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On the spectral resolution of a differential operator II

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Abstract

We call $H(x,y,\lambda)$ which generates the resolution of the identity of the self-adjoint operator T arising from the formally self-adjoint differential operator

$$M = \begin{pmatrix} -D^2+p & r \\ r & -D^2+q \end{pmatrix}$$

and the prescribed boundary conditions, the spectral matrix (or the resolution matrix). $H^F(\cdot)$ is the resolution matrix corresponding to the Fourier case *i.e.*, the case when $Q=\begin{pmatrix} p & r \\ r & q \end{pmatrix}=0$. In the present paper we obtain (i) a connection between $H(x,y,\lambda)$ and $H^F(x,y,\lambda)$, as λ tends to infinity; (ii) an equiconvergence theorem and (iii) an expansion theorem in generalized Fourier integrals.

Key words: Spectral resolution, resolution matrix, generalized orthogonal relation, majorizing a matrix, summable, closed, generalized Fourier integral, generalized Parseval relation, spectral representation theorem.

1. Introduction

Consider the differential system

$$M\,U=\lambda\,U$$

where

$$M\equiv \begin{pmatrix} -D^2+p(x) & r(x) \\ r(x) & -D^2+q(x) \end{pmatrix},\quad D\equiv\frac{\mathrm{d}}{\mathrm{d}x},\quad U=\begin{pmatrix} u \\ v \end{pmatrix} \tag{1.1}$$

λ-complex and $p(x)$, $q(x)$, $r(x)$ are real, $C_{1-k}(a,b)$, $(k=0,1)$ - class functions, integrable in (a,b), finite or infinite. By $C_k(\alpha,\beta)$ - class functions we mean the set of functions (real or complex-valued) which are k times continuously differentiable with respect to the variable x, defined in an open finite or infinite interval (α,β).

Let U be the solution of (1.1) and ϕ_l, ϕ_j, $l=1,2$, $j=3,4$ be the boundary condition vectors at $x=a$, $x=b$ (*i.e.* solutions of (1.1) which together with their first derivatives take prescribed constant values at a and b). We choose the boundary conditions at a and b as

$$[U,\phi_l]=[U,\phi_j]=0,\ [\phi_1,\phi_2]=[\phi_3,\phi_4]=0. \tag{1.2}$$

where $[\cdot]$ is the bilinear concomitant of the vectors. The bilinear concomitant of the two vectors $U=\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $V=\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ is defined by

$$[U,\ V]=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}+\begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$$

Then (1.1) together with (1.2) gives rise to a self-adjoint eigenvalue problem considered by Chakravarty (vide Chakravarty and Roy Paladhi[1]).

The differential operation M defines on $C_2(-\infty,\infty)$ an operator T_0 symmetrical in $L_2(-\infty,\infty)$, - the minimal unclosed differential operator, the closure T_1, of which is the minimal differential operator defined by M. Let T, determined by the prescribed set of linearly independent boundary conditions assumed in the problem, be the self-adjoint extension of T_1. Then T is 'generated' by M.

If $E(\lambda)$ be the spectral resolution (or the resolution of the identity of the operator T), then T is connected with $E(\lambda)$ by means of the relation

$$T=\int_{-\infty}^{\infty}\lambda\, d\, E(\lambda) \tag{1.3}$$

It is well-known that every resolution of the identity $E(\lambda)$ determines a self-adjoint operator T by (1.3) and the spectral theorem shows that every self-adjoint operator T admits an expression (1.3) by means of a resolution of the identity $E(\lambda)$ uniquely determined by T. In the Appendix, an alternative method of showing that T is self-adjoint is given.

By using a method entirely different from the ones usually adopted for solving eigenvalue problems, Levitan and Sargsyan[2] and Levitan[3] obtained the asymptotic formula for the spectral function, the equiconvergence theorem, the expansion theorem for the scalar Sturm-Liouville equation

$$y''+(\lambda-q(x))\,y=0. \tag{1.4}$$

Their technique is to consider in conjunction with this, the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} - q(x)u = \frac{\partial^2 u}{\partial t^2}$$

$$u(x,t) \Big|_{t=0} = f(x)\,; \frac{\partial u(x,t)}{\partial t} \Big|_{t=0} = 0 \tag{1.5}$$

and use the Fourier cosine transform in the sequel.

As can be easily seen one cannot replace the Fourier cosine transform by the Fourier sine transform in the investigation so presented by them. Our object is to obtain corresponding results for the matrix system (1.1). The corresponding Cauchy type problem is

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} - Q(x)\,U \tag{1.6}$$

with boundary conditions

$$U(x,t) \Big|_{t=0} = f(x) \tag{1.7}$$

$$\frac{\partial U(x,t)}{\partial t} \Big|_{t=0} = h(x) \tag{1.8}$$

where

$$f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \; h(x) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \neq 0, \; Q(x) = \begin{pmatrix} p & r \\ r & q \end{pmatrix}$$

We make use of the Fourier sine transform, it being not possible to apply the Fourier cosine transform in the investigation that follows.

Let

$$\phi_r(x,\lambda) = \begin{pmatrix} u_r \\ v_r \end{pmatrix} \quad r = 1,2$$

be the solutions of (1.1) satisfying at $x = 0$, the conditions

$$\begin{aligned} (u_1, v_1, u_1', v_1') \Big|_{x=0} &= (1,0,0,0) \\ (u_2, v_2, u_2', v_2') \Big|_{x=0} &= (0,1,0,0) \end{aligned} \tag{1.9}$$

and $\theta_r(x,\lambda)$ two other solutions of (1.1), connected with ϕ_r by the relations

$$[\,\phi_r, \theta_k\,] = \delta_{rk}, \; [\theta_1, \theta_2] = 0, \; r,k = 1,2\,. \tag{1.10}$$

Then ϕ_r, θ_r constitute a linearly independent set of solutions.

Let

$H(x,y,\lambda)$ be the matrix

$$H(x,y,\lambda)=\begin{cases}\int_0^\lambda [\phi(x,\lambda)\,d\xi(\lambda)\,\phi^T(y,\lambda)+\phi(x,\lambda)\,d\eta(\lambda)\,\theta^T(y,\lambda) \\ \quad +\theta(x,\lambda)\,d\eta(\lambda)\,\phi^T(y,\lambda)+\theta(x,\lambda)\,d\zeta(\lambda)\,\theta^T(y,\lambda)\,];\ \lambda>0 \\ -\int_\lambda^0 [\phi(x,\lambda)\,d\xi(\lambda)\,\phi^T(y,\lambda)+\phi(x,\lambda)\,d\eta(\lambda)\,\phi^T(y,\lambda) \\ \quad +\theta(x,\lambda)\,d\eta(\lambda)\,\phi^T(y,\lambda)+\theta(x,\lambda)\,d\zeta(\lambda)\,\theta^T(y,\lambda)\,];\ \lambda>0 \\ 0 \qquad ;\ \lambda=0.\end{cases} \tag{1.11}$$

where

$$\phi=\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix},\quad \theta=\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\ \text{and}\ \ \xi\equiv(\xi_{ij}(\lambda))$$

$\eta\equiv(\eta_{ij}(\lambda))$, $\zeta\equiv\zeta_{11}(\lambda)\times I$ (I, the (2×2) unit matrix), ξ_{ij}, η_{ij}, ζ_{11} are non-decreasing functions of λ. Then $H(\cdot)$ generates the spectral resolution $E(\lambda)$ of the operator T generated by M. We call H for brevity the 'resolution matrix' the corresponding spectral representation formula is given by (5.4) in Chakravarty and Roy Paladhi[1] (p. 158).

Let Λ, Λ' be the intervals (a,b) and (c,d) respectively so that $H(x,y,\Lambda)=H(x,y,b)$ $-H(x,y,a)$, with a similar meaning for $H(x,y,\Lambda')$. Then

$$\int_{-\infty}^{\infty} H(x,t,\Lambda)\,H^T(t,y,\Lambda')\,dt=\pi\,H(x,y,\Lambda\cap\Lambda') \tag{1.12}$$

'the generalized orthogonal relation' holds.

$H(x,y,\lambda)$ is symmetric in the sense that

$$H(x,y,\lambda)=H^T(y,x,\lambda).$$

In particular,

$$\int_{-\infty}^{\infty} H(x,t,\Lambda)\,H^T(t,y,\Lambda)\,dt=\pi\,H(x,y,\Lambda) \tag{1.13}$$

Let

$$H(x,y,\Lambda)=\begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}$$

Then the first element in (1.13) is given by

$$\pi\,H_{11}(x,y,\Lambda)=\int_{-\infty}^{\infty}[H_{11}(x)\,H_{11}(y)+H_{12}(x)\,H_{12}(y)\,]\,dt,$$

where $H_{rk}(x)=H_{rk}(x,t,\Lambda)$.

