J. Indian Inst. Sci. 65(B), Nov. 1984, pp. 245-253

Rayleigh waves in magneto-thermoelastic rotating media with thermal relaxation

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Abstract

Following a linear theory of magneto-thermoelasticity with thermal relaxation, the propagation of Rayleigh waves in a semi-infinite, homogeneous, isotropic, electrically and thermally conducting body permeated by a magnetic field parallel to the boundary surface is investigated. It is assumed that the entire elastic medium is rotating with a uniform angular velocity. Frequency equation is obtained and is analysed for small and large values of the frequency and large values of spin velocity and the magnetic field. The dependence between the angular frequency and the surface wave speed is graphically shown.

Key words: Spin velocity, thermal relaxation time, perturbation.

1. Introduction

The study of coupled bulk magnetothermoelastic surface waves has been the subject of many works by applied physicists and theoretical mechanicians alike. Nayfeh and Nemat-Naseer[1] have analysed propagation pattern of Rayleigh surface waves in a thermoelastic half-space. Explicit expressions have been obtained for various parameters that characterize these waves. Tomita and Shindo[2] have shown that the variation of Rayleigh wave speed against the magnetic pressure number is perceptible taking the thermal relaxation time parameters zero. Roy Choudhuri and Debnath[3-4] have considered the plane wave problem in a rotating medium. They have shown that the rotation causes the medium to be dispersive and anisotropic. The objective of the present paper is to consider a problem of surface waves in a thermoelastic medium permeated by a primary uniform magnetic field, rotating with a uniform angular velocity. A detailed numercial work is undertaken to find out the nature of dependence of surface wave speed on angular velocity. It is observed that even ten-fold increase in the spin velocity has only little effect on the Rayleigh wave speed. However, the change is appreciable in a particular range of frequency. The computations are carried out for the large angular frequencies.

2. Formulation and basic equations

We consider a semi-infinite homogeneous, isotropic, thermally and electrically-conducting elastic solid permeated by a primary magnetic field. $\underline{B}_0$ (B_1, B_2, B_3). The entire elastic medium is rotating uniformly with an angular velocity $\underline{\Omega} = \Omega\, \underline{\omega}$, where $\underline{\omega}$ is the unit vector representing the direction of the axis of rotation. The displacement equation is given by

$$\rho\,[\underline{u} + \underline{\Omega} \times (\underline{\Omega} \times \underline{u}) \times 2\underline{\Omega} \times \underline{u}\,] = (\lambda + \mu)\,\nabla\,(\,\nabla \cdot \underline{u}) + \mu\,\nabla^2\,\underline{u} +$$

$$\underline{J} \times \underline{B} - \beta\,\nabla\,T \tag{1}$$

where the terms $\underline{\Omega} \times (\underline{\Omega} \times \underline{u})$ and $2\underline{\Omega} \times \dot{\underline{u}}$ are centripetal and coriolis accelerations respectively. $\underline{J} \times \underline{B}$ is electromagentirc force. J is the current density, $\underline{B} = \underline{B}_0 + \underline{b}$ is the total magnetic field, $\underline{b}$ (b_x, b_y, b_z) is the perturbed magnetic field assumed to be small, T is the increase in temperature above the reference temperature T^*

The generalized heat conduction equation with thermal relaxation time is

$$\kappa\,\nabla^2\,T = \rho\,C_v\,(\,\dot{T} + \tau\,\ddot{T}) + \beta\,T^*\,(\,\dot{\Delta} + \tau\,\ddot{\Delta}\,) \tag{2}$$

Here, τ is the thermal relaxation time, Δ is the dilatation, κ is the coefficient of thermal conductivity and C_v is the specific heat of solid at constant volume.

The electromagnetic field is governed by the Maxwell equations with the displacement current and charge density neglected

$$\text{curl}\ \underline{H} = \underline{J},\ \text{curl}\ \underline{E} = -\,\partial\,\underline{B}/\,\partial t,\ \text{div}\ \underline{B} = 0 \tag{3}$$

where $\underline{B} = \mu_e\,\underline{H}$ μ_e is the magnetic permeability.

