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Level-fair derivations in context-free rewriting systems

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Abstract

The family of languages $\mathscr{L}(k\text{-EOS})$ generated by EOS systems under k-level fairness of derivations is defined and shown to be the same for all integers $k \geq 1$. $\mathscr{L}(k\text{-EOS})$ is proved to be equal to the family of languages generated by EOL systems.

Key words: Fairness, EOS and EOL systems.

1. *Introduction*

It is well known that regulating the rewriting in a rewriting system can significantly affect the language generated. The literature gives many different mechanisms for regulating the rewriting, *e.g.*, matrix, time-varying and programmed grammars, control languages and tabled systems[1].

The concept of fairness as a mechanism for regulating the rewritings has been a study of recent interest and investigation in formal language theory[2, 3, 4]. Porat *et al*[2] have provided a characterization of fairly terminating context-free grammars by the property of non-variable doubling. The same characterization has been shown to remain valid under certain canonical derivations by Porat and Francez[3]. Rangarajan and Arunkumar[4] have dealt with fairness of derivations in synchronized EOL systems and proved that such systems are fairly terminating.

In this paper, we choose for our study the EOS system[5] which is simply a context-free grammar in which the terminal symbols can also be rewritten. It is well known that this feature does not alter the generative capacity of context-free grammars[5].

We define the notion of level-fairness of derivations, by associating with each symbol of a sentential form derived in the EOS system, a non-negative integer representing its depth in the generation tree and by restricting the derivations of the EOS system such that the level difference in the generation tree is no more than some finite integer k. We call such derivations as k-level fair derivations.

We note that the rule-fairness studied[2,3] relates to the question 'Are all applicable rules applied equally often?', while the level fairness considered here relates to the question 'Are the rules applied to all parts of the string equally often?'.

We prove that the generative power of EOS systems under k-level fairness of derivations is the same for all integral values of $k \geq 1$. Thus we define a class of languages generated by the EOS systems under k-fair derivations called the k-EOS languages. We also show that this class indeed equals the family of EOL languages.

2. k-Level fairness

In this section, we introduce the notion of k-level fair derivations of the EOS system and define the class of k-EOS languages. The reader is referred to Porat and Francez[3] and Kleijn and Rozenberg[5] for unexplained terms.

Definition 2.1: Let $G = (\Sigma, P, S, \Delta)$ be a rewriting system, such that Σ is a finite non-empty alphabet which is the total alphabet of $G, \Delta \subseteq \Sigma$ is the terminal alphabet of G, $S \in \Sigma \backslash \Delta$ is the axiom of G and $P \subseteq \Sigma \times \Sigma^*$ is a finite set of productions.

(a) G is a context-free grammar if
(i) $P \subseteq (\Sigma \backslash \Delta) \times \Sigma^*$
(ii) for $u, v \in \Sigma^*$, $u \Longrightarrow_G v$ if $u = u_1 b u_2$, $v = u_1 \beta u_2$,

for some $u_1, u_2 \in \Sigma^*$ and $(b, \beta) \in P$.

(b) G is an EOS system if
(i) $P \subseteq \Sigma \times \Sigma^*$
(ii) $\Longrightarrow_G$ is defined as under ((a) (ii))

(c) G is an EOL system if
(i) P is defined as under ((b) (i))
(ii) for u, $v \in \Sigma^*$, $u \Longrightarrow_G v$ if $u = b_1 \ldots b_n$

$$v = \beta_1 \ldots \beta_n \text{ where } b_i \in \Sigma \text{ and } (b_i, \beta_i) \in P \text{ for all } i \in \{1, \ldots n\}.$$

For a grammar G,$\Longrightarrow_G^*$ is the reflexive and transitive closure of $\Longrightarrow$. The language of G is defined by

$$L(G) = \{w \in \Delta^* : S \Longrightarrow_G^* w\}.$$

The families of languages generated by context-free grammars, EOS systems and EOL systems are denoted by $\mathcal{L}(\text{CF})$, $\mathcal{L}(\text{EOS})$ and $\mathcal{L}(\text{EOL})$ respectively.