Apply first the Cauchy inequality, then the Schwarz inequality and finally the obvious inequality $2\,|ab| \leq a^2 + b^2$, for real a, b. Then, it follows that

$$H_{rs}(x,y,\Lambda) \leq \frac{1}{2}\,[\,H_{rr}(x,x,\Lambda) + H_{ss}(y,y,\Lambda)\,],\ r,s = 1,2.$$

Hence

$$H(x,y,\Lambda) \ll \frac{1}{2}\,[\,h(x,x,\Lambda) + h^T(y,y,\Lambda)\,] \tag{1.14}$$

where

$$h\,(x,x,\Lambda) = \begin{pmatrix} H_{11}\,(x) & H_{11}\,(x) \\ H_{22}\,(x) & H_{22}\,(x) \end{pmatrix}$$ and '$\ll$' represents

that the matrix on the right hand side 'majorizes' the matrix on the left (Mirsky[4], p. 328). Since

$$\pi\,H_{11}(x,x,\Lambda) = \int_{-\infty}^{\infty}[\,H_{11}^2\,(t) + H_{12}^2\,(t)\,]\,dt$$

with a similar result for $H_{22}(x)$, therefore $H_{rr}(x,x,\Lambda)$ are positive.

Also $H_{12}^2\,(x,x,\Lambda) \leq H_{11}\,(x)\ H_{22}(x)$.

Hence the symmetric matrix $H(x,x,\Lambda)$ is positive in the sense that the corresponding quadratic form is positive.

2. Some auxiliary formulae

In conjunction with the system (1.1) we consider the Cauchy type equation (1.6) with boundary conditions (1.7), (1.8). Then following Levitan and Sargsyan[2], we can use the Riemann method of integration of (1.6) to show that the solution $U(x,t;f,g)$ of (1.6)-(1.8) is given by[5]

$$\begin{aligned} U\,(x,t;f,g) = {} & \frac{1}{2}\,[f(x+t) + f(x-t) + g(x+t) - g(x-t)\,] + \\ & + \frac{1}{2}\int_{x-t}^{x+t}[\,W(x,t,s)\,f(s) - T(x,t,s)\,g(s)\,]\,ds \end{aligned} \tag{2.1}$$

where $g(x) = \int^{x} h(y)\,dy$ and $W(x,t,s)$, $T(x,t,s)$ are two known 2×2 matrices called the Riemann matrices for the system.

Let ϕ_j, θ_j be those given by (1.9) and (1.10). Replacing $f(x)$ by 0 and $g(x)$ by $\int \phi_j(s,\lambda)\,ds$ so that $U\,(x,t) = 1/\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t\ \phi_j\,(x,\lambda)$ now satisfies (1.1), it follows from (2.1) by the uniqueness of the solution of the Cauchy type problem, that

$$\begin{aligned} \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,t\ \phi_j\,(x,\lambda) = {} & \frac{1}{2}\int_{x-t}^{x+t}[I + \Omega\,(x,t,s)\,]\,\phi_j\,(s,\lambda)\,ds - \\ - \frac{1}{2}\,\{\,\Omega\,(x,t,s)\Big|_{s=x+t} & \int_0^{x+t}\phi_j\,(y,\lambda)\,dy - \Omega\,(x,t,s)\,\Big|_{s=x-t}\int_0^{x}\phi_j\,(y,\lambda)\,dy\,\} \end{aligned} \tag{2.2.}$$

where $\Omega(x,t,s)=\int_0^s T(x,t,y)\,dy$ and I is the 2×2 unit matrix.

Let $g_\epsilon(t)$, a scalar function, be defined as follows

(i) $g_\epsilon(t)$ is odd: $g_\epsilon(t)=-g_\epsilon(-t)$;
(ii) $g_\epsilon(t)\neq 0,\ 0<|t|<\epsilon$
$\qquad = 0$, otherwise;

(iii) $g_\epsilon(t)$ has a piece-wise continuous, piece-wise monotone derivative *i.e.* $g_\epsilon(t)\in C_p^1$.

Let $\sqrt{\lambda}=\mu$ and let $\Psi_\epsilon(\mu)$ be the Fourier sine transform of $g_\epsilon(t)$ *i.e.*

$$\Psi_\epsilon(\mu)=\int_0^\infty g_\epsilon(t)\sin\mu t\,dt=\int_0^\epsilon g_\epsilon(t)\sin\mu t\,dt \tag{2.3}$$

Multiplying both sides of (2.2) by $g_\epsilon(t)$ integrate over $(0,\epsilon)$ with respect to t. Then using (2.3) and changing the order of integration in the resulting integrals on the right hand side we obtain, after some manipulation,

$$\frac{1}{\sqrt{\lambda}}\Psi_\epsilon(\sqrt{\lambda})\,\phi_j(x,\lambda)=\frac{1}{2}\int_{x-\epsilon}^{x+\epsilon}P(x,s,\epsilon)\,\phi_j(s,\lambda)\,ds \tag{2.4}$$

where

$$P(x,s,\epsilon)=(P_{ij}(x,s,\epsilon))$$

$$=\int_{|x-s|}^{\epsilon}[I+\Omega(x,t,s)-A\,\Omega(x,t,x+t)-B\,\Omega(x,t,x-t)]\,g_\epsilon(t)\,dt \tag{2.5}$$

A, B being defined as follows:

$A=1$, if $s\in(0,x+\epsilon)$; $\quad B=1$, if $s\in(0,x-\epsilon)$
$\quad=0$, otherwise $\qquad\qquad\quad =0$, otherwise.

Similarly,

$$\frac{1}{\sqrt{\lambda}}\Psi_\epsilon(\sqrt{\lambda})\,\theta_j(x,\lambda)=\frac{1}{2}\int_{x-\epsilon}^{x+\epsilon}P(x,s,\epsilon)\,\theta_j(s,\lambda)\,ds \tag{2.6}$$

Thus for a fixed x, each component of

$$\frac{2}{\sqrt{\lambda}}\Psi_\epsilon(\sqrt{\lambda})\,\phi_j(x,\lambda)$$

is the 'ϕ'-Fourier transform of a vector equal to

$P_j(x,s,\epsilon)=\begin{pmatrix}P_{j1}\\P_{j2}\end{pmatrix}$, $j=1,2$ in $(x-\epsilon,x+\epsilon)$ and zero outside the interval; with a similar result for the components of

$$\frac{2}{\sqrt{\lambda}}\Psi_\epsilon(\sqrt{\lambda})\,\theta_j(x,\lambda).$$

Then from (2.4), (2.6), the relations obtained from these by changing x to y and the generalized Parseval relation[1] (p. 151), we obtain, in view of (1.11),

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda}\,\Psi_\epsilon^2(\sqrt{\lambda})\,\mathrm{d}_\lambda\,H(x,y,\lambda)=\frac{1}{4}\int_{\Delta xy}P(x,s,\epsilon)\,P^T(y,s,\epsilon)\,\mathrm{d}s \tag{2.7}$$

where

$$\Delta_{xy}=(x-\epsilon,\ x+\epsilon)\cap(y-\epsilon,\ y+\epsilon).$$

In particular,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda}\,\Psi_\epsilon^2(\sqrt{\lambda})\,\mathrm{d}_\lambda\,H(x,x,\lambda)=\frac{1}{4}\int_{x-\epsilon}^{x+\epsilon}P(x,s,\epsilon)\,P^T(x,s,\epsilon)\,\mathrm{d}s \tag{2.8}$$

3. Preliminary estimates

In the following we obtain certain lemmas which involve the matrix H and which will be used subsequently to obtain deeper results. The method of procedure is similar to that used by Levitan and Sargsyan[2] (pp. 23-26) and we shall indicate only those steps where we considerably differ.

Lemma 3.1: Let p, q, r be integrable over any finite interval. ϵ_0 an arbitrary positive number and (x_0, x_1) an arbitrary finite interval on the real line. There there exists a constant matrix $C \equiv C(\epsilon_0, x_0, x_1)$ depending on the arguments shown, such that for x, y lying in (x_0, x_1),

$$\int_{-\infty}^{0} e^{\epsilon_0\sqrt{|\lambda|}}\,\mathrm{d}_\lambda\,H(x,y,\lambda)\ \leqslant\ C$$

holds.

In particular,

$$\int_{-\infty}^{0} e^{\epsilon\sqrt{|\lambda|}}\,\mathrm{d}_\lambda\,H(x,y,\lambda)$$

is finite for arbitrary finite ϵ, x, y. Also $H_{ij}(x,y,-\infty)<\infty$.

Put $g_\epsilon(t) = 1/\epsilon^2$ for $0<t<\epsilon$

$= -1/\epsilon^2$ for $-\epsilon<t<0$

$= 0$ otherwise.

Then

$$\frac{1}{\sqrt{\lambda}}\,\Psi_\epsilon(\sqrt{\lambda})=\frac{1}{2}\left(\frac{\sin 1/2\epsilon\sqrt{\lambda}}{1/2\,e\sqrt{\lambda}}\right)^2.$$

Then the lemma follows in the Levitan-Sargsyan[2] (pp. 25-26) manner by utilizing the results (1-14) and (2.8).

