J. Indian Inst. Sci. 65(B), Nov. 1984, pp. 263-270

On a distributional generalized Stieltjes transformation

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Abstract

A complex inversion formula for the generalized Stieltjes transform of a function $f(t) \in L(0,\infty)$ defined by

$$F(s) \stackrel{\Delta}{=} \Gamma(\rho) s^{m\rho-1} \int_0^\infty \frac{f(t)}{(s^m + t^m)^\rho} dt \; (m, \rho > 0)$$

converges for complex s-plane cut from the origin along the negative real axis is extended to a certain class of generalized functions interpreting convergence in the weak distributional sense.

Key words: Generalized Stieltjes transformation, inversion formula, topological linear space.

1. Introduction

The conventional Stieltjes transform of a function $f(t) \in L(0,\infty)$ defined by the integral

$$F(s) \stackrel{\Delta}{=} \int_0^\infty \frac{f(t)}{(s+t)} dt \tag{1.1}$$

converges for complex s-plane cut from the origin along the negative real axis. The generalized form of the transform (1.1) is given by the convergent integral

$$F(s) \stackrel{\Delta}{=} \int_0^\infty \frac{f(t)}{(s+t)\rho} dt \tag{1.2}$$

in which $\rho \in R$ is fixed, $-\pi < \arg s < \pi$, the principle value of $(s+t)^{-\rho}$ is taken and $(l+t)^{-\rho} f(t) \in L(0,\infty)$ is assumed. The generalizations of the transform (1.1) have been made by various mathematicians from time to time. Some generalizations of the transform (1.1) have been studied in the distributional sense by Ghosh[1], Tiwari[2] and others. The transform (1.1) and (1.2) have also been studied in the distributional sense by Pandey[3] and Pathak[4] respectively. The transform (1.2) has also been extended to generalized functions both by direct approach

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and the method of adjoints and the resulting extension are correlated and inversion formulae are also developed, as the application of fractional integration to these transforms by Erdelyi[5]. The simple generalization of (1.1) given by the integral

$$F(s) \triangleq \Gamma(\rho)\, s^{m\rho-1} \int_0^\infty \frac{f(t)}{(s^m+t^m)^\rho}\, dt \quad (m, \rho > 0) \tag{1.3}$$

converges for complex s in the cut plane has not so far been extended to generalized functions.

The inversion formula for (1.3) is given by the following:

Theorem 1.1: Let $t^{mu-m} f(t) \in L(0,\infty)$ and let $f(x)$ be of bounded variation in the neighbourhood of the point $x=t$. Let $f(s)$ be defined by (1.3) and $s^{ml-m} F(s) \in L(0,\infty)$ where $l=\sigma+iT$. Then

$$1/2\,[f(t+0)+f(t-0)] = (2\pi i)^{-1} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} Q(l)\, M(l)\, t^{-ml+m-l}\, dl \tag{1.4}$$

where

$$Q(l) = \frac{m}{\Gamma(1-l)\,\Gamma(\rho-1+l)}$$

and

$$M(l) = \int_0^\infty s^{ml-m}\, F(s)\, ds$$

provided

$$1-\rho < Re(l) < 1$$

and

$M(l)$ is absolutely convergent.

Proof: This can be easily proved by using Mellin's inversion theorem.

Pandey[3] and Pathak[4] extended the real and complex inversion formulae for the Stieltjes transforms (1.1) and (1.2) respectively to the same space of generalized functions. But the object of the present paper is to extend the inversion formula (1.4) for the more generalized Stieltjes transform (1.3) to a different space of generalized functions interpreting the convergence in the weak distributional sense. Also we adopt a different technique to those of Pandey[3] and Pathak[4]. The notation and terminology follow that of Zemanian[6].

Let I be the open interval $(0,\infty)$ and $D(I)$ the space of smooth functions on I having compact supports. The symbol $D'(I)$ stands for the space of distributions defined over the testing function space $D(I)$. The topology of $D(I)$ is that which makes its dual the space of

$D'(I)$ of Schwartz distributions. $\epsilon(I)$ denotes the space of smooth functions on I. Its dual $\epsilon'(I)$ is the space of distributions with compact supports on I.

