so that ϕ_1, ϕ_2, ϕ_3 are independent of the azimuthal angle, and satisfy the same set of equations of § 2.

In the present case, let there be a point source at $(0, -\eta)$, then

$$\phi_2 \sim \frac{P_n(\cos\theta)}{R_0'^{n+1}} \text{ as } R_0' = \{ r^2 + (y+\eta)^2 \}^{1/2} \to 0, n = 0, 1, 2, \ldots \tag{6.1}$$

where r is the distance from the y-axis and $\theta = \tan^{-1}\left(\frac{r}{y+\eta} \right)$.

Let us assume

$$\phi_1 = \int_0^\infty A(k) \exp(-ky) J_0(kr)\, dk, \tag{6.2}$$

$$\phi_2 = \frac{P_n(\cos\theta)}{R_0'^{n+1}} + \int_0^\infty \{ B(k)\cosh k(n+y) + C(k)\sinh k(h+y)\} J_0(kr)dk, \tag{6.3}$$

$$\phi_3 = \int_0^\infty D(k)\exp(ky) J_0(kr)\, dk. \tag{6.4}$$

The following integral representation is necessary,

$$\begin{aligned} \frac{P_n(\cos\theta)}{R_0'^{n+1}} &= \frac{1}{n!}\int_0^\infty k^n \exp\{ -k(y+\eta) \} J_0(kr)dk, y>-\eta \\ &= \frac{(-1)^n}{n!}\int_0^\infty k^n \exp k(y+\eta) J_0(kr)\, dk,\ y<-\eta. \end{aligned} \tag{6.5}$$

Using this integral representation and proceeding somewhat similar to § 3, we can obtain

$$\phi_1 = \int_0^\infty \frac{k^n}{n!}\left[\exp(-k\eta) + (-1)^n(1-s_2)\frac{k+K}{K} - \sinh kh \exp\{ -k(\mathrm{h}-\eta) \} \right]$$

$$\times \exp(-ky) J_0(kr)\, dk +$$

$$+ \oint_0^\infty \left[\frac{k-s_2(k+K)}{K}\sinh kh - \cosh kh \right] \exp(-ky)\frac{V}{\Delta} J_0(kr)\, dk,$$

$$\phi_2 = \frac{P_n(\cos\theta)}{R_0'^{n+1}} + \frac{(-1)^n}{n!}\ \frac{s_2-1}{K}\int_0^\infty (k+K)k^n \exp\{ -k(h-\eta) \}$$

$$\cosh k(h+y) J_0(kr)\, dk$$

$$+ \oint_0^\infty \left[\frac{1}{K}\{ s_2(k+K)-k \} \cosh k(h+y) + \sinh k(h+y) \right]\frac{V}{\Delta} J_0(kr)\, dk,$$

$$\phi_3 = \frac{(-1)^n}{n!}\int_0^\infty k^n \exp\{ k(\eta+y) \} J_0(kr)dk +$$

$$\oint_0^\infty \frac{V}{\Delta}\exp k(h+y) J_0(kr)dk$$

where Δ is given by (3.20) and

$$V = \frac{(1-s_1)}{n!}\,(k-K)\,k^n \exp(-k\eta)$$

$$+\ \frac{(1-s_2)(-1)^n}{K\quad n!}\,(k+K)\,k^n\,(K-k+s_1 k)\sinh kh +$$

$$s_1 K \cosh kh\,\}\,\exp\{-k\,(h-\eta)\,\}.$$

By substituting $\rho_1 = \rho_2 = \rho_3$ it is verified that

$$\phi_1 = \phi_2 = \phi_3 = \frac{P_n(\cos\theta)}{R_0'^{\,n+1}}$$

which is obviously the potential in an infinite fluid due to a point source at $(0,-\eta)$.