The generalized Ohm's law is

$$\underline{J} = \sigma\,[\,\underline{E} + (\,\partial\,\underline{u}/\,\partial\,t + \underline{\Omega} \times \underline{u}) \times \underline{B}] \tag{4}$$

For the Rayleigh surface waves, we shall deal with the half-space defined by $z \geq 0$, where we assume that both the surface tractions and the temperature gradient vanish on the plane $z = 0$. The solution of the problem can be expressed as

$$\begin{aligned}
\underline{u} &= (p_0, q_0, r_0)\,\exp\,[\,-\,\alpha z + i\omega t + ikx\,] \\
T &= T_0\,\exp\,[-\,\alpha z + i\omega t + ikx] \\
\underline{J} &= (J_1, J_2, J_3)\,\exp\,[-\,\alpha z + i\omega t + ikx] \\
\underline{B} &= (b_1, b_2, b_3)\,\exp\,[-\,\alpha z + i\omega t + ikx] \\
\underline{\Omega} &= [\Omega_1, \Omega_2, \Omega_3]\,\exp\,[-\,\alpha z + i\omega t + ikx] \\
\underline{E} &= [E_1, E_2, E_3]\,\exp\,[-\,\alpha z + i\omega t + ikx]
\end{aligned} \tag{5}$$

and for the Rayleigh waves we require α to have positive real part. The solutions (5) represent plane harmonic waves which propagate in the positive x-direction and these waves decay exponentially with the depth in the positive z-direction.

Substitution from (5) into (3) and (4) yields

$$\underline{J} = (J_1, J_2, J_3) = \left[\frac{b_2 \alpha}{\mu_e} \quad \frac{b_2 \alpha - b_3 ik}{\mu_e}, \frac{b_2 ik}{\mu_e} \right] \tag{6}$$

as div $\underline{B} = 0$ leads to $b_1 \, ik = \alpha b_3$ for $t \geq 0$ (7)

$$J_1 = \sigma [E_1 + i\omega (B_3 q_0 - B_2 r_0) + B_3 (p_0 \Omega_3 - r_0 \Omega_1) - B_2 (q_0 \Omega_1 - p_0 \Omega_2)]$$

$$J_2 = \sigma [- b_3 \frac{\omega}{k} + i\omega (B_1 r_0 - B_3 p_0) + B_1 (q_0 \Omega_1 - p_0 \Omega_2)$$

$$- B_3 (r_0 \Omega_2 - q_0 \Omega_3)] \tag{8}$$

$$J_3 = \sigma [(\frac{\omega}{k}) b_2 - E_1 \frac{b_1}{b_3} + i\omega (B_2 p_0 - B_1 q_0) + B_2 (r_0 \Omega_2 - q_0 \Omega_3)$$

$$- B_1 (p_0 \Omega_3 - r_0 \Omega_1)]$$

Here $\underline{E} = [E_1, - b_3 \frac{\omega}{k}, \frac{\omega}{k} b_2 - E_1 \frac{b_1}{b_3}]$ (9)

Eliminating $\underline{J}$ from (8) by using (6), and the first equation of which defines E_1, we get,

$$p_0 [- i\omega B_3 - B_1 \Omega_2] + q_0 [B_1 \Omega_1 + B_3 \Omega_3] + r_0 [i\omega B_1 - B_3 \Omega_2]$$

$$= b_3 \frac{\omega}{k} - \frac{b_1 \alpha + b_3 ik}{\sigma \mu_e} \tag{10}$$

$$p_0 [- \frac{i\alpha}{k} (B_3 \Omega_3 + B_2 \Omega_2) + i\omega B_2 - B_1 \Omega_3]$$

$$+ q_0 [- \frac{i\alpha}{k} (i\omega B_3 - B_2 \Omega_1) - i\omega B_1 - B_2 \Omega_3]$$

$$+ r_0 [- \frac{i\alpha}{k} (i\omega B_2 - B_3 \Omega_1) + B_1 \Omega_1 + B_2 \Omega_2]$$

$$= b_2 \frac{ik}{\sigma \mu_e} - \frac{\omega}{k} - \frac{i\alpha^2}{k \sigma \mu_e} \tag{11}$$