Put $g_\epsilon(t,a) = g_\epsilon(t)\cos at$, where a is an arbitrary real number and

$$\Psi_\epsilon(\sqrt{\lambda},a)=\int_0^\epsilon g_\epsilon(t,a)\sin\sqrt{\lambda}t\,\mathrm{d}t.$$

We note that $g_\epsilon(t,a)$ considered as a function of t, satisfies all the conditions imposed on $g_\epsilon(t)$ and therefore by choosing $g_\epsilon(t)$ as before in the proof of lemma 3.1, it follows that

$$|\Psi_\epsilon(\sqrt{\lambda},a)| \leq \frac{1}{\epsilon^2} \cdot \frac{\cosh \epsilon\sqrt{|\lambda|}}{\sqrt{|\lambda|}}, \text{ where } \lambda < 0.$$

Put

$$\Gamma(x,y,a,\epsilon) = \int_{-\infty}^{0} \frac{1}{\lambda} \Psi_\epsilon^2(\sqrt{\lambda},a)\, d_\lambda H(x,y,\lambda). \tag{3.1}$$

Then

$$\Gamma(x,y,a,\epsilon) \leqslant \frac{1}{\epsilon^4\delta^2}\int_{-\infty}^{0} \cosh^2(\epsilon\sqrt{|\lambda|})\, d_\lambda H(x,y,\lambda)$$

where $|\lambda| > \delta > 0$, δ being a positive number however small it may be.

Hence, by lemma 3.1,

$$\Gamma(x,y,a,\epsilon) \leqslant C \tag{3.2}$$

where $C \equiv C(\epsilon, x_0, x_1)$ are various constants depending on the arguments shown and $x,y \in (x_0, x_1)$ as in lemma 3.1.

Let us change $\Psi_\epsilon(\sqrt{\lambda})$ to $\Psi_\epsilon(\sqrt{\lambda}, a)$, $g_\epsilon(t)$ to $g_\epsilon(t,a)$ and let the matrix $P((x,s,\epsilon)$ change to $P(x,s,a,\epsilon)$. Then the formula (2.7) can be written as

$$\int_0^\infty \frac{1}{\lambda} \Psi_\epsilon^2(\sqrt{\lambda},a)\, d_\lambda H(x,y,\lambda)$$
$$= \pi/4 \int_{\Delta xy} P(x,s,a,\epsilon)\, P^T(y,s,a,\epsilon)\, ds - \Gamma(x,y,a,\epsilon). \tag{3.3}$$

For, $\lambda > 0$, let $\lambda = \mu^2$, $H(x,y,\lambda) = H_1(x,y,\mu)$ and for fixed x, y, H_1 is continued to the negative half-line as a matrix having each element an odd function of μ. Then from (3.3) at $x=y$.

$$\int_{-\infty}^{\infty} \frac{1}{\mu^2} \Psi_\epsilon^2(\mu,a)\, d_\mu H_1(x,x,\mu) = \frac{\pi}{2}\int_{x-\epsilon} P(x,s,a,\epsilon)\, P^T(x,s,a,\epsilon)\, ds$$
$$- \Gamma(x,x,a,\epsilon). \tag{3.4}$$

Since

$$\Psi_\epsilon(\mu,a) = \frac{1}{2}\left[\Psi_\epsilon(\mu+a) + \Psi_\epsilon(\mu-a)\right],$$

it follows on substitution of the particular value of $\Psi_\epsilon(\sqrt{\lambda})$ as obtained in the course of proof of lemma 3.1, that for $0 < \delta \leq \mu - a \leq \mu \leq 1$,

$$\frac{\sin^2 1/2\,(\mu-a)\epsilon}{(\mu-a)^2\epsilon^2} \leq \frac{\sin^2 1/2(\mu-a)\epsilon}{\mu(\mu-a)^2\epsilon^2} \leq \frac{1}{\delta}\,\frac{\sin^2 1/2(\mu-a)\epsilon}{\mu(\mu-a)\epsilon^2} \leq \frac{1}{2\delta\mu}\quad \Psi_\epsilon(\mu,a).$$

Hence for fixed ν

$$\int_0^{\gamma} \left[\frac{\sin 1/2(\mu-a)\epsilon}{1/2\,(\mu-a)\,\epsilon} \right] d_\mu H_\Gamma(x,x,\mu) \ll \frac{4}{\delta^2} \int_{-\infty}^{\infty} \frac{1}{\mu^2} \Psi_\epsilon^2(\mu,a)\, d_\mu H_1(x,x,\mu$$

Putting $\epsilon = 1$ and then proceeding in the Levitan-Sargsyan[2] manner (pp. 25-26) we obtain by using (3.4), (3.2) and (2.7) the following

Lemma 3.2: Let p, q, r be integrable in every finite interval and let (x_0, x_1) be an arbitrary interval on the real line. Then for $x, y, \epsilon\, (x_0, x_1)$ $H_1(x, y, \mu+\nu) - H_1(x, y, \mu) \ll C$, where ν is fixed and $C \equiv C(x_0, x_1)$ is a constant matrix depending on the arguments shown.

In our further discussions we require more stringent conditions on $g_\epsilon(t)$ defined in section 2. These are in addition to the conditions (i), (ii) and are as follows:

a) $g_\epsilon(t)$ and its indefinite integral $h_\epsilon(t)$, $|t| < \epsilon$, are infinitely differentiable; all such derivatives being equal to zero for $|t| \geq \epsilon$.

b) $g_\epsilon(t)$, $h_\epsilon(t)$ are uniformly bounded with respect to ϵ, t and $h_\epsilon(t) = 0$ for $|t| = \epsilon$.

Then

$$\frac{\Psi_\epsilon \sqrt{\lambda})}{\sqrt{\lambda}} = \int_0^\epsilon g_\epsilon(t) \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}}\, dt = -\int_0^\epsilon h_\epsilon(t \quad \cos\sqrt{\lambda}\, t\, d$$

Hence,

$$\left| \frac{\Psi_\epsilon(\sqrt{\lambda})}{\sqrt{\lambda}} \right| = O\left[e^{(\sqrt{|\lambda|}+1)\epsilon}\right]$$

and as in Levitan-Sargsyan[2] (p. 28), we have by lemma 3.1, the following

Lemma 3.3: If p, q, r are integrable over any finite interval and (x_0, x_1) an arbitrary finite interval, then for all $x, y, \epsilon\, (x_0, x_1)$

$$\int_{-\infty}^{0} |(\Psi_\epsilon(\sqrt{\lambda}))\sqrt{\lambda}|\, d_\lambda H(x, y, \lambda) << C,$$

where $C \equiv C(x_0, x_1)$ is a constant matrix depending on the arguments shown.

4. Asymptotic relations

In the present section we establish asymptotic relations involving the matrix $\Omega(x, t, s)$ which occur in section 2. We represent the sum of the moduli of the elements of a matrix A by $|A|$.

Let $Q(x)$ defined in (1.8), satisfy

$$\int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma \leq Ct^{a+1}, \qquad \text{say} \tag{4.1}$$

$a > 0$ is a constant.

When $p(x)$, $q(x)$, $r(x)$ are integrable over $(-\infty,\infty)$, the condition (4.1) is satisfied. In fact, in this case

$$\frac{1}{t^{a+1}} \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma = o(1)$$

as t tends to infinity (for fixed x). Then the following inequality holds[5].

$$\mid T(x,t,s) \mid \leq \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma \exp \left[\frac{1}{2} \cdot t \int_{x-t}^{x+t} \mid Q(\sigma) \mid d\sigma \right] \tag{4.2}$$

where T is the 2×2 matrix which occurs in (2.1).

Then from (4.1), (4.2) and the definition of $\Omega\,(x,t,s)$ in terms of T, it follows that

$$\begin{aligned} &\mid \Omega\,(x,t,s) - A\;\Omega\,(x,t,x+t) - B\,\Omega\,(x,t,x-t) \mid \\ &\leq 3 \int_{0}^{x+t} \mid T\,(x,t,y) \mid dy \\ &\leq C t^{a+1}, \end{aligned} \tag{4.3}$$

$0 < x-t < s < x+t$, A, B are those defined in the formula (2.5) and $C \equiv C(x_0, x_1)$, a positive constant depending on the arguments shown, where $t \leq t_0$, a fixed number and $x \in (x_0, x_1)$, t may take sufficiently small values and $a \equiv a(x_0, x_1)$ is a constant > 0.

For convenience of presentation, we introduce the following definition:

A matrix A is $O(f)$ or $o(f)$, where f is a scalar, if each element of A is $O(f)$ or $o(f)$ in the usual sense. In particular $A = o\,(1)$ means that each element of A tends to zero.

We establish the following lemma:

Lemma 4.1: Let $Q(x)$ satisfy (4.1) and let

$$\alpha\,(x,s,\nu) = \nu \int_{|x-s|} \left[\Omega\,(x,t,s) - A\;\Omega\,(x,t,x+t) - B\;\Omega\,(x,t,x-t) \right] \sin \nu t \, dt$$

A, B being defined as in (2.5).

Then

$$\int_0^{\mu} \alpha\,(x,s,\nu)\, d\nu = o(1),$$

as μ tends to infinity, uniformly in every finite domain of definition of x and s.