Remark: It is easy to see that $\mathcal{L}(\text{CF}) = \mathcal{L}(\text{EOS})$[5].

We now introduce the notion of d-words. A d-word is a sentential form in which the depth of each symbol in the generation tree of the word is also represented.

Definition 2.2: (i) A d-word over an alphabet Σ is a sequence $\langle a_1, d_1\rangle \langle a_2, d_2\rangle \ldots \langle a_n, d_n\rangle$ where for $1 \leq i \leq n$, $a_i \in \Sigma$ and $d_i \in \mathbb{P}$, the set of non-negative integers.

(ii) Given $\alpha = a_1 a_2 \ldots a_n \in \Sigma^*$, we denote by $\langle \alpha | i\rangle$ the word $\langle a_1, i\rangle \langle a_2, i\rangle \ldots \langle a_n, i\rangle$ for some $i \in \mathbb{P}$.

(iii) Given a d-word $x = \langle a_1, d_1\rangle \ldots \langle a_n, d_n\rangle$ we define

$$\max d(x) = \max d_i (1 \le i \le n)$$

$$\min d(x) = \min d_i (1 \le i \le n)$$

and $\quad w(x) = a_1 a_2 \ldots a_n$

(iv) Given an integer $k \ge 0$, a d-word x is k-level fair iff $\max d(x) - \min d(x) \le k$.

Definition 2.3: (i) Given an EOS system $G = (\Sigma, P, S, \Delta)$ a d-word $x\langle a,i\rangle y \Longrightarrow x\langle \alpha | i+1\rangle y$ for some d-words x, y iff $(a, \alpha) \in P$. Such a derivation is called a d-derivation.

(ii) A d-derivation sequence $x_0 \Longrightarrow x_1 \Longrightarrow \ldots \Longrightarrow x_n$ is k-level-fair iff each of the d-words $x_i (0 \le i \le n)$ is k-level-fair. We denote it by $x_0 \underset{k}{\Longrightarrow}^* x_n$.

(iii) The language generated by G under k-level fairness is defined by $L_k(G) = \{w(x) | \langle S, 0\rangle \Longrightarrow_k^* x \text{ and } w(x) \in \Delta^*\}$.

Example 2.1: Let $G = (\Sigma, P, S, \Delta)$ where $\Sigma = \{S, a, b, c\}$; $\Delta = \{a, b, c\}$; $P = \{S \to ab, b \to bc\}$ be an EOS system, then

$$L_k(G) = \{abc^i | 0 \le i \le k\}.$$

This example illustrates that the language generated by a grammar G, under k-level-fair derivations can depend on the value of k.

Example 2.2: Let $G = (\{S, a, B\}, \{S \to a, a \to BB, B \to a\}, S, \{a\})$ be an EOS system. We see that $L_1(G) = \{a^{2^n} | n \ge 0\}$.

Remark: For any EOS system G, $L_0(G)$ is finite. Hence, in the rest of the paper, we will consider k-level fair derivations for only positive integer values of k.

Theorem 2.1: For any $k \ge 1$, given an EOS system $G = (\Sigma, P, S, \Delta)$ there exists an EOS system $G' = (\Sigma, P', S, \Delta)$ such that $L_k(G') = L(G)$.

Proof: Let $P' = P \cup \{a \to a | a \in \Sigma\}$. Clearly $L(G) = L_k(G')$ for any positive integer k.

This theorem states that for any positive integer k, the EOS systems under k-fair derivations have at least the generative power of the CF grammars.

Theorem 2.2: For any $k \ge 1$, given an EOS system $G = (\Sigma, P, S, \Delta)$ there exists an EOS system $G' = (\Sigma', P', S', \Delta)$ such that $L_{k+1}(G) = L_k(G')$.

Proof: Let $\Sigma' = \{\bar{a}, \bar{\bar{a}} | a \in \Sigma\} \cup \Delta$, $S' = \bar{S}$. P' contains the following rules:
(i) $\bar{a} \rightarrow \bar{\alpha}$ if $a \rightarrow \alpha \in P$
(ii) $\bar{a} \rightarrow \bar{\bar{a}}$ for all $a \in \Sigma$
(iii) $\bar{a} \rightarrow a$
$\bar{\bar{a}} \rightarrow a$ for all $a \in \Delta$.

