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On μ-separated sets and density of sets in uniform space

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Abstract

Let $(\chi,\mathscr{U})$ be a uniform space and μ be an outer measure on the power set of the set χ satisfying certain axioms. We introduced the definitions of μ-separated sets and density of sets with respect to the measure μ. We have shown that almost all the results on μ-separated sets and density of sets in R^n or metric space or topological group can be extended in uniform space.

Key words: μ-separated sets, density of sets, σ-finiteness, Vitali's axiom, outer regularity.

1. Introduction

In this paper, we have extended the notions of μ-separated sets and density of a set with respect to an outer measure μ defined on the power set of the set X in the uniform space $(X,\mathscr{U})$. Lahiri[1] extended these notions to topological groups for an invariant measure. These notions have been extended to metric space[2], to measure space[3] and to Romanovski space[4,5]. In these cases, extensive use has been made of Vitali type Theorem[6] and of regularity condition in some cases. In our case, we have assumed that μ is σ-finite and it satisfies outer regularity axiom (see 2.7) and Vitali axiom (see 2.6). In section 5, we have constructed an outer measure on a metric space satisfying the axioms (iii), (iv) and (v) of 2.8.

2. Preliminary definitions and assumptions

Let $(\chi,\mathscr{U})$ be a uniform space and let μ be an outer measure on the power set of the set X. We denote by υ a fixed collection of symmetric members of $\mathscr{U}$ which are open in the product space $X \times X$ such that υ forms a base for the uniformity $\mathscr{U}$.

2.1 Definition: Let A be a subset of X. If $A \times A \subset V$ for some V in υ, we say that diameter of A is less than V and write $\delta(A) < V$.

2.2. Definition: Let $\{A_n : n \in D, \geq\}$ be a net of subsets of X. If for every V in ν, there is an element n_0 in D such that $\delta(A_n) < V$ for all n in D with $n \geq n_0$, we say that diameters of A_n tend to zero and $\delta(A_n) \to 0$.

2.3. Definition: A net $\{E_n : n \in D, \geq\}$ of subsets of X is said to converge to the point $\xi \in X$ if $\xi \in \cap \{E_n : n \in D\}$ and $\delta(E_n) \to 0$.

2.4. Definition: For $V \in v$ and $x \in X$, let

$$S(x, V) = \{y : y \in X \text{ and } (x, y) \in V\}$$

$$\bar{S}(x, V) = \{y : y \in X \text{ and } (x, y) \in \bar{V}\}$$

$S(x, V)$ and $\bar{S}(x, V)$ are called open and closed balls respectively with x as centre and radius V. We also write $V[x]$ and $\bar{V}[x]$ for $S(x, V)$ and $\bar{S}(x, V)$ respectively. It may be noted that $\bar{V}[x]$ need not be the closure of $V[x]$.

2.5. Definition: Let E be a subset of X and $\mathcal{F}$ be a family of closed balls in X. We say that the family $\mathcal{F}$ covers the set E in the sense of Vitali if for every point $x \in E$ there is a net of closed balls in $\mathcal{F}$ converging to x.

2.6. Vitali axiom: If a family $\mathcal{F}$ of closed balls in X is a Vitali converging of a set $E \subset X$ with $\mu E < +\infty$, then for every positive number ϵ, there exists a countable family of pairwise disjoint closed balls $\{F_i\}$ in $\mathcal{F}$ such that

$$\Sigma_i \, \mu F_i < \mu E + \epsilon \text{ and } \mu(E | U_i F_i) = 0.$$

2.7. Outer regularity axiom: For every set $E \subset X$ and for every $\epsilon > 0$, there exists an open set $G \supset E$ such that $\mu \, G \leq \mu \, E + \epsilon$.

2.8. Conventions and assumptions:

(i) Sets under consideration are subsets of X unless otherwise stated.

(ii) $\tilde{A}$ means the complement of the set A in X.

(iii) Outer measure μ is σ-finite and satisfies Vitali axiom and outer regularity axiom.

(iv) Every open set in X is μ-measurable.

(v) For every x in X there is a member V in v such that $\mu \, \bar{V}[x] < +\infty$.

From Vitali axiom the following result may be deduced which we call Vitali Theorem.