Equation (1), using (5) gives,

$$p_0\,[-\rho\,(\omega^2+\Omega_2^2+\Omega_3^2)+(\lambda+2\mu)\,k^2-\mu\alpha^2]+q_0\,[\rho(\Omega_1\,\Omega_2-2i\omega\Omega_3)]$$

$$+\,r_0\,[\rho(\Omega_1\Omega_3+2i\omega\Omega_2)+(\lambda+\mu)\,ik\alpha]=-\frac{B_3}{\mu_e}\,(b_1\alpha+ikb_3)$$

$$-\,\frac{B_2}{\mu_e}\,ikb_2-\beta\,T_0\,ik \tag{12}$$

$$p_0\,[\rho\,(\Omega_1\,\Omega_2+2i\omega\,\Omega_3)]+q_0\,[-\rho\,(\omega^2+\Omega_1^2+\Omega_3^2)-\mu\,(\alpha^2-k^2)]$$

$$+\,r_0\,[\rho\,(\Omega_2\,\Omega_3-2i\omega\Omega_1)]=\frac{iB_1\,b_2\,k}{\mu_e}-\frac{B_3\,b_2\,\alpha}{\mu_e} \tag{13}$$

$$p_0\,[\rho\,\Omega_1\,\Omega_3-2i\omega\,\rho\,\Omega_2\,(\lambda+\mu)\,ik\alpha]+q_0\,[\rho\,(\Omega_2\,\Omega_3+2i\omega\Omega_1)]$$

$$+\,r_0\,[-\rho\,(\omega^2-\Omega_1^2+\Omega_2^2)-(\lambda+\mu)\,\alpha^2-\mu\,(\alpha^2-k^2)]$$

$$=\,1/\mu_e\,[B_2b_2\alpha+B_1\,(b_1\alpha+b_1ik)] \tag{14}$$

Equation (2) leads to

$$p_0\,[-\beta\,T^*\,k\omega-\beta\,T^*\,\tau\,i\omega^2k]+r_0\,[\beta\,T^*\,\tau\omega^2\alpha-\beta\,T^*\,\alpha i\omega]$$

$$=\,T_0\,[k\,(\alpha^2-k^2)-\rho\,C_v\,(i\omega-\tau\omega^2)] \tag{15}$$

Equations (10) to (15) constitute a system of six equations with six unknowns p_0, q_0, r_0, b_1, b_2, T_0 and b_3 being related to b_1 by (7). We assume $\Omega_1=\Omega_2=0$ and $\Omega_3=\Omega$, set the applied and perturbed magnetic fields to be $(0,B_2,0)$ and $(0,b_2,0)$.

We nondimensionalize the equations by introducing

$$\chi=\frac{\omega}{\omega^*},\ \xi=\frac{k\,C_1}{\omega^*},\ \epsilon_T\ \frac{T^*\,\beta^2}{\rho^2\,C_v\,C_1^2},\epsilon_v=\frac{\rho\,C_v}{\beta}$$

$$\epsilon=\frac{\omega^*\,\nu_H}{C_1^2},\ \Omega_0=\frac{\Omega}{\omega^*}$$

$$\epsilon'=\frac{\rho\,\tau\,C_v\,C_1^2}{K},\ R_H=\frac{B_2^2}{\rho\,C_1^2\,\mu_e},\nu_H=\frac{1}{\mu_e\,\sigma}\ \text{and}\ s^2=\frac{C_2^2}{C_1^2} \tag{16}$$

where $$C_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad \omega^* = \frac{\rho C_v C_1^2}{\kappa}$$