Now putting $2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$ and rotating the contour in the integrals involving $H_0^{(1)}(kr)$ in the first quadrant and in the integrals involving $H_0^{(2)}(kr)$ in the fourth quadrant, we can reduce the integrals into suitable forms from which the farfield behaviour of ϕ_1, ϕ_2, ϕ_3 as $r \to \infty$ have the following forms

$$\phi_1 \sim \pi i\,\Big[\,\frac{1}{K}\,\{\,K_1 - s_2(K_1+K)\,\}\,\sinh K_1 h - \cosh K_1 h\,\Big] \times$$

$$\times\,\Big(\frac{V}{\Delta'}\Big)_{k=K_1} \exp(-K_1 y)\,H_0^{(1)}(K_1 r)\,+$$

$$+\ \pi i\,\Big[\,\frac{1}{K}\,\{\,k_0 - s_2(k_0+K)\,\}\,\sinh k_0 h - \cosh k_0 h\,\Big]\,\Big(\frac{V}{\Delta'}\Big)_{k=k_0}$$

$$\exp(-k_0 y)\,H_0^{(1)}(k_0 r),$$

$$\phi_2 \sim \pi i\,\Big[\,\frac{1}{K}\,\{\,s_2(K_1+K) - K_1\,\}\,\cosh K_1(h+y) + \sinh K_1(h+y)\,\Big]$$

$$\Big(\frac{V}{\Delta'}\Big)_{k=K_1} H_0^{(1)}(K_1 r)$$

$$+\ \pi i\,\Big[\,\frac{1}{K}\,\{\,s_2(k_0+K) - k_0\,\}\cosh k_0(h+y) + \sinh k_0(h+y)\,\Big]$$

$$\Big(\frac{V}{\Delta'}\Big)_{k=k_0} H_0^{(1)}(k_0 r),$$

$$\phi_3 \sim \pi i \exp\{K_1(h+y)\}\left(\frac{V}{\Delta'}\right)_{k=K_1} H_0^{(1)}(K_1 r) + \pi i \exp\{k_0(h+y)\}$$

$$\left(\frac{V}{\Delta'}\right)_{k=k_0} H_0^{(1)}(k_0 r)$$

7. Multipoles submerged in the lower fluid

In this case $\phi_1 \sim \frac{P_n(\cos\psi)}{R_0^{n+1}}$ as $R_0 = \{r^2+(y-\eta)^2\}^{1/2} \to 0 \qquad n = 0, 1, 2,$

where $\psi = \tan^{-1}(r/(y-\eta))$. It can be shown that

$$\phi_1 = \frac{P_n(\cos\psi)}{R_0^{n+1}} + \frac{(-1)^n}{n!}\int_0^\infty k^n \exp\{-k(y+\eta)\} J_0(kr)\,dk -$$

$$-1/K \oint_0^\infty [\{s_2(k+K)-k\}\sinh kh + \cosh kh]\frac{V_1}{\Delta}\exp(-ky) J_0(kr)\,dk,$$

$$\phi_2 = \oint_0^\infty [1/K\{s_2(k+K)-k\}\cosh k(h+y) + \sinh k(h+y)]\frac{V_1}{\Delta} J_0(kr)\,dk$$

$$\phi_3 = \int_0^\infty \frac{V_1}{\Delta}\exp\{k(h+y)\} J_0(kr)\,dk$$

where $V_1 = 2K\frac{(-1)^n}{n!}k^n \exp(-k\eta)$. As $r \to \infty$,

$$\phi_1 \sim -\pi i\, 1/K [\{s_2(K_1+K)-K_1\}\sinh K_1 h + \cosh K_1 h]\left(\frac{V_1}{\Delta'}\right)_{k=K_1}$$

$$\exp(-K_1 y) H_0^{(1)}(K_1 r),$$

$$\phi_2 \sim -\pi i\frac{1}{K}[\{s_2(k_0+K)-k_0\}\sinh k_0 h + \cosh k_0 h]\left(\frac{V_1}{\Delta'}\right)_{k=k_0} \times$$

$$\times \exp(-k_0 y) H_0^{(1)}(k_0 r),$$

$$\phi_3 \sim \pi i\left(\frac{V_1}{\Delta'}\right)_{k=k_1}\exp K_1(h+y) H_0^{(1)}(K_1 r) + \pi i\left(\frac{V_1}{\Delta'}\right)_{k=k_0}$$