2.9. Vitali Theorem: Let E be a subset of X with $\mu \, E < +\infty$ and let $\mathcal{F}$ be a family of closed balls in X which covers the set E in the sense of Vitali. Then for every $\epsilon > 0$ there exists a finite family of pairwise disjoint closed balls $\{F_1, F_2, \ldots, F_n\} \subset \mathcal{F}$ such that

$$\Sigma_{i=1}^{n} \, \mu(E \cap F_i) > \mu \, E - \epsilon \text{ and } \Sigma_{i=1}^{n} \, \mu \, F_i < \mu \, E + \epsilon.$$

3. μ-separated sets

3.1. Definition: Two sets E_1 and E_2 are said to be μ-separated if for every $\epsilon > 0$, there exist open sets G_1, G_2 such that $E_1 \subset G_2$, $E_2 \subset G_2$ and $\mu(G_1 \cap G_2) < \epsilon$.

From the above definition we see that if E_1 and E_2 are μ-separated, then any subset of E_1 is μ-separated from any subset of E_2.

3.2. Lemma: For any set E, there exists a μ-measurable set $A \supset E$ such that $\mu A = \mu E$. The set A is called a μ-measurable cover of E.

3.3. Lemma: Let E be any μ-measurable set and let ϵ be any positive number. Then there exists an open set $G \supset E$ such that $\mu(G | E) < \epsilon$.

3.4. Theorem: A set E is μ-measurable if and only if E and $\widetilde{E}$ are μ-separated.

3.5. Corollary: Let S be a μ-measurable set and $A \subset S$, $B = S/A$. If A and B are μ-separated, then A and B are μ-measurable.

3.6. Theorem: If the sets E_1 and E_2 are μ-separated, then $\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2$.

3.7. Theorem: Let E_1 and E_2 be any two sets with μE_1 and μE_2 finite. If $\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2$, then E_1 and E_2 are μ-separated.

The above results can be proved in the usual way. For the proofs of the results of this section, Vitali axiom is not necessary.

4. Density of sets

4.1. Definition: Let $E \subset X$, $\xi \in X$ and $V \in v$

Write $\Delta(\xi, V) = \{ \bar{W}[\xi]: W \in v \text{ and } W \subset V \}$,

$$\Delta(\xi) = \{ \bar{W}[\xi]: W \in v \}.$$

Let $D^*(E, \xi; V) = \text{Sup} \{ \frac{\mu(E \cap F)}{\mu F} : F \in \Delta(\xi, V) \}$,

$$D_*(E, \xi; V) = \inf \{ \frac{\mu(E \cap F)}{\mu F} : F \in \Delta(\xi, V) \}.$$

[If $\mu F = 0$, we take $\frac{\mu(E \cap F)}{\mu F} = 0$]

$$D^*(E, \xi) = \inf \{ D^*(E, \xi; V) : V \in v \},$$

$$D_*(E, \xi) = \text{Sup} \{ D_*(E, \xi; V) : V \in v \}.$$

$D^*(E, \xi)$ and $D_*(E, \xi)$ are called upper and lower densities of E at ξ. If $D^*(E, \xi) = D_*(E, \xi)$, the common value is denoted by $D(E, \xi)$ and is called the density of E at ξ. If $D(E, \xi) = 1$, we call ξ a density point of E and if $D(E, \xi) = 0$, ξ is called a dispersion point of E.

For U and V in v, let us define $U \geq V$ is $U \subset V$. It is easy to see that $v, \geq)$ becomes a directed set. For a member V_0 in v we write

$$v(V_0) = \{ V : V \in v \text{ and } V \subset V_0 \}.$$

4.2. Lemma: Let E be a set and $\xi \in E$ and λ be a positive number. If $D_*(E, \xi) < \lambda$, then there exists a $V_0 \in v$ and a net $\{F_v: V \in v(V_0)\}$ of closed balls with ξ as centre and converging to ξ such that

$$\frac{\mu(E \cap F_v)}{\mu F_V} < \lambda \qquad \text{for all } V \in v(V_0).$$

4.3. Lemma: Let E be a set and $\xi \in E$ and let $0 < \lambda < 1$. If $D^*(E, \xi) > \lambda$, there exists a $V_0 \in v$ and a net $\{F_V: V \in v(V_0)\}$ of closed balls with ξ as centre and converging to ξ such that

$$\frac{\mu(E \cap F_v)}{\mu F_v} > \lambda \text{ for all } \quad V \in v(V_0).$$

The proofs of the Lemmas are easy and omitted.