2. The testing function space $J_{\alpha,\beta}(I)$

Let β be a suitably restricted real number and let $\mu_\beta(t)$ be the function

$$\mu_\beta(t) = t^\beta \qquad 0<t<1,$$
$$\qquad = 1 \qquad t\geq 1.$$

An infinitely differentiably complex valued function $\phi(t)$ defined over $I=(0,\infty)$ belongs to the testing function spaces $J_{\alpha,\beta}(I)$ if

$$\gamma_k(\phi) \triangleq \gamma_{\alpha,\beta,k}(\phi) \triangleq \sup_{0<t<\infty} (1+t^m)^\alpha \quad | \quad \mu_\beta(t)\, t^{mk}\, (t^{1-m}\, \mathrm{d}/\mathrm{d}t)^k\, \phi(t) \quad | < \infty$$

for all $k=0,1,2,\ldots$ and α is a fixed real number less than or equal to ρ. We assign to $J_{\alpha,\beta}(I)$ the topology generated by the collection of seminorms $\{\gamma_k\}_{k=0}^{\infty}$. $J_{\alpha,\beta}(I)$ is sequentially complete Hausdorff locally convex topological linear space. The dual $J'_{\alpha,\beta}(I)$ of $J_{\alpha,\beta}(1)$ is also sequentially complete. The members of $J'_{\alpha,\beta}(I)$ are called generalized functions. The topology of $D(I)$ is stronger than that induced on it by $J_{\alpha,\beta}(I)$. The restriction of any $f \in J'_{\alpha,\beta}(I)$ to $D(I)$ is in $D'(I)$.

3. The distributional generalized Stieltjes transformation

For a complex s not negative or zero.

$$K(s,t) = \frac{\Gamma(\rho)\, s^{m\rho-1}}{(s^m+t^m)^\rho} \quad (m,\rho>0) \tag{3.1}$$

belongs to $J_{\alpha,\beta}(I)$ where $\alpha \leq \rho$ and $m\alpha + \beta \geq m\rho$.

Therefore, the distributional Stieltjes transformation $F(s)$ of an arbitrary element $f \in J'_{\alpha,\beta}(I)$ for $\alpha \leq \rho$ and $m\alpha + \beta \geq m\rho$ is defined by

$$F(s) \triangleq \langle f(t), K(s,t) \rangle \tag{3.2}$$

where s belongs to the complex plane cut along the negative real axis including the origin.

Theorem 3.1: For an arbitrary $f \in J'_{\alpha,\beta}(I)$, $\alpha \leq \rho$ and $m\alpha + \beta \geq m\rho$ let $F(s)$ be defined by 3.2). Then for $r=0,1,2,\ldots$

$$\left(\frac{d}{\mathrm{d}s}\right)^r F(s) = \langle f(t), \frac{\partial^r}{\partial s^r} K(s,t) \rangle$$

where $K(s,t)$ is as defined by the equation (3.1).

Proof: Using the fact that for $\alpha \leq \rho$ and $m\alpha + \beta \geq m\rho$

$$\gamma_k \left[\frac{\partial^r}{\partial s^r} K(s,t)\right] < \infty \, (k, r = 0,1,2, \ldots)$$

where s belongs to the complex plane cut from the origin along the negative real axis and by using Cauchy's integral formula one can prove the result.

4. Complex inversion formula

Before going to prove the main theorem, we shall first prove some lemmas.

Lemma 4.1: The function u^{ml-m} as a function of u is a member of $J_{\alpha,\beta}(I)$ if $m-m\alpha-\beta \leq m \, Re\,(l) < m-m\alpha$.

Proof: It is clear that u^{ml-m} is differentiable function of u.

Consider

$$\sup_{0<u<\infty} (1+u^m)^\alpha \left| \mu_\beta(u)\, u^{mk} \left(u^{l-m}\frac{d}{du}\right)^k u^{ml-m} \right|$$

$$= \sup_{0<u<\infty} (1+u^m)^\alpha \left| \mu_\beta(u)\, u^{mk}\, m^k (l-1)(l-2) \ldots (l-k)\, u^{ml-mk-m} \right|$$

$$= \sup_{0<u<\infty} (1+u^m)^\alpha \left| m^k \mu_\beta(u) (l-1)(l-2) \ldots (l-k)\, u^{ml-m} \right| < \infty$$

under the condition stated in Lemma 4.1.