$$\exp\{k_0(h+y)\} H_0^{(1)}(k_0 r).$$

8. Multipoles submerged in the upper fluid

In this case

$$\phi_3 \sim \frac{P_n(\cos\chi)}{R_1^{n+1}} \text{ as } R_1 = \{ r^2 + (2h+y-\eta)^2 \}^{1/2} \to 0 \quad n=0, 1, 2 \ldots$$

where $\chi = \tan^{-1}\{ r/(2h+y-\eta) \}$. The velocity potentials are given by

$$\phi_1 = -\frac{2s_2}{n!} \int_0^\infty k^n \sinh kh \exp\{-k(h+y-\eta)\} J_0(kr)\, dk +$$

$$+ \int_0^\infty \left[\frac{1}{K} \{ k - s_2(k+K) \} \sinh kh - \cosh kh \right] \frac{V_2}{\Delta} \exp(-ky) J_0(kr)\, dk,$$

$$\phi_2 = \frac{2s_2}{n!} \int_0^\infty k^n \exp\{-k(h-\eta)\} \cosh k(h+y) J_0(kr)\, dk +$$

$$+ \int_0^\infty \left[\frac{1}{K} \{ s_2(k+K) - k \} \cosh k(h+y) + \sinh k(h+y) \right] \frac{V_2}{\Delta} J_0(kr)\, dk,$$

$$\phi_3 = \frac{P_n(\cos\chi)}{R_1^{n+1}} + \int_0^\infty \frac{k^n}{n!} \exp\{ k(y+\eta) \} J_0(kr)\, dk +$$

$$+ \int_0^\infty \frac{V_2}{\Delta} \exp\{ k(h+y) \} J_0(kr)\, dk,$$

where $V_2 = -2s_2 \dfrac{k^n}{n!} \exp\{-k(h-\eta)\} [(K-k+s_2K) \sinh kh + s_2 K \cosh kh]$.

As $r \to \infty$ we can show that

$$\phi_1 \sim \pi i \left[\frac{1}{K} \{ K_1 - s_2(K_1+K) \} \sinh K_1 h - \cosh K_1 h \right] \times$$

$$\times \left(\frac{V_2}{\Delta'} \right)_{k=K_1} \exp(-K_1 y) H_0^{(1)}(K_1 r)$$

+ a similar expression with K_1 replaced by k_0,

$$\phi_2 \sim \pi i \left[\frac{1}{K} \{ s_2(K_1+K) - K_1 \} \cosh K_1(h+y) + \sinh K_1(h+y) \right] \times$$

$$\times \left(\frac{V_2}{\Delta'} \right)_{k=K_1} H_0^{(1)}(K_1 r)$$

+ a similar expression with K_1 replaced by k_0,

$\phi_3 \sim \pi i \left(\dfrac{V_2}{\Delta'} \right)_{k=K_1} \exp\{ K_1(h+y) \} H_0^{(1)}(K_1 r)$ + a similar expression with K_1 replaced by k_0.

9. Conclusion

Integral representations of the potential function in different fluids of a three-layered fluid medium are obtained. When the upper medium is taken to be vacuo earlier results for the case of a two fluid medium are recovered (*cf.* Chakrabarti and Mandal[8]). Again, when the three-fluid medium is reduced to an infinite one-fluid medium by making the densities equal, the corresponding results for the infinite one-fluid medium are readily recovered. Also, the extension of the problem to the case where the lower fluid is of finite depth H, say, instead of infinity is not difficult, although the final result will be more complicated.

It may be noted that in the construction of the line source potentials in the present paper by the image method, an infinite set of image sources due to the two surfaces of separation has been introduced. Usefulness of this image method can be demonstrated as follows.