$$C_2^2 = \mu/\rho$$

Equations (10) to (15) now take the form

$$p_0^*(x_5 - s^2\eta^2) + q_0^*(-2i\chi\Omega_0) + r_0^* ix_6\eta + T_0^* i\xi^2 \epsilon_T \epsilon_v + b_2^* iR_H \xi^2 = 0$$

$$p_0^*(2i\chi\Omega_0) - q_0^*(s^2\eta^2 + x_4) = 0$$

$$p_0^* x_6\eta + r_0^*(-(x_1 - \eta^2)) + T_0^*(-\eta\epsilon_v \epsilon_T \xi) + b_2^*(-\eta R_H \xi) = 0$$

$$p_0^* i\chi\xi + q_0^*(-\Omega_0\xi) + r_0^*(-\chi\eta) + b_2^*(x_2 + i\epsilon_H \xi\eta^2) = 0$$

$$p_0^*(-x_3\xi) - r_0^* i\eta x_3 + T_0^* \epsilon_T \epsilon_v \xi(x_7 - \eta^2) = 0 \tag{17}$$

where the quantities with asterisks are nondimensional,

$$\eta = \frac{C_1^2 \alpha^2}{\omega^{*2}} \text{ and}$$

$$x_1 = \chi^2 + s^2\xi^2$$
$$x_2 = \chi\xi - i\xi^3\epsilon_H$$
$$x_3 = \chi\epsilon\tau(1 + i\tau'\chi)$$
$$x_4 = \chi^2 + \Omega_0^2 - s^2\xi^2$$
$$x_5 = \xi^2 - (\chi^2 + \Omega_0^2)$$
$$x_6 = \xi(1 - s^2)$$
$$x_7 = i\chi - \chi^2\tau' + \xi^2 \tag{18}$$

We proceed to solve (17) in the next section.

3. General solution and the boundary conditions

The equations (17) admit non-trivial solutions if and only if the determinant of the coefficients of p_0^*, q_0^*, r_0^*, T_0^* and b_2^* is identically zero, resulting in

$$
\begin{aligned}
&\xi s^4\epsilon_H\eta^{10} + (\xi s^2\epsilon_H x_4 + i\xi\epsilon_H s^4 x_3 - \xi s^4\epsilon_H x_7 - is^2 y_6)\eta^8 + (-i\xi s^2 x_3 y_0 \\
&- 4\chi^2\xi\Omega_0^2\epsilon_H - s^2 x_3 y_4 + i\xi\epsilon_H s^2 x_3 x_4 + is^2 x_7 y_6 - is^2 y_7 - \xi s^2\epsilon_H x_4 x_7 \\
&- ix_4 y_6)\eta^6 + (-4i\xi\chi^2\Omega_0^2\epsilon_H x_3 + 4\chi^2\xi\,\Omega_0^2\epsilon_H x_7 + 2\chi\Omega_0 y_2 - i\xi s^2 x_3 y_1 \\
&- i\xi x_3 x_4 y_0 - s^2 x_3 y_5 - x_3 x_4 y_4 + is^2 x_7 y_7 - is^2 y_8 + ix_4 x_7 y_6 - ix_4 y_7)\eta^4 \\
&+ (2\chi\Omega_0 y_3 - 4\chi^2\Omega_0^2 x_2 x_3 - 2\chi\Omega_0 x_7 y_2 - \xi^2 s^2 x_1 x_2 x_3 \\
&- i\xi x_3 x_4 y_1 - x_3 x_4 y_5 + is^2 x_7 y_8 + ix_4 x_7 y_7 - ix_4 y_8)\eta^2 \\
&- (2\chi\Omega_0 x_7 y_3 + \xi^2 x_1 x_2 x_3 x_4 - ix_4 x_7 y_8) = 0
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
y_0 &= \xi\epsilon_H x_6 - \epsilon_H\xi^2 \\
y_1 &= \xi^2\epsilon_H x_1 + i\xi x_2 - ix_2 x_6 \\
y_2 &= 2i\chi\Omega_0 x_2 + i\xi^2\Omega_0 R_H x_6 - 2i\xi\chi^2\,\Omega_0\,R_H - i\xi^3\Omega_0 R_H + 2\xi\chi\Omega_0\epsilon_H x_1 \\
y_3 &= i\xi^3\Omega_0 R_H x_1 - 2i\chi\Omega_0 x_1 x_2 \\
y_4 &= i\xi\epsilon_H x_5 + i\xi^2\epsilon_H x_6 - s^2 x_2 \\
y_5 &= \xi x_2 x_6 - \xi^3\chi R_H + x_2 x_5 + \chi\xi^3 R_H \\
y_6 &= -i\xi\epsilon_H x_6^2 - \chi\xi R_H s^2 - i\epsilon_H\xi s^2 x_1 - i\epsilon_H\xi x_5 + x_2 s^2 \\
y_7 &= \xi^2\chi R_H x_6 + \chi\xi R_H x_5 + \chi\xi^2 R_H x_6 - x_2 x_6^2 + i\xi\epsilon_H x_1 x_5 \\
&\qquad - x_1 x_2 s^2 - x_2 x_5 - \chi\xi^3\,R_H \\
y_8 &= x_1 x_2 x_5 + \chi\xi^3 R_H x_1
\end{aligned}
\tag{20}
$$