Proof: Let

$$\alpha(x,s,\nu) = (\alpha_j(x,s,\nu)),\ \Omega(x,t,s) = (\Omega_j(x,t,s)),$$

$j = 1,2,3,4.$

In view of the result

$$\frac{1}{\mu}\int_0^{\mu} d\nu \int_0^{\nu} \alpha_j(x,s,u)\,du - \int_0^{\mu}\left(1-\frac{\nu}{\mu}\right)\alpha_j(x,s,\nu)\,d\nu$$

obtained by integration by parts, there is no loss of generality in assuming that $\alpha_j(x,s,u)$ is always positive. It follows therefore that

$$\frac{1}{\mu}\int_0^{\mu} d\nu \int_0^{\nu} \alpha_j(x,s,u)\,du \geq \frac{1}{2}\int_0^{1/2\mu} \alpha_j(x,s,u)\,du \tag{4.4}$$

Since

$$\int_0^{\mu} d\nu \int_0^{\nu} u\sin ut\,du = -\frac{\mu\sin\mu t}{t^2} + \frac{2}{t^3}(1-\cos\mu t),$$

it follows on substitution for $\alpha_j(x,s,\nu)$ in terms of Ω_j as defined in the lemma and then changing the order of integration, that the left hand side of (4.4) is equal to

$$\begin{aligned} I &= -\int_{|x-s|}^{1}(\cdot)\frac{\sin\mu t}{t^2}\,dt + \frac{2}{\mu}\int_{|x-s|}^{1}(\cdot)\frac{dt}{t^3} - \frac{2}{\mu}\int_{|x-s|}^{1}(\cdot)\frac{\cos\mu t}{t^3}\,dt \\ &= -I_1 + I_2, \end{aligned} \tag{4.5}$$

where $(\cdot) \equiv \Omega_j(x,t,s) - A\,\Omega_j(x,t,x,+t) - B\,\Omega_j(x,t,x-t) \equiv K(x,t,s)$, say.

Now,

$$I_1 = \int_{|x-s|}^{1}(\cdot)\frac{\sin\mu t}{t^2}\,dt = \left(\int_{|x-s|}^{\eta} + \int_{\eta}\right)(\cdot)\frac{\sin\mu t}{t^2}\,dt = I_{11}+I_{12}, \text{ say,}$$

where η is an arbitrary positive number which does not exceed unity. Then by (4.3)

$$|I_{11}| \leq C\int_{|x-s|}^{\eta} t^{a-1}\,dt < C\eta^a,\ a>0,$$

where we choose η so small that for all μ, $C\eta^a \leq 1/2\zeta$, where ζ is an arbitrary positive number which may be as small as we like. Then

$$|I_{11}| < \frac{1}{2}\zeta \tag{4.6}$$

Having chosen η as such it follows from the Riemann-Lebesgue lemma that there exists a μ_0 such that for $\mu > \mu_0$

$$|I_{12}| < \frac{1}{2}\xi. \tag{4.7}$$

Thus $I_1 = o(1)$ uniformly as μ tends to infinity.

Now $|I_2| \leq 4/\mu \int^{1} |K_j(x,t,s)/t/\sin^2 \frac{1}{2}\mu t/t^2\,dt$.

Since $|K_j(x,t,s)/t| < Ct^a$, $a>0$, it follows that $K_j(x,t,s)$ as t tends to zero

Thus $|K_j(x,t,s)/t| < \epsilon$ for $t \leq \eta$.

Hence

$$|I_2| \leq \frac{4}{\mu} \left(\int_0^{\eta} + \int_{\eta}^{1} \right) | K_j(x,t,s)/t | \sin^2 \frac{1}{2} \mu t / t^2 \, dt.$$

We now argue as in Titchmarsh[6] (P.414) by utilizing the familiar integral

$\int_0^{\infty} \frac{\sin^2 x}{x^2} du = \frac{\pi}{2}$ so as to obtain $I_2 = o(1)$, uniformly, as μ tends to infinity.

The lemma therefore follows from (4.4), the uniformity following from the uniform boundedness of $\Omega_j(x,t,s) - A\,\Omega_j(x,t,x+t) - B\,\Omega_j(x,t,x-t) \equiv K_j(x,t,s)$.

Lemma 4.2: Let $h(x,s,t)$, $\beta(x,s,\nu)$ be the matrices

$$h(x,s,t) = (h_j(x,s,\nu) = \int_{\infty}^{0} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}t} d_{\lambda}\, h(x,s,\lambda)$$

$$\beta(x,s,\nu) = (\beta_j(x,s,\nu)) = \nu \int_0^1 h(x,s,t) \sin \nu t \, dt.$$

$j = 1,2,3,4$ and $Q(x)$ satisfy (4.1). Then

$$\int_0^{\mu} \beta(x,s,\nu)\, d\nu = -\frac{\pi}{2} H(x,s,-\infty) + o(1), \tag{A}$$

as μ tends to infinity, uniformly in every finite region containing x and s.

Proof: The existence of the matrix $h(x,s,t)$ is a consequence of the lemma 3.3. Let

$$H(x,s,\lambda) = (H_{rk}(x,s,\lambda)),\ r,k = 1,2.$$

It follows by integration by parts that

$$\int_0^{\mu} (1 - \frac{\nu}{\mu}\ \beta_j(x,s,\nu)\, d\nu = \frac{1}{\mu} \int_0^{\mu} d\nu \int_0^{\nu} \beta_j(x,s,u)\, du \tag{4.8a}$$

$$= \frac{1}{\mu} \int_0^1 h_j(x,s,t) \left[-\frac{\mu \sin \mu t}{t^2} + \frac{2(1-\cos\mu t)}{t^3} \right] dt$$

$$= -I_1 + I_2, \text{ say} \tag{4.8}$$

(on substitution for the $\beta_j(x,s,u)$, and change in the order of integration on evaluation of the inner integral involving $\sin ut$).

In view of (4.8a), there is no loss of generality in assuming that $\beta_j(x,s,u)$ is always positive.

Now

$$I_1 = \left(\int_0^{\delta} + \int_{\delta}^{1} \right) \frac{h_j(x,s,t)}{t} \cdot \frac{\sin \mu t}{t} dt = I_{11} + I_{12}, \text{ say}$$

lere $I_{12} = o\,(1)$, as μ tends to infinity, by the Riemann-Lebesgue lemma.

ince $\lim_{t \to 0} h_j\,(x,s,t)/\,t = -\,H_{rk}\,(x,s,-\infty)$ (4.9)

nd (cosec $\frac{1}{2}\,t - \frac{1}{2}$) $\frac{h_j\,(x,s,t)}{t}$ is integrable in $(0,\delta)$, the well-known technique adopted

or the treatment of convergence of Fourier series gives (Titchmarsh[6], pp. 403-406).

$$\frac{I_{11}}{\mu} = S_{\mu j} + 0(1), \text{ as } \mu \text{ tends to infinity,}$$

here,

$$S_{\mu j} = \frac{1}{2\pi} \int_0^\delta \frac{\sin(\mu + 1/2)t}{\sin 1/2 t} \cdot \frac{h_j\,(x,s,t)}{t}\,\mathrm{d}t$$

nd

$$S_{\mu j} + \frac{1}{2}\;H_{rk}\,(x,s,-\infty) = o(1),$$

; μ tends to infinity.

ltogether, $I_1 = -\pi/2\,H_{rk}\,(x,s,-\infty) + o(1)$ (4.10)

; μ tends to infinity.

gain,

$$I_2 = \frac{4}{\mu}\;\left(\int_0^\delta + \int_\delta^1\right) \frac{h_j\,(x,s,t)}{t} \cdot \frac{\sin^2 1/2\,\mu t}{t^2}\,\mathrm{d}t$$
$$= I_{21} + I_{22}, \text{ say.}$$

:om the continuity and consequent integrability of $h_j\,(x,s,t)/t$ over $(\delta, 1)$, it follows by the iemann Lebesgue lemma that

$$I_{22} = o(1), \text{ as } \mu \text{ tends to infinity.}$$

/ the usual technique adopted for the consideration of the summability of Fourier series 'itchmarsh[6], pp. 412-413)

$$I_{21} = 2\pi\,\sigma_j\,(\mu) + o(1), \text{ as } \mu \text{ tends to infinity,}$$

here

$$\sigma_j(\mu) = \frac{1}{2\pi} \cdot \frac{1}{\mu} \int_0^\delta \frac{\sin^2 ½\,\mu t}{\sin^2 ½\,t} \cdot \frac{h_j\,(x,s,t)}{t}\,\mathrm{d}t$$

ıd $\sigma_j(\mu) = -\,\frac{1}{2}\;H_{rk}\,(x,s,-\infty) + o(1)$, as μ tends to infinity.

hus

$$I_2 = -\,\pi\;H_{rk}\,(x,s,-\infty) + o(1) \tag{4.11}$$

; μ tends to infinity.

From (4.8), (4.10) and (4.11), we have

$$\int_0^\mu \left(1-\frac{\nu}{\mu}\right)\beta_j(x,s,\nu)\,\mathrm{d}\nu = -\frac{\pi}{2}\,H_{rk}(x,s,-\infty) + o(1) \tag{4.12}$$

as μ tends to infinity.

$\int_0^\infty \beta_j(x,s,\nu)\,\mathrm{d}\nu$ is thus summable (C,1).

Also since

$$\frac{1}{2}\int_0^{1/2\mu} \beta_j(x,s,\nu)\,\mathrm{d}\nu \le \int_0^\mu \left(1-\frac{\nu}{\mu}\right)\beta_j(x,s,\nu)\,\mathrm{d}\nu$$

$\int_0^\infty \beta_j(x,s,\nu)\,\mathrm{d}\nu$ is convergent in the usual sense.

Therefore,

$$\lim_{\mu\to\infty}\frac{1}{\mu}\int_0^\mu \nu\beta_j(x,s,\nu)\,\mathrm{d}\nu = 0$$

(Hobson[7], p. 386).

The result (A) then follows from (4.12).