It is easy to note that $L_{k+1}(G) = L_k(G')$.

Corollary 2.1: For any EOS system $G = (\Sigma, P, S, \Delta)$ and any positive integer k, there exists an EOS system $G' = (\Sigma', P', S', \Delta)$ such that $L_k(G) = L_1(G')$.

Remark: This corollary shows that 1-level fair EOS systems have at least the generative power of k-level fair EOS systems for any positive integer value of k. In the next theorem we prove the converse of this result.

Theorem 2.3: For any EOS system $G = (\Sigma, P, S, \Delta)$ and for any positive integer k, there exists an EOS system $G' = (\Sigma', P', S, \Delta)$ such that $L_1(G) = L_k(G')$.

Proof: Let G' be such that $\Sigma' = \Sigma \cup \{a_1, a_2, \ldots, a_{k-1} | a \in \Sigma\}$. P' contains the following rules.

$a \rightarrow a_1$

$a_{i-1} \rightarrow a_i$ for $2 \leq i \leq k-1$

$a_{k-1} \rightarrow \alpha$ if $a \rightarrow \alpha \in P$.

It can be seen that $L_1(G) = L_k(G')$.

Corollary 2.2: Given an EOS system G and positive integers k and k' there exists an EOS system G' such that $L_k(G) = L_{k'}(G')$.

Proof: Follows directly from Corollary 2.1 and Theorem 2.3.

The above corollary states that the generative power of EOS system under k-level fair derivations is independent of the value of k. Thus we can postulate a class $\mathscr{L}(k\text{-EOS})$ of languages generated by EOS systems under k-level fair derivations.

3. k-EOS languages and EOL systems

The EOL systems use parallel rewriting in which all the symbols of a word are rewritten simultaneously. By contrast, the EOS systems use sequential rewriting, *i.e.*, symbols of a word are rewritten one after another. In this section we show that the class of k-EOS languages equals the class of EOL languages. Thus the k-level fairness constraint can simulate the effect of parallel rewriting.

Theorem 3.1: Let $G = (\Sigma, P, S, \Delta)$ be an EOL system. There exists an EOS system $G' = (\Sigma', P', S, \Delta)$ such that $L(G) = L_1(G')$.

Proof: Let G' be such that $\Sigma' = \Sigma \cup \{\bar{a} | a \in \Sigma\}$. P' contains the following rules.

$a \rightarrow \bar{a}$ if $a \in \Sigma$

$\bar{a} \rightarrow \alpha$ if $a \rightarrow \alpha \in P$.

It is easy to see that $L(G) = L_1(G')$.

Theorem 3.2: Let $G = (\Sigma, P, S, \Delta)$ be an EOS system. There exists an EOL system $G' = (\Sigma', P', S, \Delta)$ such that $L(G') = L_1(G)$.

Proof: Let G' be such that, $\Sigma' = \Sigma \cup \{\bar{a} | a \in \Sigma\}$. P' contains the following rules.

$a \rightarrow \bar{a}$ if $a \in \Sigma$,

$a \rightarrow \bar{\alpha}$ if $a \rightarrow \alpha \in \ P$

$\bar{a} \rightarrow a$ if $a \in \Sigma$.

It is quite easy to show that $L(G') = L_1(G)$.

As a consequence of Theorems 3.1 and 3.2, Corollary 3.1 for any $k \geq 1$, we have

$$\mathcal{L}(\text{EOL}) = \mathcal{L}(k\text{-EOS}).$$

4. Discussion

We have related the parallel rewritings of an EOL system to the sequential rewritings of the EOS system by regulating the rewritings of an EOS system. The regulating mechanism takes the form of a fairness constraint over the levels of the symbols in the generation tree. This gives the necessary increase in the generative power of the EOS systems.

Porat and Francez[3] have applied similar notion of fairness to the non-terminals of a context-free grammar derivation. Since the fairness constraint is applied only to the non-terminal symbols, the generative power of the grammar does not increase.

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