In the simple case of a line source at $(0, \eta)$ in a single layer of finite constant depth, there exists an infinite set of image sources due to the free surface and the bottom. Without using the whole set of images, Thorne[1] used only the image source due to the free surface and constructed the potential function as

$$\phi = \log\frac{R_0}{R_0'} + 2\int_0^\infty \left\{ \frac{\cosh k(h-\eta)\cosh k(h-y)}{K\cosh kh - k\sinh kh} - \frac{\exp(-kh)}{k}\sinh k\eta \sinh ky\right\} \frac{\cos kx}{\cosh kh}\, dk \tag{9.1}$$

where R_0 is the distance from the source and R_0' is the distance from the image source. However, if we introduce all the image sources then we obtain

$$\phi = \log\frac{R_0}{R_0'} + \sum_0^\infty (-1)^j\left(\log\frac{R_j}{R_j'} + \log\frac{R_{-j}}{R_{-j}'}\right) + 2\int_0^\infty \frac{\sinh k(h-\eta)\cosh k(h-y)}{K\cosh kh - k\sinh kh}\,\frac{\cos kx}{\cosh kh}\,dk. \tag{9.2}$$

By using the representation

$$\log\frac{x^2+\alpha^2}{x^2+\beta^2} = 2\int_0^\infty \frac{1}{k}\{\exp(-\beta k)-\exp(-\alpha k)\}\cos kx\, dk$$

it can be shown that (9.2) reduces to (9.1). Thus the sum of the image potentials (excepting the image at $(0, -\eta)$) in (9.2) can be expressed as an integral and can be combined with the integral in (9.2) to give the integral in (9.1).

This naturally will motivate one to construct the potential functions in a layered medium by a similar technique used by Thorne[1]. However, this will lead to the appearance of some divergent integrals in the resulting expressions of the potential functions. To demonstrate this, we now take a simple case where we consider the construction of potentials in two superposed infinite fluids with a line source present in the lower fluid at $(0, \eta)$. By using the image method (there is only one image due to the surface of separation), Gorgui and Kassem[3] obtained the following result

$$\phi_1 = \log R_0 - \frac{1-s}{1+s} \log R_0' - \frac{2(1-s)}{1+s} \oint_0^\infty \frac{\exp\{-k(y+\eta)\}}{\Delta} \cos kx\,dk, y>0,$$

$$\phi_2 = \frac{2}{1+s} \log R_0 + \frac{2(1-s)}{1+s} \oint_0^\infty \frac{\exp\{k(y-\eta)\}}{\Delta} \cos kx\,dk,\ y<0, \tag{9.3}$$

where $\Delta = (1-s)\,k-(1+s)\,K$, s being the ratio of the densities of the upper and lower fluids respectively. One may note that the integrals in (9.3) are convergent but ϕ_i's become unbounded at infinity although grad ϕ_i's remain bounded. We can also construct ϕ_1, ϕ_2 by the method[1] as

$$\phi_1 = \log \frac{R_0}{R_0'} + \int_0^\infty X \exp(-ky) \cos kx\,dk, y>0$$

$$\phi_2 = \int_0^\infty Y \exp(ky) \cos kx\,dk,\ y<0,$$

where X, Y can be obtained from the two SS conditions. The resulting expressions for ϕ_1, ϕ_2 are

$$\phi_1 = \log \frac{R_0}{R_0'} + 2 \oint_0^\infty \left\{ \frac{s\,(k+K)}{k} - 1 \right\} \frac{\exp\{-(y+\eta)\}}{\Delta} \cos kx\,dk, y>0,$$

$$\phi_2 = -2 \int_0^\infty \frac{\exp\{k(y-\eta)\}}{k} \cos kx\,dk - 2 \oint_0^\infty \left\{ \frac{s(k+K)}{k} - 1 \right\} \tag{9.4}$$

$$\frac{\exp\{k(y-\eta)\}}{\Delta} \cos kx\,dk,\ y<0.$$

It is obvious that the integrals in (9.4) are divergent as the integrands have a pole at $k=0$. However, the expressions in (9.4) can be identified with those in (9.3) if one is willing to replace the divergent integrals

$$\int_0^\infty k^{-1} \exp\{-k\,(y+\eta)\} \cos kx\,dk \text{ and } \int_0^\infty k^{-1} \exp\{-k\,/\,y+\eta)\} \cos kx\,dk$$

appearing in (9.4) by the unbounded functions log R_0' and log R_0 respectively. In fact, the appearance of divergent integrals in (9.3) is not unexpected and this reflects the unbounded nature of the potential functions. We may point out here that in a single layer fluid, this unbounded nature of the potentials does not exist.

Thus to avoid the appearance of divergent integrals in the potentials, the image method used here seems to be convenient.

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