4.4. Theorem: Almost all points of a set E are density points of E.

Proof: First suppose that $\mu E < +\infty$.

Let $\{\lambda_\nu\}$ be a strictly increasing sequence of positive numbers converging on 1. For each positive integer ν, let A_ν denote the set of points of E where the lower density of E is less than λ_ν. Take any positive integer n and consider the set A_n. If $x \in A_n$, then by Lemma 4.2, there exists a net $\{F_v: V \in v(V_0)\}$ of closed balls with centre at x and converging on x such that for all $V \in v(V_0)$

$$\frac{\mu(E \cap F_v)}{\mu F_v} < \lambda_n \tag{1}$$

Let $\mathscr{F}$ denote the family of all closed balls $\{F_v\}$ thus associated with the points of the set A_n. Then the family $\mathscr{F}$ covers the set A_n in the sense of Vitali. Choose any $\epsilon > 0$. By Vitali Theorem there exists a finite family of pairwise disjoint closed balls $\{F_1, F_2, \ldots\ldots, F_N\} \subset \mathscr{F}$ such that

$$\Sigma_{i=1}^{N} \mu(A_n \cap F_i) > \mu A_n - \epsilon \text{ and } \Sigma_{i=1}^{N} \mu F_i < \mu A_n + \epsilon \tag{2}$$

Using the relations (1) and (2), we get

$$\mu A_n - \epsilon < \Sigma_{i=1}^{N} \mu(A_n \cap F_i) \leq \Sigma_{i=1}^{N} \mu(E \cap F_i)$$

$$< \lambda_n \Sigma_{i=1}^{N} \mu F_i < \lambda_n (\mu A_n + \epsilon).$$

$$\text{or } (1 - \lambda_n)\, \mu A_n < (1 + \lambda_n)\, \epsilon$$

$$\text{or} \quad \mu A_n < \left(\frac{1 + \lambda_n}{1 - \lambda_n}\right) \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\mu A_n = 0$. Now, let A denote the set of all points of E where the lower density of E is less than unity. The $A = \bigcup_{n=1}^{\infty} A_n$. So $\mu A = 0$.

The general case can be proved using the σ-finiteness of the measure μ.

4.5. Theorem: If the sets E_1 and E_2 are μ-separated, then at almost all points of one set the density of the other is zero.

4.6. Theorem: Let E_1 and E_2 be any two sets. If at almost all points of E_1 the density of E_2 is zero, then E_1 and E_2 are μ-separated.

Proceeding as above and using the Lemmas 4.2 and 4.3, the results can be proved.

Let A and B be any two sets. We denote by $A_B\,[B_A]$ the part of $A\,[B]$ where the upper density of $B\,[A]$ is positive.

4.7. Theorem: Suppose that the sets A and B are not μ-separated. Then the sets A_B and B_A have positive outer measures; also no part of A_B with positive outer measure is μ-separated from B_A and no part of B_A with positive outer measure is μ-separated from A_B.

The result can be proved in the usual way.

4.8. Theorem: Let F be any closed ball.

Then $\mu\,(A_B \cap F) = \mu\,(B_A \cap F) = \mu\,[\,(A_B \cup B_A) \cap F\,]$ (3)

Proof: If A and B are μ-separated, then by Theorem 4.5, $\mu A_B = 0$ and $\mu B_A = 0$ and (3) follows.

Suppose that A and B are not μ-separated.

Write $A_0 = A_B \cap F$, $B_0 = B_A \cap F$ and $C_0 = (A_B \cup B_A) \cap F$.

We have $\mu\, A_0 \leq \mu\, C_0$ and $\mu\, B_0 \leq \mu\, C_0$.

Assume that $\mu A_0 < \mu C_0$. Let Δ be any open ball containing F. By outer regularity of μ we can choose an open set $G \subset \Delta$ such that $A_0 \subset G$ and $\mu\, G < \mu\, C_0$.

