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New characteristic polynomial—a reliable index to detect isomorphism between kinematic chains*

R. K. DUBE AND A. C. RAO

Mechanical Engineering Department, Maulana Azad College of Technology, Bhopal 462 007, Madhya Pradesh.

Government Engineering College, Bilaspur 495 009, Madhya Pradesh.

Abstract

Graph theory is used and an nth order symmetric matrix based on 'distance' is defined to represent a simple and multiple jointed n-link kinematic chain. Based on Faddeev–Leverrier method a computer program is used for generating the coefficient of the characteristic polynomial of the matrix associated with the kinematic chain. A comparison of characteristic polynomials of the matrices detects the isomorphic chains. A method is illustrated by application to single and two degree of freedom linkages.

Keywords: Kinematic chains, graphs, distance matrices, characteristic polynomials, identical, isomorphism.

1. Introduction

Recognition and identification of equivalence of kinematic chains is generally done either by graphical methods based on visual inspection of various forms of simplified schematic diagrams or by using mathematical techniques based on the theory of graphs[1–12]. However even to-date a conclusive and easily applied test, for possible isomorphism (equivalent topology) of two kinematic chains, is lacking.

Using graph theory, Uicker and Raicu[11] have represented a simple jointed kinematic chain by an nth order symmetric zero-one adjacency matrix. When two links are connected it is indicated by one in the matrix while zero indicates that the links are not connected with each other. Later, Mruthyunjaya and Raghavan[12] have presented a generalized matrix notation to represent simple as well as multiple-jointed kinematic chains. According to this notation the value of the matrix element is taken equal to the number of joints if two links are connected with each other and taken as zero if otherwise. The characteristic polynomial of the above matrices is expected to be of use in the detection of isomorphism. But in the literature[13] examples are known where non-isomorphic graphs have identical characteristic polynomials. Thus the comparison of characteristic polynomials of adjacency matrices do not lead to a fool-proof method in detecting possible isomorphism. Here it is proposed that the characteristic polynomial of distance matrix can serve as a reliable invariant for both simple- as well as multiple-jointed kinematic chains.

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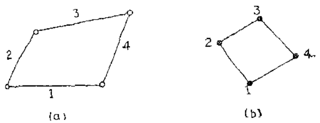

FIG. 1.

2. Distance matrix representation

In this matrix representation the elements of the matrix are taken equal to the distance *i.e.* minimum number of edges between the vertices of the graph[14]. Figures 1a and b represent respectively a four-bar chain and its graph in which the vertices correspond to links and the edges indicate the connection between vertices. The distance matrix A for the graph (fig. 1(b)), will be as follows.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

It may be noted that similar to adjacency matrix the distance matrix also comes out to be a square symmetric one.

3. Characteristic polynomial computation

Uicker and Raicu[11] have presented a recursive method for the calculation of the coefficients of characteristic polynomial. Later, Mruthyunjaya and Raghavan[12] have presented an alternative method based on Bocher's formulae having the advantage of simpler calculation. In this method, powers of matrix are used and hence is limited to a low order matrix. The reason for this is that the raising matrices to various powers, in addition to requiring a great number of calculations, in many cases result in loss of accuracy as the elements of the new matrices require more significant figures[15]. Here Faddeev–Leverrier method[16] is presented which is simple and an efficient technique for generating the polynomial coefficients when the order of the matrix is large.

Let the characteristic polynomial of the matrix A be represented by $(-1)^n$ $(\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} \ldots p_n) = 0$ where the factor $(-1)^n$ is used merely to give the terms of the polynomial the same signs that they would have if the polynomial were generated by expanding a determinant. The polynomial coefficients $p_k (k = 1, 2, 3, \ldots n)$ values are determined, by forming a sequence of matrices $B_1, B_2, B_3 \ldots . B_n$

as follows:

$$B_1 = A \quad \text{and} \quad p_1 = \text{trace } B_1$$

$$B_2 = A(B_1 - p_1 I) \quad \text{and} \quad p_2 = \frac{1}{2} \text{trace } B_2$$

$$B_3 = A(B_2 - P_2 I) \quad \text{and} \quad p_3 = \frac{1}{3} \text{trace } B_3$$

$$B_k = A(B_{k-1} - P_{k-1} I) \quad \text{and} \quad p_k = \frac{1}{k} \text{trace } B_k$$

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$$B_n = A(B_{n-1} - P_{n-1} I) \quad \text{and} \quad p_n = \frac{1}{n} \text{trace } B_n$$

where I is the identity matrix of the order of A. For example consider matrix A of the four-bar chain shown in fig. 1(a).