Hence $u^{ml-m} \in J_{\alpha,\beta}(I)$.

Lemma 4.2: For $f \in J'_{\alpha,\beta}(I)$,

$$\int_0^\infty s^{ml-m} < f(u), K(s,u) > ds$$

$$= < f(u), \int_0^\infty s^{ml-m} K(s,u)\, ds > \tag{4.1}$$

Proof: By using the technique of Riemann sums one can easily show that

$$\int_0^R s^{ml-m} < f(u), K(s,u) > ds = < f(u), \int_0^R s^{ml-m} K(s,u)\, ds > \tag{4.2}$$

Again, since

$$\int_R^\infty s^{ml-m} K(s,u)\, ds \to 0 \text{ in } J_{\alpha,\beta}(I) \text{ as } R \to \infty,$$

one can easily justify taking limits as $R \to \infty$ in (4.2) to obtain (4.1).

Lemma 4.3: Let $\phi \in D(I)$, and set

$$\psi_\rho(l) = \int_0^\infty \phi(y) \quad y^{-ml+m-1}\, dy$$

where $l = \sigma + iT$ and $f \in J'_{\alpha,\beta}(I)$ then for any fixed r with $0 < r < \infty$ and $m - m\alpha - \beta \leq m\sigma < m - m\alpha$,

$$\int_{-r}^{r} < f(u), u^{ml-m} > \psi_\rho(l)\, dT = < f(u), \quad \int_{-r}^{r} u^{ml-m} \psi_\rho(l)\, dT > \tag{4.3}$$

Lemma 4.4: If $\phi \in D(I)$, then

$$(\pi)^{-1} \int_0^\infty \phi(y) (\quad)^{m\sigma-m+1} [u \log(\frac{u}{y})^m]^{-1} [\sin(r \log(\quad)^m] \, dy \tag{4.4}$$

converges in $J_{\alpha,\beta}(I)$ to ϕ as $r \to \infty$.

The proofs of lemmas 4.3 and 4.4 can be proved on the similar lines by changing suitable variables from lemmas 3.5.1 and 3.5.2 of Zemanian[6] (pp. 64-66).

Theorem 4.1 (Complex inversion formula): Let $f(y) \in J'_{\alpha,\beta}(I)$ where $\alpha \leq \rho$, $m\alpha + \beta \geq m\rho$ and $F(s)$ be the generalized Stieltjes transform of $f(y)$ as defined by (3.2). Then for each $\phi(y) \in D(I)$, we have

$$\lim_{r\to\infty} < (2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} Q(l)\, M(l)\, y^{-ml+m-1}\, dl, \ \phi(y) >$$

$$= < f(y), \phi(y) > \tag{4.5}$$

Proof: Let $\phi \in D(I)$ and choose real numbers α and β such that $m - m\alpha - \beta \leq m\sigma < m - m\alpha$ and $\alpha \leq \rho$. Our object is to show that equation (4.5) is true. Since the integral on l is a continuous function of y, the left hand side of (4.5) without the limit notation can be written as

$$(2\pi)^{-1} \int_0^\infty \phi(y) \int_{-r}^{r} Q(l)\, M(l)\, y^{-ml+m-1}\, dT\, dy, \ l = \sigma + iT, \ r > 0.$$

As $\phi(y)$ is of bounded support and the integrand is a continuous function of (y, T), the order of integration may be changed.