We note that η_k^2, $k = 1,2,3,4,5$ are the roots of the equation (19) and we recall that η_k must all have positive real part.

Referring to equations (17), we conclude that to each η_k, $k = 1,2,3,4,5$ there corresponds a set of constants p_{ok}^*, q_{ok}^*, r_{ok}^*, T_{ok}^*, b_{2k}^* and for a fixed value of η, say, η_k, equations (17) are employed to express four of the constants in terms of the other, say q_{ok}^*. Thus, we obtain

$$
\begin{aligned}
p_{0k}^* &= A_{1k}\,q_{0k}^* \\
r_{0k}^* &= A_{2k}\,q_{0k}^* \\
T_{0k}^* &= A_{3k}\,q_{0k}^* \\
b_{2k}^* &= A_{4k}\,q_{0k}^*
\end{aligned}
\tag{21}
$$

where

$$A_{1k} = \frac{s^2 \eta_k^2 + x_4}{2i \chi \Omega_0}$$

$$A_{3k} = \frac{x_3 [A_{1k} (x_2 + i\epsilon_H \xi \eta_k^2) [i\xi (x_1 - \eta_k^2) + i x_6 \eta_k^2] + \eta_k^2 \; R_H \xi^2 \Omega_0]}{\epsilon_T \epsilon_v \xi [(x_2 + i\xi\epsilon_H \eta_k^2) [i (x_7 - \eta_k^2) - \eta_k^2 x_3] + i\chi\xi R_H \eta_k^2 (x_7 - \eta_k^2)]}$$

$$A_{2k} = \frac{i\xi x_3 A_{1k} - i\epsilon_T \epsilon_v \xi (x_7 - \eta_k^2) A_{3k}}{\eta_k x_3}$$

$$A_{4k} = \frac{\xi [\Omega_0 x_3 - i\epsilon_T \epsilon_v \chi (x_7 - \eta_k^2) A_{3k}]}{x_3 (x_2 + i\epsilon_H \xi \eta_k^2)} \tag{22}$$

By superposition, the general solution may now be written as

$$(p^*, q^*, r^*, T_0^*, b_2^*) = \sum_{k=1}^{5} (p_{ok}^*, q_{ok}^*, r_{ok}^*, T_{ok}^*, b_{2k}^*) \times$$

$$\exp [-\eta_k z^* + i\chi t^* + i\xi x^*] \tag{23}$$

where x^* and z^* are nondimensional space coordinates and t^* is nondimensional time coordinate.