When t is complex, $h(x,s,t)$ is an entire function of t and hence in particular infinitely differentiable in the neighbourhood of $t=0$. The uniformity in (A) then follows from the boundedness of $h(x,s,t)$ and $\partial h/\partial t$.

5. The Fourier system: The asymptotic formula for $H(x,y,\lambda)$

The Fourier system corresponding to the given system (1.1) is the system (1.1) with $p(x)=q(x)=r(x)=0$ *i.e.* $Q(x)=0$. If ϕ_r^F, θ_r^F are the ϕ, θ of the Fourier system satisfying (1.9), (1.10), it is easy to verify that the matrix $H_1^F(x,y,\mu)$ for the Fourier system corresponding to the matrix $H_1(x,y,\mu)$ for the general has the simple representation

$$H_1^F(x,s,\mu) = 1/\pi\cdot\frac{\sin\mu(x-s)}{x-s}\cdot I \tag{5.1}$$

where I is the unit 2×2 matrix.

Also $H^F(\cdot)$ behaves in the same way with respect to the Fourier system as the $H(\cdot)$ with respect to the given system (1.1). That is $H^F(\cdot)$ generates the resolution of the identity of the operator T^F which corresponds to the Fourier differential operator.

Let $f(x)=\begin{pmatrix} f_1\\ f_2\end{pmatrix}\epsilon\, L_2(-\infty,\infty)$ and let $V_\epsilon(x,s)$ be defined as follows.

$$V_\epsilon(x,s) = P(x,s,\epsilon) \text{ for } s\;\epsilon\;(x-\epsilon, x+\epsilon)$$
$$= 0 \qquad \text{otherwise,}$$

where $P(x, s, \epsilon)$ is given by (2.5). Then using (2.4), (2.6) and the generalized Parseval relation stated in section 2, we obtain after some easy calculations that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\lambda}} \Psi_\epsilon(\sqrt{\lambda})\, d_\lambda \int_{-\infty}^{\infty} H(x,s,\lambda) f(s)\, ds = \frac{1}{2} \int_{x-\epsilon}^{x+\epsilon} P(x,s,\epsilon) f(s)\, ds \tag{5.2}$$

Now $\{e^{inx}\}$, $-\infty < n < \infty$, is closed over $(-\pi, \pi)$ and hence form a complete orthogonal system on $(-\pi, \pi)$. Therefore the vector $\begin{pmatrix} e^{inx} \\ e^{-inx} \end{pmatrix}$, $-\infty < n < \infty$, is closed over $(-\pi, \pi)$; in other words these vectors form a complete orthogonal system in the subspace of those $\begin{pmatrix} f \\ g \end{pmatrix} \epsilon$ $L_2(-\pi,\pi) \oplus L_2(-\pi,\pi)$ satisfying $g(-x) = f(x)$.

Since the vector f in (5.2) is arbitrary, we choose

$$f(s) = \begin{pmatrix} e^{ins} \\ e^{-ins} \end{pmatrix}, \quad -\infty < n < \infty,\ x-\epsilon \leq s \leq x+\epsilon$$
$$= 0 \qquad \text{otherwise;}$$

then it follows from (5.2) that

$$\int_{-\infty}^{\infty} 1/\sqrt{\lambda}\ \Psi_\epsilon(\sqrt{\lambda})\, d_\lambda H(x,s\lambda) = \pi/2\ P(x,s,\epsilon), \text{ for } |x-s| \leq \epsilon$$
$$= 0, \text{ for } |x-s| > \epsilon \tag{5.3}$$

The convergence of the integral on the left of (5.3) is ensured from lemma 3.3 and the fact that $1/\mu\ \Psi_\epsilon(\mu) = O(1/\mu^2)$, as μ tends to infinity.

As in section 3, put $g_\epsilon(t, a) = g_\epsilon(t) \cos at$,

where a is an arbitrary positive number. Then form (5.3) it follows as before

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\lambda}} [\Psi_\epsilon(\sqrt{\lambda}+a) + \Psi_\epsilon(\sqrt{\lambda}-a)]\, d_\lambda H(x,s,\lambda) = \pi\ P(x,s,a,\epsilon), \quad \text{if } |x-s| \leq \epsilon$$
$$= 0 \quad \text{if } |x-s| > \epsilon \tag{5.4}$$

where

$$P(x,s,a,\epsilon) = \int_{|x-s|}^{\epsilon} g_\epsilon(t,a) [I + \Omega(x,t,s) - A\,\Omega(x,t,x+t) - B\,\Omega(x,t,x-t)]\, dt$$

and I is the 2×2 unit matrix, A, B being those as defined in (2.5).

Then noting as before that when $\lambda > 0$, $\lambda = \mu^2$, $H(x,y,\lambda) = H_1(x,y,\mu)$ and for fixed x, y, H_1 is continued to negative μ as a matrix whose elements are all odd functions, it follows from (5.4) that

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, d_\mu H_1(x,s,\mu)$$
$$= \pi P(x,s,a,\epsilon) - 2\int_{-\infty}^{0} \int_{0}^{\epsilon} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d_\lambda H(x,s,\lambda)\,dt;\ |x-s| \le \epsilon$$
$$= -2\int_{-\infty}^{0} \int_{0} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d_\lambda H(x,s,\lambda)\,dt;\ |x-s| > \epsilon \tag{5.5}$$

For the Fourier case we have $Q(x) = 0$. Also from (4.2), $T(x,t,s) = 0$ and consequently $\Omega(x,t,s) = 0$. Thus for the Fourier case we have from (5.5)

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, d_\mu H_1^F(x,s,\mu) = \pi I \int_{|x-s|}^{\epsilon} g_\epsilon(t,a)\,dt;\quad |x-s| \le \epsilon$$
$$= 0 \qquad\qquad |x-s| > \epsilon \tag{5.6}$$

where we utilize the fact for the Fourier case the negative part of the spectrum is absent (compare Chakravarty and Sengupta[8]).

Put
$$g(x,s,a,\epsilon) = P(x,s,a,\epsilon) - I\int_{|x-s}^{\epsilon} g_\epsilon(t,a)\,dt$$
$$= \int_{|x-s|}^{\epsilon} [\Omega(x,t,s) - A\,\Omega(x,t,x+t) - B\,\Omega(x,t,x-t)]\, g_\epsilon(t,a)\,dt$$

and $\Phi(x,s,\mu) = H_1(x,s,\mu) - H_1^F(x,s,\mu)$.

Then from (5.5) and (5.6)
$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, d_\mu \Phi(x,s,\mu)$$
$$= \pi g(x,s,a,\epsilon) - 2\int_{-\infty}^{0} \int_{0}^{\epsilon} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dt\, d_\lambda H(x,s,\lambda);\ |x-s| \le \epsilon$$
$$= -2\int_{-\infty}^{0} \int_{0}^{\epsilon} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dt\, d\lambda\, H(x,s,\lambda);\qquad |x-s| > \epsilon. \tag{5.7}$$

By the Parseval theorem for Fourier sine transforms applied to each element of $1/\mu\,\alpha(x,s,\mu)$ defined in lemma 4.1 and

$$\Psi_\epsilon(\mu,a) = \int_0^\epsilon g_\epsilon(t,a) \sin\sqrt{\lambda}t \tag{5.8}$$

defined in section 3, and noting that $\Psi_\epsilon(\cdot)$ is odd, we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, \alpha(x,s,\mu)\, d\mu = \pi g(x,s,a,\epsilon);\ \text{if } |x-s| \le \epsilon$$
$$= 0 \qquad \text{if } |x-s| > \epsilon \tag{5.9}$$

Again, by changing the order of integration which is easily verifiable from lemma 3.3, and the conditions imposed on g_ϵ therein we have

$$\int_{-\infty}^{0} \int_{0}^{\epsilon^{\mathsf{r}}} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \, dt \, d_\lambda H(x,s,\lambda) = \int_{0}^{\epsilon} g_\epsilon(t,a) \, h(x,s,t) \, dt \tag{5.10}$$

where $h(x,s,t)$ is defined as in lemma 4.2.

Again by the Parseval theorem for the Fourier sine transform applied to (5.8) and each element of $1/\mu \, \beta(x,s,\mu)$ defined in lemma 4.2 and from consideration that $\Psi_\epsilon(\cdot)$ is odd, we obtain for $0 < \epsilon \leq 1$,

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a) \, \beta(x,s,\mu) \, d\mu = \pi \int_{0}^{\epsilon} g_\epsilon(t,a) \, h(x,s,t) \, dt. \tag{5.11}$$

Hence, from (5.7), (5.9), (5.10) and (5.11) it follows that

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a) \, d_\mu \, \Phi^*(x,s,\mu) = 0 \tag{5.12}$$

where

$$\Phi^*(x,s,\mu) = \Phi(x,s,\mu) - \int_{0}^{\mu} \alpha(x,s,\nu) \, d\nu + \frac{2}{\pi} \int_{0}^{\mu} \beta(x,s,\nu) \, d\nu. \tag{5.13}$$

Since a is arbitrary and Ψ_ϵ is continuous, (5.12) can be written in the form

$$\int_{-\infty}^{\infty} \chi_\epsilon(\mu) \, d_\mu \, \Phi^*(x,s,\mu) = 0 \tag{5.14}$$

where $\chi_\epsilon(\mu) = \dfrac{1}{\mu} \Psi_\epsilon(\mu)$ which is the Fourier cosine transform of $\{-h_\epsilon(t)\}$ introduced in connection with lemma 3.3. As in lemma 3.2 it follows easily that for fixed ν,

$$\Phi^*(x,s,\mu+\nu) - \Phi^*(x,s,\mu) \ll C \tag{5.15}$$

where C is a constant matrix $C \equiv C(x_0, x_1)$, (x_0, x_1) being any fixed interval which x and s vary.