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

$$B_1 = A \text{ and } P_1 = \text{trace } B_1 = 0.$$

$$B_2 = A(B_1 - p_1 I)$$

or

$$B_2 = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 4 & 6 & 4 & 2 \\ 2 & 4 & 6 & 4 \\ 4 & 2 & 4 & 6 \end{bmatrix} \text{ and } p_2 = \frac{1}{2} \text{ trace } B_2 = 12$$

$$B_3 = A(B_2 - P_2 I)$$

or

$$B_3 = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 6 & 4 & 2 & 4 \\ 4 & 6 & 4 & 2 \\ 2 & 4 & 6 & 4 \\ 4 & 2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix} \right)$$

or

$$B_3 = \begin{bmatrix} 12 & 4 & -4 & 4 \\ 4 & 12 & 4 & -4 \\ -4 & 4 & 12 & 4 \\ 4 & -4 & 4 & 12 \end{bmatrix} \text{ and } p_3 = \frac{1}{3} \text{ trace } B_3 = 16$$

$$B_4 = A(B_3 - p_3 I)$$

or

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } p_4 = \frac{1}{4} \text{ trace } B_4 = 0.$$

On substituting values of p_1, p_2, p_3 and p_4 in the expression $(-1)^n (\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \ldots p_n)$, the characteristic polynomial for the four-bar chain (fig. 1(a)) is given by $\lambda^4 - 12\lambda^2 - 16\lambda$.

Faddeev and Faddeeva[17] have proved that for nth order matrix $B_n = P_n I$. This can also be observed in the above example. Therefore, one can always obtain p_n simply as $p_n = b_{11} = b_{22} = \ldots = b_{nn}$ where the b_{ii} are identical elements composing the trace of B_n.

The Fortran source program is written for the generation of the coefficients of the characteristic polynomial by using the Faddeev–Leverrier method. It consists of 30 statements and takes less than 3 seconds on HP 1000 for computing the characteristic coefficients of a nine-link chain.

For the class of matrices representing large linkages, the present method is reliable and the calculations involved are simple and well suited for computation.

4. Application

To show that the characteristic polynomial can be used as an index of isomorphism and that the method of computation of the characteristic coefficients presented in this paper is valid for the multiple-jointed chains also, two six-link multiple-jointed chains (fig. 2) are taken. Their respective graphs are drawn in fig. 3. It is found that both the six-link

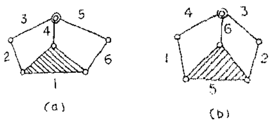

FIG. 2.

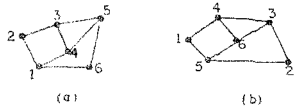

FIG. 3.

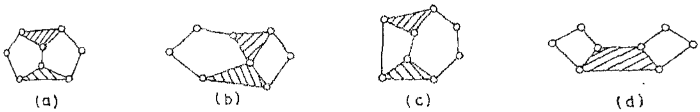

FIG. 4.

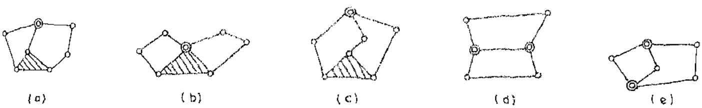

FIG. 5.

multiple-jointed chains have the same characteristic polynomial, *viz.*, $\lambda^6 - 36\lambda^4 - 118\lambda^3 - 114\lambda^2 - 10\lambda + 7$. Thus these chains are isomorphic.

The characteristic polynomials for all the possible seven-link chains (fig. 4—simple jointed chains and fig. 5—multiple-jointed chains) are calculated. The characteristic polynomials are worked out as different for all the possible chains shown in figs 4 and 5. Hence the chains are non-isomorphic.

For all the existing eight- and nine-link simple jointed kinematic chains the characteristic polynomials are computed to evaluate nonisomorphic chains and the results are found to be in agreement with the results of Freudenstein and Dobrjanskyj[18] and Crossley[19]

To show that the present test is more reliable in detecting isomorphism than the test based on characteristic polynomials of the adjacency matrix, consider three seven-link chains given by Balaban and Harary[13]. These chains and their respective graphs are represented in figs. 6–8. On computation the characteristic polynomials of adjacency matrices of graphs represented by figs. 6(b), 7(b) and 8(b) are obtained as the same, *i.e.* $\lambda^7 - 11\lambda^5 - 10\lambda^4 + 16\lambda^3 + 16\lambda^2$. Whereas on applying the present test the characteristic polynomials based on distance matrices of these three nonisomorphic graphs are obtained as $\lambda^7 - 61\lambda^5 - 268\lambda^4 - 442\lambda^3 - 302\lambda^2 - 68\lambda$ in the case of fig. 6(b),

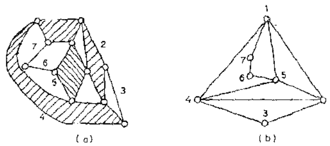

FIG. 6.

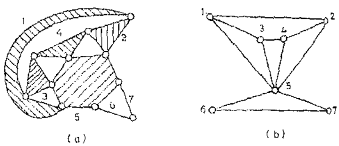

FIG. 7.

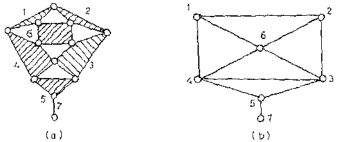

FIG. 8.

$\lambda^7-51\lambda^5-226\lambda^4-392\lambda^3-296\lambda^2-80\lambda$ in the case of fig. 7(b) and $\lambda^7-66\lambda^5-292\lambda^4-478\lambda^3-318\lambda^2-68\lambda$ in the case of fig. 8(b). Thus the new characteristic polynomials of the nonisomorphic graphs are different.

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