Let $E = \Delta / G$. Then E and G are μ-separated. So, same is true for the sets $E \cap C_0$ and $G \cap A_B$. Again, $E \cap C_0 \subset F$ and $E \cap A_B \subset \Delta / F$; so the sets $E \cap C_0$ and $E \cap A_B$ are μ-separated. Hence, $E \cap C_0$ is μ-separated from the set $(G \cap A_B) \cup (E \cap A_B) = \Delta \cap A_B$. The sets Δ and $\tilde{\Delta}$ are μ-separated; hence, so are the sets $E \cap C_0$ and $\tilde{\Delta} \cap A_B$. Therefore $E \cap C_0$ is μ-separated from $(\Delta \cap A_B) \cup (\tilde{\Delta} \cap A_B) = A_B$. Clearly $E \cap C_0 \subset A_B \cup B_A$ and $\mu\,(E \cap C_0) > 0$.

Write $E_1 = E \cap C_0 \cap A_B$ and $E_2 = E \cap C_0 \cap B_A$. Then $E_1 \cup E_2 = E \cap C_0$. So either $\mu E_1 > 0$ or $\mu\, E_2 > 0$ or both.

Let $\mu\, E_1 > 0$. Since $E_1 \subset A_B$, by Theorem 4.4, at almost all points for E_1, the density of A_B is unity. This contradicts Theorem 4.5. If $\mu\, E_2 > 0$, then it contradicts Theorem 4.7. Thus in any case we arrive at a contradiction. So $\mu\, A_0 = \mu\, C_0$.

Similarly, we can show that $\mu\, B_0 = \mu\, C_0$.

4.9. Corollary: Suppose that the sets A and B are not μ-separated. Then at almost all points of the set $A_B\,[B_A]$ the density of $B_A\,[A_B]$ is unity.

4.10. Theorem: Let E be any set. Then E has density either zero or unity almost everywhere.

4.11. Theorem: Let A be any set. If A contains almost all its points of density, then A is μ-measurable.

The above results can be proved in the usual way using the previous results.

4.12. Theorem: Let A and B be two sets. Suppose that at each point of B the density of A is unity. Then at any point $\alpha \epsilon \chi$,

$$D^*(A, \alpha) \geq D^*(B, \alpha) \text{ and } D_*(A, \alpha) \geq D_*(B, \alpha).$$

Proof: We prove the result by the following steps.

(I) Let E denote a measurable cover of A. Write $B_1 = B \cap E$ and $B_2 = B \cap \widetilde{E}$. Since E and $\widetilde{E}$ are μ-separated, it follows that A is μ-separated from $\widetilde{E}$. So A has density zero at almost all points of $\widetilde{E}$. This gives that $\mu B_2 = 0$ and so $\mu B = \mu B_1$. Since $B_1 \subset E$, we have $\overline{\mu B} = \mu B_1 \leq \mu E = \mu A$.

(II). Let G be any open set. $A_0 = G \cap A$ and $B_0 = G \cap B$. We show that

$$\mu B_0 \leq \mu A_0 \tag{4}$$

If B_0 is void, then clearly (4) holds. Let $x \epsilon B_0$.

Then we can choose a member V in υ such that

$$\mu \bar{V}[x] < +\infty \text{ and } V[x] \subset G.$$

Let F be any closed ball with x as centre and $F \subset V[x]$.

Then $A_0 \cap F = A \cap G \cap F = A \cap F$. So,

$$\frac{\mu(A_0 \cap F)}{\mu F} = \frac{\mu(A \cap F)}{\mu F}$$

This gives that A_0 has unit density at each point of B_0. So by step (I) we have

$$\overline{\mu B_0} = \mu A_0$$

(III) Let $\alpha \epsilon X$ and let V be any member of υ with $\mu \bar{V}[x] < +\infty$. Take any closed ball I with α as centre and $F \subset V[\alpha]$.

For each positive integer n, we can choose an open set $G_n \supset F$ such that $\mu G_n < \mu F + 1/n$

We can choose the sets $G_1, G_2, G_3, \ldots$ such that $G_1 \supset G_2 \supset G_3 \supset \ldots$ Let $E = \bigcap_{n=1}^{\infty} G_n$.