We now use the following modified version of the Marchenko Tauberian theorem (Marcenko[9]; also Levitan and Sargsyan[2]; pp. 90-92, special p. 92):

Theorem A: Let f be the set of infinitely differentiable functions in $(-\infty, \infty)$ and let

$$\int_{-\infty}^{\infty} E_f(\lambda) \, d\{\rho(\lambda) - \sigma(\lambda)\} = 0,$$

where,

$$E_f(\lambda) = \int_{-\infty}^{\infty} f \cdot e^{-i\lambda x} \, dx \text{ and } \rho(\lambda), \sigma(\lambda) \; (-\infty < \lambda < \infty)$$

are nondecreasing and continuous on the left such that one of the functions $\sigma(\lambda)$, say, satisfies

$$\lim_{|a| \to \infty} \{\sigma(a+\lambda_0) - \sigma(a)\} = \sigma^*(\lambda_0) < \infty, \text{ for some } \lambda_0 > 0$$

(or $\rho(\lambda)$, $\sigma(\lambda)$ are odd functions and $\sigma(\lambda)$ has a bounded derivative). Then

$$\lim_{N\to\infty} \{\rho(N) - \sigma(N)\} = 0.$$

Applying this theorem, we obtain from (5.14),

$$\lim_{\mu\to\infty} \Phi^* (x, s, \mu) = 0 \tag{5.16}$$

uniformly in every finite interval containing x, s.

Therefore, from (5.13), we obtain by using (5.16) and the lemma 4.1 and 4.2, the following theorem.

Theorem: If $Q(x)$ satisfy (4.1), then

$$\lim_{\mu\to\infty} [H_1 (x,s,\mu) - H_1^F (x,s,\mu)] = H (x, s, -\infty),$$

for each fixed x, s, the result holding uniformly in every finite domain in which x, s vary, H_1^F being given by (5.1).

Put $x = s = 0$. Then from the initial conditions it follows that the matrices θ and ϕ in (1.11) are the zero and the unit matrices respectively. Hence $H_1 (x, s, \mu)$ reduces in this case to the matrix $\rho(\mu)$ whose elements in terms of m_{ij}, M_{ij} are given explicitly by

$$\rho_{rr} (\mu) = \lim_{\nu\to 0} \int_0^\mu \operatorname{im} \left[\frac{m_{rr} (\lambda)\, M_{rr} (\lambda) + m_{rs}^2 (\lambda)}{m_{11} (\lambda) - M_{11} (\lambda)} \right] du;\ \lambda = u + iv$$

and

$$\rho_{rs} (\mu) = \lim_{\nu\to 0} \int_0^\mu \operatorname{im} \left[\frac{m_{rs} (\lambda)\, M_{rr} (\lambda) + m_{ss} (\lambda)\, M_{rs} (\lambda)}{m_{11} (\lambda) - M_{11} (\lambda)} \right] du$$

$\rho_{rs} (\mu) = \rho_{sr}^{\cdot} (\lambda)$, $\quad m \neq s = 1,2.$

Then from the theorem we have the representation

$$\lim_{\mu\to\infty} [\rho(\mu) - \frac{\mu}{\pi} \cdot I] = \rho (-\infty)$$

where I is the unit 2×2 matrix.

(For discussion involving m_{ij}, M_{ij}, etc., and introduction of the matrix $H (x, s, \mu)$ in the explicit form (1.11), (see [1]).

Incidentally, we remark that the Titchmarsh [10] (p.43) spectral function $k(\lambda)$ is obtained by putting $x = y = 0$ in $\theta(x, y, \lambda)$, the spectral function which occurs in Levitan and Sargsyan [2] (p.22 formula (2.1.13)). To verify this the formula (2.1.13) of Levitan and Sargsyan has to be obtained in the same way as in authors' paper [1], by deriving the explicit form of the Green's function for the Sturm-Liouville equation in the singular case $[0, \infty)$, $H(x, y, \lambda)$ similarly yields ($k_{rs} (\lambda)$), the generalization of k (λ) in the matrix case for the interval $[0, \infty)$.

In what follows we obtain an equiconvergence theorem for the eigenfunction expansion associated with the system (1.1) and therefrom deduce a general expansion theorem involving an arbitrary vector $f(x) \in L_2(-\infty,\infty)$.

6. Some preliminary investigations

Let

$$S(x,\lambda) = \int_{-\infty}^{\infty} H(x,s,\lambda) f(s)\, ds$$

$$S^F(x,\lambda) = \int_{-\infty}^{\infty} H^F(x,s,\lambda) f(s)\, ds \tag{6A}$$

$H^F(\cdot)$ being the resolution matrix in the Fourier case.

For $\lambda = \mu^2$, let $H(x,y,\lambda) = H_1(x,y,\mu)$; $H^F(x,y,\lambda) = H_1^F(x,y,\mu)$;

$S(x,\lambda) = S_1(x,\mu)$, $S^F(x,\lambda) = S_1^F(x,\mu)$ for $\mu > 0$.

For fixed x,y, as before let $H_1(x,y,\mu)$ be continued to the negative half-line as an odd function.

Similarly for $H_1^F(x,y,\mu)$.

Also for $\mu < 0$, let $S_1(x,\mu) = -S_1(x,-\mu)$ with a similar meaning for $S_1^F(x,\mu)$.

As in section 3, let

$g_\epsilon(t,a) = g_\epsilon(t)\cos at$, a an arbitrary positive number; it follows from (5.2) that

$$\frac{1}{2}\pi \int_{-\infty}^{\infty} 1/\sqrt{\lambda}\,[\,\Psi_\epsilon(\sqrt{\lambda}+a) + \Psi_\epsilon(\sqrt{\lambda}-a)\,]\, d_\lambda\, S(x,\lambda) = \frac{1}{2}\int_{x-\epsilon}^{x+\epsilon} P(x,s,a,\epsilon)\, f(s)\, ds \tag{6.1}$$

where $f(x) \in L_2(-\infty\,\infty)$, $P(x,s,a,\epsilon)$ has the same meaning as in (5.4) and

$$\Psi_\epsilon(\mu,a) = \frac{1}{2}\,[\,\Psi_\epsilon(\mu+a) + \Psi_\epsilon(\mu-a)\,] = \int_0^\epsilon g_\epsilon(t,a)\,\sin\mu t\, dt \tag{6.2}$$

the Fourier sine transform of $g_\epsilon(t,a)$.

Therefore, (5.2) can be written as

$$\int_{-\infty}^{\infty} \frac{1}{\mu}\,\Psi_\epsilon(\mu-a)\, d_\mu\, S_1(x,\mu) =$$

$$\pi \int^{x+\epsilon} P(x,s,a,\epsilon) f(s)\, ds - 2\int_{-\infty}^{0} [\int_0^\epsilon g_\epsilon(t,a)\,\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}t}\, dt\,]\, d_\lambda\, S(x,\lambda). \tag{6.3}$$

The corresponding formula for the Fourier case ($Q(x) = 0$) is

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, d_\mu S_1^F(x,\mu) = \pi \int_{x-\epsilon}^{x+\epsilon} I\Big[\int_{|x-s|}^{\epsilon} g_\epsilon(t,a)\, dt \Big] f(s)\, ds. \quad (6.4)$$

Put

$$R(x,\mu) = S_1(x,\mu) - S_1^F(x,\mu).$$

Then from (6.3) and (6.4),

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, d_\mu R(x,\mu) = \pi \int_{x-\epsilon}^{x+\epsilon} g(x,s,a,\epsilon) f(s)\, ds$$

$$- 2 \int_{-\infty}^{0} \Big[\int_0^\epsilon g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dt \Big] d_\lambda S(x,\lambda) \quad (6.5)$$

where $g(x,s,a,\epsilon)$ is the same as in section 5.

By a change in the order of integration

$$\int_{x-\epsilon}^{x+\epsilon} g(x,s,a,\epsilon) f(s)\, ds = \int_0^{\epsilon} g_\epsilon(t,a)\, y(x,t)\, dt \quad (6.6)$$

where

$$y(x,t) = \begin{pmatrix} y_1(x,t) \\ y_2(x,t) \end{pmatrix} = \int_{x-t}^{x+t} [\Omega(x,t,s) - A\,\Omega(x,t,x+t) - B\,\Omega(x,t,x-t)] f(s) ds \quad (6.7)$$

$\Omega(x,t,s)$ and the constants A, B being those of section 2.

