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Short Communication

A tractable pair of the Cauchy singular integral equations

A. CHAKRABARTI
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012, India.

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Abstract

A simplified analysis is presented for the solution of a special pair of singular integral equations with Cauchy-type kernels.

Key words: Cauchy-type kernel, Carleman equation, barrier equation, inverse operator.

1. Introduction

A special pair of singular integral equations as given by

$$af + T(bg) = f_1 \tag{1}$$

and

$$g + T(d(cf+g)) = f_2, \tag{2}$$

where $T\phi = \frac{1}{\pi}\int_{-1}^{1} \phi(t)\mathrm{d}t/(t-x)$,

and a, b, c, d, f_1 and f_2 are differentiable functions, in $-1 < x < 1$ is considered for their solution in closed form when $c = k_1 a$ and $d = k_2 b$, k_1, k_2 being constants.

While the pair of equations (1) and (2) can be cast into the form

$$\sum_{j=1}^{2} [a_{ij}\phi_j + b_{ij}T\phi_j] = F_i (i = 1, 2)$$

with suitable choice of the functions a_{ij}, b_{ij}, ϕ_j and F_i, for which a general theory has been worked out by Peters[1] involving the reduction of the problem to a non-linear barrier equation, it is hard to obtain tractable solution of these equations even in the special case when $c = k_1 a$ and $d = k_2 b$ with k_1, k_2 as constants. With this as the background we are motivated to study the above pair of equations for the special case mentioned.

By using an analysis similar to that described in the earlier work[2-4] involving the utility of the inverse operator T^{-1} we have reduced the above pair of equations, in the case when $c = k_1 a$ and $d = k_2 b$, to a pair of independent Carleman equations from which the solution can be derived relatively easily by using standard results. The present analysis thus avoids the occurrence of a non-linear barrier equation as described by Peters[1].

The following basic questions about the structure of the problem at hand then arise:

1. Can the manipulations be made still simpler?
2. Does there exist a general class of such pairs of singular integral equations as given by (1) and (2), for which reduction to independent Carleman equations is possible?

A study of these two questions is taken up at the end.

2. The analysis

By using equation (1), equation (2) can be expressed in the case when $c = k_1 a$ and $d = k_2 b$, as

$$g + k_2 T(bg) - k_2 k_1 T(bT(bg)) = f_2 - k_1 k_2 T(bf_1) \tag{3}$$

which can be rewritten, by using the inverse operator T^{-1} (cf. ref. 3), as

$$\begin{aligned} & g(1-x^2)^{-1/2} - k_2 T^{-1}(bg(1-x^2)^{-1/2}) \\ & + k_1 k_2 T^{-1}(bT(bg)\,(1-x^2)^{-1/2}) \\ & = -T^{-1}\psi_2 + k_1 k_2 T^{-1}(bf_1(1-x^2)^{-1/2}) + \frac{\lambda}{(1-x^2)^{1/2}}, \end{aligned} \tag{4}$$

where λ is an arbitrary constant, with

$$f_2 = T((1-x^2)^{1/2}\psi_2), \tag{5}$$

and

$$T^{-1} f = \frac{\lambda_0}{(1-x^2)^{1/2}} - \frac{1}{(1-x^2)^{1/2}} T(f(1-x^2)^{1/2}), \tag{6}$$

λ_0 being another arbitrary constant.

Operating both sides of equation (4) by the operator T we obtain the following integral equation for the function g:

$$\begin{aligned} & T(g(1-x^2)^{-1/2}) - k_2 gb(1-x^2)^{-1/2} + k_1 k_2 b(1-x^2)^{-1/2} T(bg) \\ & = -\psi_2 + k_1 k_2 bf_1(1-x^2)^{-1/2}. \end{aligned} \tag{7}$$

Setting

$$\phi = g(1-x^2)^{-1/2}, \tag{8}$$

equation (7) can be cast into the form

$$\begin{aligned} & (1-x^2)^{1/2} b^{-1} T\phi + k_1 k_2 T(b\phi(1-x^2)^{1/2}) - k_2\phi(1-x^2)^{1/2} \\ & = k_1 k_2 f_1 - (1-x^2)^{1/2} b^{-1}\psi_2, \end{aligned} \tag{9}$$

which has the structure

$$AT\phi+T(B\phi)+C\phi = \theta(x), \tag{10}$$

where

$$A = b^{-1}(1-x^2)^{1/2},\ B = k_1k_2b(1-x^2)^{1/2},$$
$$C = -k_2(1-x^2)^{1/2},\ \theta = k_1k_2f_1-(1-x^2)^{1/2}b^{-1}\psi_2. \tag{11}$$

We observe that the T-part of the operator in the left of equation (10), *i.e.*, the part $AT\phi+T(B\phi)$ with $AB = k_1k_2(1-x^2)$ is the same operator studied[3] for its inverse through an operator L, where

$$L = A(1-x^2)^{-1/2}T\phi = b^{-1}T\phi. \tag{12}$$

Using this operator L we find that the T-part of the operator in equation (10) can be expressed as

$$AT\phi+T(B\phi) = -T\left[\frac{1-x^2}{A}\,(L^2-AB(1-x^2)^{-1}\phi\right], \tag{13}$$

so that, by using equations (11) and (13), we can rewrite equation (10) as

$$(1-x^2)^{1/2}T^{-1}\left[\frac{(1-x^2)^{1/2}}{A}(L^2-k_1k_2)\phi\right]-(1-x^2)^{1/2}k_2\phi = \theta(x)+\mu_0 \tag{14}$$

with μ_0 as an arbitrary constant, if equation (6) is also utilised. Dividing both sides of equation (14) by $(1-x^2)^{1/2}$ and operating by T once again, we ultimately arrive at the equation for ϕ as given by