Then $F \subset E \subset G_n$. So $\mu F \leq \mu E \leq \mu G_n < \mu F + 1/n$.

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \mu G_n = \mu E = \mu F.$$

This gives that

$$\mu(A \cap F) = \lim_{n\to\infty} \mu(G_n \cap A)$$

$$\text{and } \mu(B \cap F) = \lim_{n\to\infty} \mu(G_n \cap B)$$

By step (II), $\mu(G_n \cap B) \leq \mu(G_n \cap A)$ for each n.

So, $\mu(B \cap F) \leq \mu(A \cap F)$

$$\text{or} \quad \frac{\mu(B\cap F)}{\mu F} \leq \frac{\mu(A\cap F)}{\mu F}$$

This gives that

$$D^*(A,\alpha) \geq D^*(B,\alpha) \text{ and } D_*(A,\alpha) \geq D_*(B,\alpha).$$

5. Construction of an outer measure on a metric space which is σ-finite, outer regular and satisfies Vitali's axiom

Let (X, d) be a metric space such that every closed ball in X is connected. For any positive number r, let $W_r = \{(x,y): x, y \in X \text{ and } d(x,y) < r\}$ and let $v = \{W_r: r > 0\}$. Denote by $\mathscr{U}$ the collection of all subsets of $X \times X$ such that if $U \in \mathscr{U}$, then $W_r \subset U$ for some $r > 0$. It is easy to verify that $\mathscr{U}$ is a uniformity on X and v is a base for $\mathscr{U}$.

Since for each $x \in X$ and $r > 0$, $W_r[x] = \{y: y \in X \text{ and } d(x,y) < r\} = B(x,r)$ an open ball in X with x as centre and radius r, it follows that the topology induced by the uniformity $\mathscr{U}$ on X is identical with the topology induced by the metric d. Clearly each member of v is symmetric and open in the product space $X \times X$. Thus the open and closed balls in the uniform space $(X, \mathscr{U})$ are same as those in the metric space (X, d).

Denote by $\mathscr{C}$ the collection of all countable families of pairwise disjoint closed balls in X.

.et the function $\phi:[0,\infty) \to [0,\infty)$ satisfy the following axiom.

A_1): φ is strictly increasing, continuous and $\phi(0) = 0$.

For any closed ball F with radius r we define $\lambda(F) = \varphi(r)$. For any family $\mathscr{F} \in \mathscr{C}$ we write $(\mathscr{F}) = \Sigma\{\lambda(F): F \in \mathscr{F}\}$.

[ow, for any non-void open set G we define

$$\lambda(G) = \sup\{\lambda(\mathscr{F}): \mathscr{F} \in \mathscr{C} \text{ and } U\{F: F \in \mathscr{F}\} \subset G\}.$$

'learly $\lambda(G) > 0$. We also define $\lambda(\phi) = 0$.

We assume that the function λ satisfies the following axioms.

(A_2): If F_0 is a closed ball, $\mathscr{F} \in \mathscr{C}$ is finite and $\cup\{F: F \in \mathscr{F}\} \subset \operatorname{int}(F_0)$, then $\lambda(\mathscr{F}) < \lambda(F_0)$.

(A_3): If F is a closed ball and $\{G_n\}$ is a countable family of open sets with $F \subset \cup_n G_n$, then $\lambda(F) \leq \Sigma_n \lambda(G_n)$.

(A_4): There is a positive number M such that $\lambda(\hat{F}) \leq M\lambda(F)$ for all closed balls F in X, where $\hat{F}$ denotes the closed balls concentric with F and with radius five times that of F.

We now define the mapping $\mu: \mathscr{P}(X) \to [0, \infty]$ as follows:

Let E be any subset of X. We define

$$\mu(E) = \inf\{\lambda(G): G \text{ is open and } E \subset G\}.$$

We can verify the following:

(i) The mapping μ is an outer measure and $\mu G = \lambda G$ for any open set G and $\mu F = \lambda F$ for any closed ball F.

(ii) μ is a metric outer measure and so every open set is μ-measurable. Further μ is σ-finit and outer regular.

(iii) The outer measure μ satisfies Vitali's axiom.

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