Put

$$A(x,\nu) = \begin{pmatrix} A_1(x,\nu) \\ A_2(x,\nu) \end{pmatrix} = \int_0^{\epsilon} \nu y(x,t) \sin \nu t\, dt \quad (6.8)$$

Then applying the Parseval theorem for the Fourier sine transform to each element of (6.8) and (6.2), it follows that

$$\frac{1}{2} \int_0^{\infty} \frac{1}{\mu} [\Psi_\epsilon(\mu+a) + \Psi_\epsilon(\mu-a)]\, A(x,\mu)\, d\mu = \frac{\pi}{2} \int_0^{\epsilon} g_\epsilon(t,a)\, y(x,t)\, dt. \quad (6.9)$$

Hence since $\Psi_\epsilon(\mu)$ is an odd function of μ, we have from (6.6) and (6.9)

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu-a)\, A(x,\mu)\, d\mu = \frac{\pi}{2} \int_{x-\epsilon}^{x+\epsilon} g(x,s,a,\epsilon) f(s)\, ds \quad (6.10)$$

Again by a change in the order of integration

$$\int_{-\infty}^{0} \Big[\int_0^{\epsilon} g_\epsilon(t,a) \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dt \Big] d_\lambda S(x,\lambda) = \int_0^{\epsilon} g_\epsilon(t,a)\, Z(x,t)\, dt \quad (6.11)$$

where
$$Z(x,t) = \begin{pmatrix} Z_1(x,t) \\ Z_2(x,t) \end{pmatrix} = \int_{-\infty}^{0} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \, d_\lambda S(x,\lambda) \tag{6.12}$$

$\lambda = \mu^2, \qquad 0 \leq t \leq 1.$

Since
$$\frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} = \int_0^t \cos (\sqrt{\lambda}\, u)\, du$$

therefore
$$\frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \leq t \leq 1, \quad \text{for } 0 \leq t \leq 1.$$

Hence $(\sin \sqrt{\lambda} t)/\sqrt{\lambda}$ is uniformly continuous in each bounded point set. Therefore, by Radon's definition of Stieltjes integrals (see Bochner [11], p. 307), $Z(x,t)$ exists uniformly in t.

$$\text{Put} \quad \frac{B(x,\mu)}{\mu} = \int_0^1 Z(x,t) \sin \mu t \, dt. \tag{6.13}$$

Let us apply the Parseval equality for the Fourier sine transforms to each term of the vector $B(x,\mu)$ defined in (6.13) and the relation (6.2), use the equality (6.11) and argue as before with $\Psi_\epsilon(\mu)$. We then ultimately obtain

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu - a)\, B(x,\mu)\, d\mu$$
$$= \frac{\pi}{2} \int_{-\infty}^{0} \left[\int_0^\epsilon g_\epsilon(t,a) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \, dt \right] d_\lambda S(x,\lambda). \tag{6.14}$$

Putting $R^*(x,\mu) = R(x,\mu) - \int_0^\mu A(x,\nu)\, d\nu + 2/\pi \int_0^\mu B(x,\nu)\, d\nu$

it follows from (6.5), (6.10) and (6.14) that

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu - a)\, d_\mu R^*(x,\mu) = 0, \qquad 0 \leq \epsilon \leq 1. \tag{6.15}$$

In view of the continuity of Ψ_ϵ and arbitrariness of a, (6.15) is equivalent to

$$\int_{-\infty}^{\infty} \frac{1}{\mu} \Psi_\epsilon(\mu)\, d_\mu R^*(x,\mu) = 0. \tag{6.16}$$

7. Some lemmas

We now establish some lemmas which are utilized in the proof of the equiconvergence theorem and the generalized Fourier integral theorem.

Lemma 7.1: Let $Q(x)$ satisfy the relation (4.1) and $A(x,\nu)$ be given by (6.8). Then

$$\int_0^\mu A(x,\nu)\, d\nu = o(1)$$

as μ tends to infinity, and for fixed ν.

$$\int_{\mu}^{\mu+\nu} A\,(x,u)\,\mathrm{d}u = o(1)$$

as μ tends to infinity; both the results hold uniformly in every finite interval containing x.

Proof: Integrating by parts,

$$I = \frac{1}{\mu}\int_0^{\mu}\mathrm{d}\nu\int_0^{\nu} A_j\,(x,u)\,\mathrm{d}u = \Big(\int_0^{1/2\mu} + \int_{1/2\mu}^{\mu}\Big)\Big(1-\frac{\nu}{\mu}\Big)\,A_j\,(x,\nu)\,\mathrm{d}\nu \tag{7.1}$$

so that without loss of generality we can assume $A_j\,(x,u)$ positive and hence the following inequality

$$I \geq \frac{1}{2}\int_0^{1/2\mu} A_j\,(x,\nu)\,\mathrm{d}\nu,\ j=1,2. \tag{7.2}$$

On changing the order of integration, we have, by using the definition of $A\,(x,\nu)$,

$$I = \frac{1}{\mu}\int_0^1 y_j\,(x,t)\left[-\frac{\mu\sin\mu t}{t^2} + \frac{2\,(1-\cos\mu t)}{t^3}\right]\mathrm{d}t = -I_1 + I_2, \quad \text{say.} \tag{7.3}$$

Now,

$$I_1 = \int_0^1 y_j\,(x,t)\,\frac{\sin\mu t}{t^2}\,\mathrm{d}t = \left(\int_0^{\delta} + \int_{\delta}^{1}\right)\frac{y_j(x,t)}{t}\cdot\frac{\sin\mu t}{t}\,\mathrm{d}t$$

$$= I_{11} + I_{12}, \quad \text{say.} \tag{7.4}$$

By the relation (4.3), it follows by the Minkowsky inequality

$$\left|\frac{y_j(x,t)}{t}\right| \leq C t^{a+1/2} \tag{7.5}$$

Thus

$$|I_{11}| \leq C\int_0^{\delta} t^{a-1/2}\,\mathrm{d}t = K\,\delta^{a+1/2} < \eta, \text{ say}$$

where we choose η as small as we please with δ, independently of x. Having so chosen δ, I_{12} tends to zero as μ tends to infinity uniformly in x over any finite interval (x_0, y_0), say, by the Riemann-Lebesgue lemma.

Thus altogether $I_1 = o(1)$, as μ tends to infinity uniformly in x over any finite interval.

For the estimate of I_2 we argue in the same manner as we did in lemma 4.1 by following Titchmarsh[6] (p. 414). Thus $I_2 = o(1)$, uniformly as μ tends to infinity.

The first part of the lemma then follows from (7.2) and (7.3). The second part is evident from the first part and the lemma therefore completely follows.

Lemma 7.2: Let $f\,(s)$ be any vector which belongs to $L_2\,(-\infty,\infty)$ and $S\,(x,\lambda)$,

$Z(x,t)$ and $B(x,\nu)$ be defined respectively by (6A), (6.12) and (6.13).

Then for every fixed x,

$$\lim_{\mu\to\infty} 2/\pi \int_0^{\mu} B(x,\nu)\, d\nu = -S(x,-\infty),$$

and if ν is fixed.

$$\int_{\mu}^{\mu+\nu} B(x,u)\, du = o(1)$$

as μ tends to infinity: both the results hold uniformly in every finite interval containing x.

Proof: Integrating by parts, utilization of the expression for $B_l(x,\nu)$ and a subsequent integration by parts we obtain

$$I = \int_0^{\mu} \left(1 - \frac{\nu}{\mu}\right) B_l(x,\nu)\, d\nu = \frac{1}{\mu}\int_0^{\mu} d\nu \int_0^{\nu} B_l(x,u)\, du$$

$$= \frac{1}{\mu}\int_0^{1} Z_l(x,t)\left[-\frac{\mu \sin \mu t}{t^2} + \frac{2(1-\cos \mu t)}{t^3}\right] dt$$

$$= -I_1 + I_2, \quad \text{say} \tag{7.6}$$

where $l = 1,2$.

The function $Z_l(x,t)/t$ exists at $t=0$ and $Z_l(x,t)\,t$ is an analytic function of t and therefore integrable over $(0,\delta)$, where $\delta > 0$ is arbitrary and we write

$$I_1 = \left(\int_0^{\delta} + \int_{\delta}^{1}\right) Z_l(x,t)\, \frac{\sin \mu t}{t^2}\, dt = I_{11} + I_{12}, \text{ say.}$$

Then by the Riemann-Lebesgue lemma $I_{12} = o(1)$, as μ tends to infinity uniformly in x lying in any fixed finite interval. Also, a Fourier series type of analysis as contained in Titchmarsh[6] (pp. 404-406) leads to

$$\frac{I_{11}}{\pi} = -\frac{1}{2} S_l(x,-\infty) + o(1)$$

as μ tends to infinity and x fixed.

We proceed with the estimate for I_2 in the same way as in lemma 4.2 so as to finally obtain

$$I = -\frac{\pi}{2} S_l(x,-\infty) + o(1),$$

as μ tends to infinity for fixed x.

The first part of the lemma follows by exploiting, as before, a criterion giving a necessary and sufficient condition for the $(C,1)$ summability of $\int_a^{\infty} f(t)\, dt$ (Hobson[7], p. 386). The uniformity of the limit for every x in any given finite interval containing x follows from the continuity of $Z(x,t)/t$ as well as that of its partial derivative with respect to t.

The second part is an immediate consequence of the first.

Lemma 7.3: Let $f(x) \in L_2(-\infty, \infty)$. Then

$$S_1(x, \mu+\nu) - S_1(x, \mu) = o(1),$$

as μ tends to infinity, uniformly in any fixed interval containing x, where ν is fixed and $S_1(x, \mu)$ is as defined at the beginning of section 6.