$$(L^2-k_1k_2)\phi-A(1-x^2)^{-1/2}T(k_2\phi) = A(1-x^2)^{-1/2}T(\theta(1-x^2)^{-1/2}),$$

which, by using equation (12), turns out to be the same as

$$(L^2-k_2L-k_1k_2)\phi = \mu(x), \tag{15}$$

where, using equation(11), we have that

$$\mu(x) = b^{-1}T(k_1k_2f_1(1-x^2)^{-1/2}-b^{-1}\psi_2). \tag{16}$$

Equation (15) can finally be reduced to two independent Carleman type singular integral equations, if the operator $L^2-k_2L-k_1k_2$ is factorized as $(L-\alpha_1)\,(L-\alpha_2)$, where

$$\alpha_{1,2} = \frac{k_2\pm(k_2^2+4k_1k_2)^{1/2}}{2} \tag{17}$$

and if we write

$$(L-\alpha_2)\phi = \phi_1,\ (L-\alpha_1)\phi = \phi_2, \tag{18}$$

so that

$$(L-\alpha_1)\phi_1 = \mu = (L-\alpha_2)\phi_2, \tag{19}$$

the function ϕ being given by

$$\phi = \frac{1}{\alpha_2-\alpha_1}\,(\phi_2-\phi_1). \tag{20}$$

The two equations (19) simplify, on using the operator T^{-1}, to the equations

$$\bar{\phi}_1 + \alpha_1 T(b\bar{\phi}_1) = \bar{\theta}(x), \tag{21}$$

and

$$\bar{\phi}_2 + \alpha_2 T(b\bar{\phi}_2) = \bar{\theta}(x), \tag{22}$$

with

$$\bar{\phi}_j = \phi_j(1-x^2)^{-1/2}, \text{ and}$$
$$\bar{\theta}(x) = -T(\mu b(1-x^2)^{-1/2}). \tag{23}$$

The solution of the two Carleman equations, (21) and (22), can be expressed by using standard results[5], and we find that

$$\bar{\phi}_j = -\frac{\bar{\theta}}{1+\alpha_j^2 b^2} + \frac{e^{\omega_j}}{(1+\alpha_j^2 b^2)^{1/2}} T[\alpha_j b(1+\alpha_j^2 b^2)^{-1/2} e^{-\omega_j}\bar{\theta}] + \frac{\nu_j e^{\omega_j}}{(1+\alpha_j^2 b^2)^{1/2}}, (j = 1,2) \tag{24}$$

where ν_js are arbitrary constants, and

$$\omega_j \equiv \omega_j(x) = Tx_j, \; x_j = \frac{1}{2i}\ln\frac{1-i\alpha_j b}{1+i\alpha_j b}. \tag{25}$$

The function ϕ can be determined by using equation (20) and the main unknown functions f and g are easily obtainable if the equations (8) and (1) are made use of.

3. Some basic questions and their answers

Looking at the simple analysis presented, we now answer the questions posed:

Q.1. Does there exist a more simpler approach than what is presented above, to arrive at the decoupling of the two equations (1) and (2), in the special case when $c = k_1 a$ and $d = k_2 b$?

Answer: The main result used in the above analysis is the relationship

$$T[(1-x^2)^{-1/2} T(f(1-x^2)^{1/2})] = -f, \tag{26}$$

obtainable from equation (6).

Then multiplying equation (3) by $(1-x^2)^{-1/2}$ and applying the operator T gives

$$T\phi - k_2 b\phi + \frac{k_2 k_1}{(1-x^2)^{1/2}} bT[b(1-x^2)^{1/2}\phi] = -\psi_2 + \frac{k_2 k_1 b f_1}{(1-x^2)^{1/2}}. \tag{27}$$

Multiplying equation (27) by (b^{-1}) and applying T once again gives another equation, after utilizing (26), which can be cast into the form of equation (15) by a further multiplication by b^{-1}.

The route of arriving at equation (15) thus becomes a very simple one and can be realised only when an analysis similar to the one already presented has been shown to exist first.

The importance of the main analysis lies not just in the fact that equation (15) is obtainable but that the pair of equations (1) and (2) can be cast into the form of equation (10) which has two basic parts: (i) A T-part and (ii) a T-free part, of which the T-part can be handled for its inversion by using our earlier analysis[3] or a variant of that.

Q.2. Does there exist a general class of the pair of equations (1) and (2) which can be reduced to two independent Carleman type equations whose inversions are simple and straightforward?

Answer: Assume that we can find a suitable constant m and a suitable function $p(x)$ such that

$$maf+g = p[mbg+d(cf+g)]. \tag{28}$$

Then, equating the coefficients of f and g from both the sides of (28) yield

$$ma = pdc \tag{29}$$

and

$$1 = p(mb+d). \tag{30}$$

Eliminating p from (29) and (30) gives rise to the equation

$$m^2+\frac{d}{b}m-\frac{dc}{ab} = 0 \tag{31}$$

and this represents a quadratic equation for the determination of the constant m, for a variety of choices of the functions $a,b,c,$and d in equations (1) and (2).

The choice of the function p is then provided by equation (29).

For the special case under consideration we have that

$$c = k_1a \text{ and } d = k_2b,$$

and equation (31) simplifies to

$$m^2+k_2m-k_1k_2 = 0. \tag{32}$$

The two roots m_1 and m_2 of equation (32) and the two corresponding functions p_1 and p_2 given by (29) finally help in obtaining from equations (1) and (2) the following two independent Carleman type equations

$$p_j\chi_j+T\chi_j = m_jf_1+f_2,\ (j = 1,2) \tag{33}$$

with

$$p_j = m_ja/dc, \text{ and } \chi_j = m_jbg+d(cf+g),\ (j = 1,2). \tag{34}$$

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