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Short Communication

A note on the nonsingular factorization algorithm for certain classes of matrices

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Abstract

Two classes of matrices whose factorization can be obtained 'on inspection' are identified by making use of the Nonsingular Factorization Algorithms (NFA). Two algorithms for writing down the factorization are given.

Key words: Nonsingular factorization algorithm (NFA), subcolumn vectors, compatible subcolumn.

1. Introduction

The nonsingular factorization technique called the NF algorithm (NFA) was first developed by Ahmed and Chen[1]. This technique can be used immediately to Kronecker matrices. In the case of non-Kronecker matrices this technique can be applied skillfully if the dimensions of the matrices are reasonably small. The purpose of this note is to identify two classes of matrices whose factorization can be obtained immediately 'on inspection'. In Section 1, we briefly describe the NFA. For further details the reader is referred to the original paper of Ahmed and Chen[1]. In Section 2, we identify the classes of matrices mentioned above. In Section 3, we give two algorithms to write down the factorization.

2. The NF algorithm (NFA)

2.1 *Some basic definitions*

We consider only matrices over the real field R. Let $M = (m_{ij})_{m\times n}$, be a nonzero matrix. Following Ahmed and Chen[1], we define a submatrix S of M as that matrix consisting of the elements of M which are located at the intersection of a set of rows and columns of M. A subcolumn vector is a $(n\times 1)$ vector consisting of elements located at the intersection of a column and a set of rows of M. Two subcolumn vectors are compatible if they are formed using the same set of rows. Also if S_1 is a $(p\times q)$ submatrix of M where $P\leq m$ and $q\leq n$ and if S_2 is a $(p\times r)$ submatrix of S_1 where $r\leq q$ then the $(q-r)$ compatible subcolumn vectors of

S_1 which do not belong to the set consisting of column vectors of S_2 are called the exterior subcolumn vectors of S_2 with respect of S_1. We can define subrow vectors and exterior subrow vectors in a similar manner.

2.2 The algorithm

(i) Choose a $(p \times n)$ submatrix S from the $(m \times n)$ matrix M:

$$S = \begin{bmatrix} \text{Row } i_1 \text{ of } M \\ \text{Row } i_2 \text{ of } M \\ \cdot \\ \text{Row } i_p \text{ of } M \end{bmatrix} \qquad 1 \leq i_1 < i_2 \dots < i_p \leq m$$

(ii) Choose a $(p \times p)$ nonsingular and nontrivial submatrix S of S:

$$\hat{S} = \begin{bmatrix} s_{1,1} & s_{1,2} & \dots & s_{1,p} \\ s_{2,1} & s_{2,2} & \dots & s_{2,p} \\ \cdot & & & \\ s_{p,} & s_{p,2} & \dots & s_{p,p} \end{bmatrix} = (\hat{s}_{ku})_{p \times p} \qquad \hat{S} = \begin{bmatrix} \text{col} & \text{col} & \dots & \text{col} \\ j_1 & j_2 & \dots & j_p \\ \text{of} & \text{of} & \dots & \text{of} \\ S & S & \dots & S \end{bmatrix}$$

where $1 \leq j_1 < j_2 < \dots < j_p \leq n$.

If, for all $p \geq 2$, no $(p \times p)$ nonsingular submatrix can be found out then stop. The rank of M is then 1 and so no factorization is necessary.

Otherwise go to (iii).

(iii) Form a set Λ consisting of $(n-p)$ exterior subcolumn vectors of $\hat{S}$ with respect to S

(iv) Check whether there exists at least one $C \epsilon \Lambda$ which is linearly dependent with respect to column vectors of $\hat{S}$ for $q<p$. If there is one such C then write S as the product of $\hat{S}$ and $[\hat{S}^{-1} S]$. Go to step (v).
If there is no such C, return to (ii).

(v) Write $M = LR$ where L and R are constructed using the following rule

$$l_{ik}, j_u \equiv \hat{s}_{ku} = m_{i_k}, j_u \text{ for } 1 \leq k, u \leq p$$

$$l_{ii} = 1 \text{ for all } i \neq i_1, i_2, \dots, i_p$$

and all the other l_{ij} are zero. R is the matrix got by replacing the row vectors $i_1, i_2, \dots, i_p$ of M by the row vectors 1,2, ... p of $[\hat{s}^{-1} S]$. The remaining vectors of M remain the same. For further details and examples, see Ahmed and Chen[1].

3. Certain types of matrices whose factorizations can be found on inspection

3.1 Even type

Let $M_E = M = (m_{ij})_{2n \times 2n}$ where $m_{ij} = 0$ if $j > n$ and i odd.

For instance, the matrices

$$M_1 = \begin{bmatrix} 5 & 12 & 0 & 0 \\ 2 & 1 & 3 & 8 \\ 9 & 7 & 0 & 0 \\ 5 & 3 & 2 & 1 \end{bmatrix} \text{ and } \quad M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 8 & 9 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 10 & 12 & 13 & 14 & 0 & 2 \\ 9 & 7 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

are in M_E.