Proof: It follows from the generalized Parseval relation (see authors' paper[1], p.151)

$$\int_{\mu}^{\mu+\nu} \left[\sum_{i,j=1}^{2} E_{1i} E_{1j} \, d\xi_{ij} + 2 \sum_{i,j=1}^{2} E_{1i} E_{2j} \, d\eta_{ij} + \sum_{i,j=1}^{2} E_{2i}^2 \, d\zeta_{11} \right] = o(1) \tag{7.7}$$

as μ tends to infinity and ν fixed. Here E_{ij}, ξ_{ij}, η_{ij} and ζ_{11} are those which occur in the theorem.

By substituting for the explicit form of the matrix $H_1(x, y, \mu)$ as given in (1.11), it follows from lemma 3.2 that

$$\int_{\mu}^{\mu+\nu} [\, \phi(x, \lambda) \, d\xi(\lambda) \, \phi^T(y, \lambda) + \phi(x, \lambda) \, d\eta(\lambda) \, \theta^T(y, \lambda)$$

$$+ \theta(x, \lambda) \, d\eta(\lambda) \, \phi^T(y, \lambda) + \theta(x, \lambda) \, d\zeta(\lambda) \, \theta^T(y, \lambda) \,] << C,$$

where ν is fixed; x, y lie in the fixed interval (x_0, x_1) and $C \equiv C(x_0, x_1)$ is a constant matrix depending on x_0, x_1 only.

Then putting $x = y$,

$$\int_{\mu}^{\mu+\nu} \left[\sum_{i,j=1}^{2} u_i u_j \, d\xi_{ij} + 2 \sum_{i,j=1}^{2} u_i x_j \, d\eta_{ij} + \sum_{i=1}^{2} x_i^2 \, d\zeta_{11} \right] = o(1) \tag{7.8}$$

with a similar result with u replaced by v and x replaced by y. (u_j, v_j are elements of the matrix ϕ and x_i, y_j those of θ and ξ_{ij}, η_{ij}, ζ_{11} are those of ξ, η, ζ that occur in the explicit form of the resolution matrix $H_1(x, y, \mu)$.

If $S_1(x, \mu) = \begin{pmatrix} S_{11}(x, \mu) \\ S_{12}(x, \mu) \end{pmatrix}$, we have

$$S_{11}(x, \mu+\nu) - S_{11}(x, \mu)$$

$$= \int_{\mu}^{\mu+\nu} [\, \sum_{i,j=1}^{2} u_i E_{1j} \, d\xi_{ij} + \sum_{i,j=1}^{2} u_i E_{2j} \, d\eta_{ij} + \sum_{i,j=1}^{2} x_i E_{1j} \, d\eta_{ij} + \sum_{i=1}^{2} x_i E_{2i} \, d\zeta_{11} \,] \tag{7.9}$$

with similar expression for $S_{12}(x, \mu+\nu) - S_{12}(x, \mu)$.

We now make use of the inequality (Hardy *et al*[12] section 29, p. 33).

$$\Sigma\, a_{\mu\nu}\, x_\mu y_\nu \leq (\Sigma\, a_{\mu\nu}\, x_\mu x_\nu)^{1/2}\ (\Sigma a_{\mu\nu} x_\mu\, x_\nu)^{1/2},$$

where $a_{\mu\nu} = a_{\nu\mu}$ and $\Sigma\, a_{\mu\nu} x_\mu\, x_\nu$ is a positive quadratic form (with real but not necessarily positive coefficients) and the Schwarz inequalitv in (7.9).

Then from (7.7) and (7.8) it follows that for a fixed ν,

$$S_{11}\,(x, \mu+\nu) - S_{11}\,(x, \mu) = o(1),$$

as μ tends to infinity uniformly, x lying in a fixed interval $(x_0,\ x_1)$ say.

Similarly for $S_{12}\,(x, \mu+\nu) - S_{12}\,(x, \mu)$.

The lemma therefore follows.

8. The equiconvergence theorem and the expansion theorem

It follows from lemma 7.3 and the same lemma applied to $S_1^F\,(x, \mu)$, that for fixed ν,

$$R\,(x, \mu+\nu) - R\,(x, \mu) = o(1),$$

as μ tends to infinity uniformly in any fixed interval.

Therefore, by lemma 7.1 and lemma 7.2, it follows from (6.16), that for fixed ν

$$R^*\,(x, \mu+\nu) - R^*\,(x, \mu) = o(1) \tag{8.1}$$

as μ tends to infinity uniformly in every finite interval of variation of x.

Now $\chi_\epsilon\,(\mu) = 1/\mu\ \Psi_\epsilon\,(\mu)$ is the Fourier cosine transform of $\{-h_\epsilon\,(t)\}$, introduced in the proof of lemma 3.3.

The condition of the Marchenko-Tauberian theorem (Theorem A, section 5) being satisfied, it follows from (6.15) that

$$\lim_{\mu\to\infty} R^*\,(x, \mu) = 0.$$

uniformly in each finite interval over which x varies. We thus obtain the equiconvergence theorem

Theorem I: If $Q(x)$ satisfies (4.1) and $f\,(x) = \begin{pmatrix} f_1\,(x) \\ f_2\,(x) \end{pmatrix} \epsilon\ L_2\,(-\infty, \infty)$,

then

$$\lim_{\mu\to\infty} \int_{-\infty}^{\infty} \{\, H_1\,(x,s,\mu) - H_1^F\,(x,s,\mu)\,\}\, f(s)\, \mathrm{d}s = \int_{-\infty}^{\infty} H_1\,(x,s,-\infty)\, f\,(s)\, \mathrm{d}s,$$

the result holding uniformly in every finite interval containing x.

Replacing $H_1^F(x,s,\mu)$ by its value given explicitly in (5.1), we obtain that if $f(x)$ satisfies certain local conditions as to the validity of the Fourier single integral formula (see Titchmarch[10], p. 434), then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} H_1^F(x,s,\mu) f(s)\, ds = \pi f(x)$$

uniformly for x in any finite interval.

We thus deduce the expansion formula (valid under Fourier conditions)

Theorem II: If $Q(x)$ and $f(x)$ satisfy the conditions of Theorem I, and if further $f(x)$ satisfies certain local conditions for the validity of the Fourier single integral formula, then

$$f(x) = \frac{1}{\pi} \lim_{n\to\infty} \int_{-\infty}^{\infty} \{ H_1(x,s,\mu) - H_1(x,s,-\mu) \} f(s)\, ds.$$

For the expansion formula under a different set of conditions, see authors' paper[1] (p. 158, formula 5.4).

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Appendix

We outline below an alternative method, suggested by a referee, to show that the operator T of section I is self-adjoint having recourse to deriving the spectral resolution of the operator T.

Let

$$P=-i\,(\mathrm{d}/\mathrm{d}x),\quad \hat{T}=\begin{pmatrix} P^2 & 0 \\ 0 & P^2 \end{pmatrix},\quad T=\hat{T}+W=\hat{T}+\begin{pmatrix} A(Q) & C(Q) \\ C(Q) & B(Q) \end{pmatrix}$$

where A, B, C are real-valued integrable functions and $(A(Q)f)(x)=A(x)$, etc., the Hilbert space being $H=L_2(R)\oplus L_2(R)$. Then the following lemmas hold.

Lemma 1: If $V\in L_1(R)$, $(P+i)^{-1}\,V(Q)(P+i)^{-1}$ is a trace class operator (Kato[13], p. 521) and hence is a compact (completely continuous) operator on $L_2(R)$.

This follows from the fact that $(P+i)^{-1}\,V_1(Q)$ and $V_2(Q)(P+i)^{-1}$ are both Hilbert-Schmidt operators, where $V=V_1V_2$, with $V_1, V_2\in L_2(R)$.

In particular, $(\hat{T}+1)^{-1/2}\,W(\hat{T}+1)^{-1/2}$ is a trace class operator.

Lemma 2: For each $\epsilon>0$, there exists a $b>0$, such that

$$|<Wf,f>|\leq\epsilon<\hat{T}f,f>+b<f,f> \text{ for all } f\in\operatorname{dom}\hat{T}.$$

The lemma is a consequence of the following

i) the operator families $(\hat{T}+1)^{1/2}\ (\hat{T}+n)^{-1/2};\ (\hat{T}+n)^{-1/2}\ (\hat{T}+1)^{1/2}$ are bounded for $n\geq 1$;

ii) $S-\lim_{n\to\infty}(\hat{T}+n)^{-1/2}(\hat{T}+1)^{1/2}=0$;

iii) $(\hat{T}+1)^{-1/2}\ W(\hat{T}+1)^{1/2}$ is compact (by lemma 1).

The self-adjointness of T follows by utilising lemma 2 with KLMN theorem[14] (Th. X. 17, Vol. 2).

The following points may be noted:

a) Negative eigenvalues of T, if any, can accumulate only at 0, (See[14], cor 4, p. 116, Th. XIII. 14, Vol. 4).

b) The absolutely continuous part of T is unitarily equivalent to $\hat{T}$.

This is a consequence of the Birman theorem[14] (Th. XI, 10, Vol. 3), since $(T+1)^{-1/2}\ W[T+1)^{-}1/2$ is trace class and $\operatorname{dom}(|T|+1)^{1/2}=\operatorname{dom}(\hat{T}+1)^{1/2}$.