This yields,

$$(2\pi)^{-1} \int_{-r}^{r} Q(l) \left\{ \int_0^\infty s^{ml-m} F(s)\, ds \right\} \int_0^\infty \phi(y)\, y^{-ml+m-1}\, dy\, dT$$

which by Lemma 4.2 is equal to

$$(2\pi)^{-1} \int_{-r}^{r} < f(u), \int_0 Q(l)\, s^{ml-m} K(s,u)\, ds > \int_0^\infty y^{-ml+m-1} \phi(y)\, dy\, dT$$

$$= (2\pi)^{-1} \int_{-r}^{r} < f(u), u^{ml-m} > \int_0^\infty \phi(y)\, y^{-ml+m-l}\, dy\, dT$$

provided

$$l - \rho < Re(l) < 1$$

on using a result

$$\int_0^\infty \frac{m}{\Gamma(1-l)\Gamma(\rho-1+l)}\, s^{ml-m} K(s,u)\, ds = u^{ml-m}.$$

By Lemma 4.3,

$$(2\pi)^{-1} \int_{-r}^{r} < f(u), u^{ml-m} > \phi(y)\, y^{-ml+m-1}\, dy\, dT$$

$$= < f(\mathrm{u}), (2\pi)^{-1} \int_{-r}^{r} u^{ml-m} \int_0^\infty \phi(y)\, y^{-ml+m-1}\, dy\, dT >.$$

The order of integration for the repeated integral herein may be changed because again $\phi(y$ is of bounded support and the integrand is a continuous function of (y, T). Upon doing thi we obtain,

$$< f(u), (2\pi)^{-1} \int_{-r}^{r} u^{ml-m} \int_0^\infty \phi(y)\, y^{-ml+m-l}\, dy\, dT >$$

$$= < f(u), (2\pi)^{-1} \int_0^\infty \phi(y) \int_{-r}^{r} u^{ml-m}\, y^{-ml+m-l}\, dT\, dy >$$

$$= < f(u), (\pi)^{-1} \int_0^\infty \phi(y)\, (u/y)^{m\sigma-m+1} \sin [\, r \log (u/y)^m \,]$$

$$[u \log (u/y)^m]^{-1}\, dy >.$$

The last expression tends to $<f(y), \phi(y)>$ as $r \to \infty$ because $f \in J'_{\alpha,\beta}$ and according to lemma 4.4, the testing function in the last expression converges to $\phi(y)$ in $J_{\alpha,\beta}(I)$.

This completes the proof.

5. Illustration of the inversion formula by means of numerical example

Consider the delta functional $\delta(t-k)$, concentrated at a point k, $0<k<\infty$. Since $\delta(t-k) \in \epsilon'(I)$, $I=(0,\infty)$ and $\epsilon'(I)$ is a subspace of $J'_{\alpha,\beta}(I)$, the generalized Stieltjes transform of $\delta(t-k)$ is given by

$$F(s) = < \delta(t-k), \quad \frac{= \Gamma(\rho). s^{m\rho-1}}{(s^m+t^m)\rho}$$

$$= \Gamma(\rho). s^{m\rho-1}(s^m+k^m)^{-\rho}$$

Now by inversion theorem, for any $\phi(y) \in D(I)$,

$$< (2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} Q(l)\ M(l)\ y^{-ml+m-1}\,dl,\ \phi(y) >$$

$$= < (2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} Q(l).\ \{ \int_0^\infty s^{ml-m}\ F(s)\,ds\ \}\ y^{-ml+m-1}\,dl,\ \phi(y) >$$

$$= < (2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} Q(l)\ \{ \int_0^\infty s^{ml-m}\ \frac{\Gamma(\rho)\,s^{m\rho-1}\,ds}{(s^m+k^m)^\rho}\ \}\ y^{-ml+m-1}\,dl,\ \phi(y) >$$

$$= < (2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} k^{ml-m}\ y^{-ml+m-1}\,dl,\ \phi(y) >.$$

On using a result

$$\int_0^\infty \frac{m}{\Gamma(1-l)\,\Gamma(\rho-1+l)}\ S^{ml-m} K(s,u)\,ds = u^{ml-m}$$

on further manipulation it can be shown that

$$(2\pi i)^{-1} \int_{\sigma-ir}^{\sigma+ir} k^{ml-m}\ y^{-ml+m-1}\,dl \to \delta(y-k)$$

Therefore

$$< (2\pi i)^{-1} \int_{\sigma - ir}^{\sigma + ir} k^{ml-m} y^{-ml+m-l} \, dl, \ \phi(y) >$$
$$= < \delta(y-k), \phi(y) >.$$

This illustrates the inversion formula.

Acknowledgements

The authors express their thanks to the referees for some valuable suggestions.

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