In the case of M_1, $n=2$. Although in the case of M_2, $m_{ij}=0$ for all $j>2$ and i odd n is equal to 3 since M_2 is of the order 6×6.

Applying the NFA to M of M_E we can write $M = LR$ where

$$L = \begin{bmatrix} m_{11} & 0 & m_{12} & 0 & \ldots & m_{1n} & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ m_{31} & 0 & m_{32} & 0 & \ldots & m_{3n} & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ m_{2n-1,1} & 0 & m_{2n-1,2} & 0 & \ldots & m_{2n-1,n} & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ m_{21} & m_{22} & m_{23} & \ldots & m_{22n} \\ 0 & 1 & 0 & \ldots & 0 \\ m_{41} & m_{42} & m_{43} & \ldots & m_{42n} \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ m_{2n,1} & m_{2n,2} & m_{2n,3} & \ldots & m_{2n,2n} \end{bmatrix}$$

3.2 *Odd type*

Let $M_0 = (M = (m_{ij})_{2n+1 \times 2n+1} \quad m_{ij} = 0$ if $j > n+1$ and i odd)

For instance, the matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \in M_0$$

Here $n = 1$ and the observation regarding n made in 2.1 applies to matrices M_0 also.

Applying the NFA to M of M_0 we can write M = LR where

$$L = \begin{bmatrix} m_{11} & 0 & m_{12} & 0 & .. & m_{1n} \\ 0 & 1 & 0 & 0 & \dots & 0 \\ m_{31} & 0 & m_{32} & 0 & \dots & m_{3n} \\ 0 & 0 & 0 & 1 & \dots & 0 \\ . & & & . & \dots & . \\ m_{2n+1,1} & 0 & m_{2n+1,2} & 0 & \dots & m_{2n+1,n} \end{bmatrix} \quad \text{and}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ m_{21} & m_{22} & m_{23} & \dots & m_{2,2n+1} \\ 0 & 1 & 0 & \dots & 0 \\ m_{41} & m_{42} & m_{43} & \dots & m_{4,2n+1} \\ . & . & . & \dots & . \\ m_{2n,1} & m_{2n,2} & m_{2n,3} & \dots & m_{2n,2n+1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Here, in R, the last row is not a zero vector as there is unity in the $n+1$st place.

4. Algorithms to find the L and R matrices in M_E and M_0

4.1 Algorithm for even type

Let $M = (m_{ij})_{2n\times 2n} \in M_E$

(i) Take the ith odd row of M and remove all the m_{ij} $(=0)$, $n+1 \leq j \leq 2n$. Form a new $(1\times 2n)$ row vector by inserting a zero in between every two consecutive m_{ij} s, $j \leq n$. Thus we get a row vector of the form: $(m_{i1}\ 0\ m_{i2}\ 0\ \dots\ m_{in}\ 0)$.

Take the row vector thus formed to be the ith odd row of L.

(ii) Take the ith even row of M and replace all the elements except the principal diagonal element by zeros. Replace the principal diagonal element by unity. Take the resulting row vector to be ith even row of L.

(iii) Form a $(1\times 2n)$ row vector whose ith element is unity and whose all other elements are zeros. Take this vector to be the ith odd row of R $(1 \leq i \leq n)$.

(iv) Find out the ith even row of M. Take this row to be the ith even row of R.

4.2 Algorithm for odd type

Let $M = (m_{ij})_{2n+1\times 2n+1} \in M_0$.

(i) Take the ith odd row of M and remove all the m_{ij} $(=0)$, $j > n+1$. Form a new $1\times 2n+1$ row vector by inserting a zero in between two consecutive m_{ij} s, $j \leq n+1$. Take the resulting vector to be the ith odd row of L.

(ii) Proceed as in 3.1 (ii), (iii) and (iv).

Note: R is also M_E type if $M \in M_E$ and hence to factorize R, this method can be used again.

4.3. *Numerical example 1*

Consider

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 8 & 9 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 10 & 12 & 13 & 14 & 0 & 2 \\ 9 & 7 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}, \quad \in M_E,$$

$$L = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 9 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 9 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 12 & 13 & 14 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \quad \text{where } M = LR.$$

By using the algorithm we have found the L and R matrices of $M \in M_E$.

(i) Take the first odd row of M and remove all the $m_{ij}(=0)$ $4 \leq j \leq 6$, we get (1,1,0). Form a new (1×6) row vector (1 0 1 0 0 0) by inserting a zero in between every two consecutive m_{ij} s, $j \leq 3$. Take this row vector thus formed to be the first odd row of L. Similarly other odd rows of L can be formed.

(ii) Take the first even row M, which is (5 8 9 0 1 0) and replace all the elements except the principle diagonal element 8 by zero. Now replace the principle diagonal element by unity. Take the resulting row vector to be the first even row of L. Similarly, other even rows of L can be formed.