The boundary conditions at $z^* = 0$ are

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0, \quad b_2 = 0 \text{ and } \frac{\partial \tau}{\partial z} = 0 \tag{24}$$

Using (23) in (24), we get five homogeneous equations in q_{ok}^*, $k = 1,2,3,4,5$, and for non-trivial solutions, we set the determinant of the coefficients of q_{ok}^* equal to zero, that is,

$$\begin{vmatrix} A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \\ \eta_1 A_{31} & \eta_2 A_{32} & \eta_3 A_{33} & \eta_4 A_{34} & \eta_5 A_{35} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} \end{vmatrix} = 0 \tag{25}$$

where

$$\alpha_{1j} = \eta_j A_{1j} - i\xi A_{2j},$$

$$\alpha_{2j} = i\xi (\beta_1^2 - 2) A_{1j} + \beta_1^2 \eta_j^2 A_{2j} - \xi b A_{3j}, \; j = 1 \text{ to } 5$$

$$\beta_1^2 = 1/s^2, \; b = \alpha_t T_0^* (3\lambda + 2\mu)/\mu$$

Equation (25) is the Rayleigh equation modified by the angular velocity of the medium apart from the magnetic and temperature fields.

4. Special case

In the absence of the spin velocity, we see that A_{1k} becomes unbounded and that the second of equations (17) leads to

$$\chi^2 - s^2 (\xi^2 + \eta^2) = 0 \text{ for } q_0^* \neq 0$$

which represents transverse wave motion. Following the argument given for obtaining (21), we get, for the present case

$$\begin{aligned} r_{0k}^* &= A_{1k}\, T_{0k}^* \\ b_{2k}^* &= A_{2k}\, T_{0k}^* \\ p_{0k}^* &= A_{3k}\, T_{0k}^*,\ k = 1,2,3,4. \end{aligned} \tag{26}$$

where

$$A_{1k} = -\frac{i \epsilon_T \epsilon_v \xi (x_7 - \eta_k^2)}{\eta_k x_3}$$

$$A_{2k} = -\frac{i \chi \epsilon_T \epsilon_v \xi (x_7 - \eta_k^2)}{x_3 (x_2 + i \epsilon_H \xi \eta_k^2)}$$

$$A_{3k} = -\frac{A_{1k} x_6 \eta_k + i \xi^2 \epsilon_T \epsilon_v + i R_H \xi^2 A_{2k}}{x_5 - s^2 \eta_k^2}$$

The frequency equation now takes the form

$$\begin{vmatrix} A_{21} & A_{22} & A_{23} & A_{24} \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{vmatrix} = 0 \tag{27}$$

where

$$\begin{aligned} \alpha_{1j} &= i \xi A_{1j} - A_{3j} \eta_j \\ \alpha_{2j} &= i \xi (\beta_1^2 - 2) A_{3j} + \beta_1^2 \eta_j A_{1j} - b \xi,\ j = 1,2,3,4. \end{aligned}$$

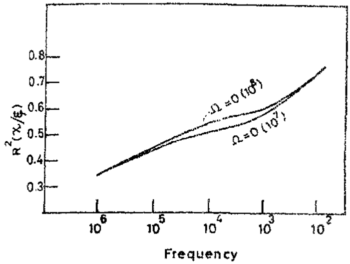

FIG. 1. Phase velocity of surface waves modified by the angular velocity.

5. Numerical results

In this section, we present some of the results obtained through analysing the problem numerically. The aim is to find out the nature of dependence of surface wave velocity on the angular velocity. Birge-Vieta method was employed to find out the complex roots of the polynomial (19). The roots were used in (25) to find out the value of the determinant. The process is iterated till the determinant values show a decreasing trend and increase afterwards. The analysis is carried out for carbon steel whose material and elastic constants are given in Maruszewski[5]. The interdependence of R_H on the Rayleigh speed is graphically shown in Tomita and Shindo[2].

The present analysis shows that there is a perceptible increase in the surface wave speed with the increase in spin velocity. The range of spin speed was restricted to the order of 10^7 and 10^8 beyond which the roots do not converge due to the limitations on the available resources.

It is observed that the shift in the wave speed is more for the range 10^3 to 10^5 of frequency and that the effect of spin velocity is not appreciable beyond this range of frequency for the assumed spin velocities (fig. 1).

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