(iii) To form the first odd row of R, take the (1×6) row vector whose first element is unity and whose all other elements are zero. Hence the first odd row of R is (1 0 0 0 0 0).

(iv) Find out the first even row of M, which is (5 8 9 0 1 0). Take this row to be the first eve[n] row of R.

Similarly, other even rows of R can be found out.

4.4. *Numerical example II*

Consider

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} \in M_0 \qquad L = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 6 & 0 & 7 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 5 \\ 0 & 1 & 0 \end{bmatrix}$$

where $M = LR$.

(i) Take the first odd row of M and remove all the $m_{ij}(=0)j>2$. We get the row vector (1 2). Form a new (1×3) row vector by inserting a zero in between two consecutive $m_{ij}j \leq 2$; we get (1 0 2). Take this vector to be the first odd row of L. Similarly, all the other odd rows are found out.

(ii) Proceed as in 3.3 (ii), (iii), and (iv).

5. Application in graph theory

1. If the adjacency matrix is of the form M_E for a graph G with $|G| = 2n$, $\chi(G) \leq n$ can be judged by inspecting the matrix.

   For example, consider the graph G in fig. 1. The chromatic number of G, $\chi(G) = 4$.

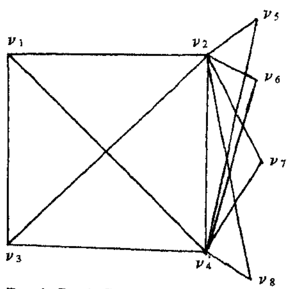

FIG. 1. Graph G.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	1	1	1	0	0	0	0
V_2	1	0	1	1	1	1	1	1
V_3	1	1	0	1	0	0	0	0
V_4	1	1	1	0	1	1	1	1
V_5	0	1	0	1	0	0	0	0
V_6	0	1	0	1	0	0	0	0
V_7	0	1	0	1	0	0	0	0
V_8	0	1	0	1	0	0	0	0

2. If the adjacency matrix is of the form M_0 for a graph G with $|G| = 2n-1$, then its chromatic number $\chi(G) \leq n$ can be judged on inspection of the matrix.

For example, consider the graph G in fig. 2. The chromatic number of G $\chi(G) = 4$.

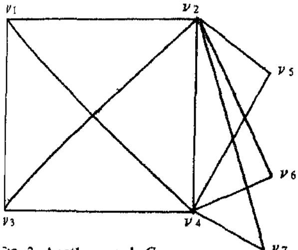

FIG. 2. Another graph G.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	1	1	1	0	0	0
V_2	1	0	1	1	1	1	1
V_3	1	1	0	1	0	0	0
V_4	1	1	1	0	1	1	1
V_5	0	1	0	1	0	0	0
V_6	0	1	0	1	0	0	0
V_7	0	1	0	1	0	0	0

6. Application of NFA in coding theory

Lemma

If $M \in M_0$, then

1. $M = L_1 R_1$ and $R_1 \in M_0$
2. $R_1 = L_2 R_2$ and $R_2 \in M_0$ and $L_2 = I$.

Proof: From algorithm 4.2 it is clear that $M = L_1 R_1$; we have to show that $R_1 \epsilon M_0$. For that $(R_1)_{ij} = 0$ if $j > n+1$ and i is odd. Consider any ith odd row of R_1. From the algorithm we see that 1 occurs in the $(i+1)/2$ th place and o in all other places. In the last row ($2n+1$st row) 1 will occur in the $n+1$st place and there are n zeros on either side. Hence, there will be at least n zeros after 1 to its right in any odd row. Hence $R_1 \epsilon M_0$, $R_1 = L_2 R_2$ and $R_2 \epsilon M_0$.

We have to finally prove that $L_2 = I$. L_2 is formed by the factorization of R_1 by the NFA. Hence, in the ith odd row of L_2 we have 1 in ith place; in the jth even row of L_2 we have 1 in the $2j$th place. Hence, $L_2 = I$ and $R_2 = R_1$. Note $M = L_1 I R_2$.

Theorem: If $M \epsilon M_0$ with any two of its rows or columns which are linearly independent and $H = [R_2 | I]_{2n+1 \times 4n+2}$ is a binary matrix then $[x'']_{4n+2 \times 1}$ is a linear code with H as parity check matrix.

Proof: Consider all vectors x such that $H\ x'' = 0$ where this equation is to be interpreted modulo 2. Clearly x can be found out as we are having $2n+1$ equations with $4n+2$ variables in which the first $2n+1$ variables are the known message symbols.

Note: From the theorem it is clear that $M \epsilon M_0$ will generate a parity check matrix by the NFA.

7. Conclusion

Two classes of matrices whose factorization can be written down immediately have been identified by means of the NFA. Two algorithms corresponding to the two classes for writing down the factorization are given. The classes of matrices considered are somewhat restricted in the sense that we can get only one factorization of a matrix in a class by applying the corresponding algorithm. But the advantage is that we can immediately write down the factorization without going through the motions of the NFA.

Reference

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