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Singularities in a two-fluid medium separated by an inertial surface

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Abstract

Velocity potentials due to the presence of a line source of variable strength in one of the fluids of a two-fluid medium that are separated by an inertial surface composed of uniformly distributed disconnected materials, are obtained. The study of internal waves at the surface separating the two fluids requires the consideration of such a source in either of the two fluids. If the density of the upper fluid is made zero, known results for a fluid with an inertial surface are recovered.

Key words: Two-fluid medium, inertial surface, potential, line source, source strength, time-harmonic wave source.

1. Introduction

When bodies are present in an one (or two)-fluid medium, surface waves (or internal waves at the surface separating two fluids) may be either generated by the movement of the bodies or reflected from them. The two cases are essentially equivalent and the motion can be described by a suitable distribution of singularities of different types on and/or within the bodies. Thus the study of the generation of waves due to singularities of different types present in fluids is of basic importance. For the case of one-fluid medium with a free surface, Thorne[1] surveyed in some detail the different types of singularities and Rhodes-Robinson[2] modified the corresponding results by taking into account the effect of surface tension at the free surface which was neglected earlier by Thorne[1].

For a two-fluid medium with upper fluid being extended infinitely upwards and the lower fluid being of finite uniform depth, Gorgui and Kassem[3] studied different types of singularities present in either of the two fluids neglecting the effect of surface tension at the surface of separation. In all these cases the motion is assumed to be small and time-harmonic with any given angular frequency.

The study of generation of waves in an ideal liquid (*e.g.* water) with an inertial surface composed of a thin uniform distribution of floating material (*e.g.* broken ice) on the surface of the liquid is of considerable interest to mathematicians as time-harmonic gravity waves of a given angular frequency cannot exist if the inertial surface is too heavy. This necessitates the study of the problems of generation of waves due to submerged singularities of different types which begin to operate in a time-dependent manner at a given instant. Recently, Rhodes-Robinson[4] studied the generation of waves due to a two-dimensional wave source of time-dependent strength submerged in a liquid of infinite depth with an inertial surface and Mandal and Kundu[5] extended this to a liquid of finite depth and also studied other types of singularities. Rhodes-Robinson[6] in an earlier paper has mentioned the case of two superposed liquids that are separated by an inertial surface while discussing the possibility of existence of time-harmonic progressive waves at the interface. This has led us to study the generation of internal waves at the interface of two superposed fluids that are separated by an inertial surface due to singularities of different types submerged in either of the fluids which begin to operate in a time-dependent manner at a given instant.

In the present paper we consider a two-dimensional wave source of time-dependent strength submerged in either of the fluids of a two-fluid medium where the upper fluid extends infinitely upwards while the lower fluid is of finite uniform depth. Explicit expressions for the velocity potentials and the shape of the inertial surface due to the wave source are obtained and analysed to some extent when the source strength is impulsive at the initial instant but zero otherwise, is constant for all time and is harmonic in time with a given angular frequency.

This problem is a generalisation of the problem considered by Mandal and Kundu[5] for an one-fluid medium. It may also be viewed to some extent as a generalisation of the work of Gorgui and Kassem[3] for two ordinary superposed fluids to two superposed fluids separated by an inertial surface.

The above discussion gives the background that motivated the work presented here. Again, potentials due to two-dimensional multipoles submerged in either of the fluids can be derived from line source potentials by elementary differentiation, and due to point sources can be obtained by extending the method used for one-fluid medium as given in Mandal and Kundu[5].

The practical relevance of the work presented in the paper lies in the fact that when one wants to consider problems of generation of internal waves at the interface of a two-fluid medium separated by an inertial surface due to small movement of bodies submerged in either or both the fluids, the resulting motion as mentioned earlier for an ordinary two-fluid medium can be described by the use of these singularities in a suitable way.

2. Statement of the problem

We are concerned with the irrotational motion under the action of gravity of two non-viscous incompressible fluids separated by an inertial surface, composed of a thin uniformly distributed disconnected materials of area density $(\rho_1 - \rho_2)\,\varepsilon$ where $s = \rho_2/\rho_1$ $(0 \leqslant s < 1)$ and ρ_1, ρ_2 are the densities of the lower and upper fluids respectively. The motion is generated by a line source submerged in either of the fluids which starts operating in a time-dependent manner from time $t = 0$ so that the motion starts from rest at time $t = 0$. The source strength $m(t)$ is assumed to be summable over all finite time intervals and to be exponentially bounded. This ensures the existence of its Laplace transform (Note that this is only a sufficient condition).

We choose the origin 0 of a rectangular Cartesian coordinate system in the mean surface of separation and axis $0y$ pointing vertically downwards into the lower fluid, zx-plane being horizontal. The velocity potentials then satisfy

$$\nabla^2 \varphi_1 = 0,\ y > 0,$$
$$\nabla^2 \varphi_2 = 0,\ y < 0, \tag{2.1}$$

except at a point of singularity, where the subscripts 1 and 2 denote the lower and upper fluids respectively. As discussed by Rhodes-Robinson[4] for an one-fluid medium with an inertial surface, $\varphi_i\ (x, y, z;\ t)\ (i = 1,2)$ can be shown to satisfy the initial conditions

$$\Phi = \frac{\partial \Phi}{\partial t} = 0 \text{ on } y = 0 \text{ at } t = 0 \tag{2.2}$$

where

$$\Phi = \varphi_1 - \varepsilon \frac{\partial \varphi_1}{\partial y} - s(\varphi_2 - \varepsilon \frac{\partial \varphi_2}{\partial y}),$$

the linearized surface of separation conditions

$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y} \text{ on } y = 0, \tag{2.3}$$

$$\frac{\partial^2}{\partial t^2}(\varphi_1 - \varepsilon \frac{\partial \varphi_1}{\partial y}) - g \frac{\partial \varphi_1}{\partial y} = s \left\{ \frac{\partial^2}{\partial t^2}(\varphi_2 - \varepsilon \frac{\partial \varphi_2}{\partial y}) - g \frac{\partial \varphi_2}{\partial y} \right\} \text{ on } y = 0 \tag{2.4}$$

where g is the gravity, the bottom condition in the lower fluid

$$\frac{\partial \varphi_1}{\partial y} = 0 \text{ on } y = h, \tag{2.5a}$$

and in the upper fluid

$$\nabla \varphi_2 \to 0 \text{ as } y \to -\infty. \tag{2.5b}$$

It may be noted that for a time-harmonic motion of circular frequency σ the condition (2.4) becomes

$$K\varphi_1 + (1 - K\varepsilon) \frac{\partial \varphi_1}{\partial y} = s \left\{ K\varphi_2 + (1 - K\varepsilon) \frac{\partial \varphi_2}{\partial y} \right\} \text{ on } y = 0, \tag{2.6}$$

where $K = \sigma^2/g$ · For $0 \leqslant K\varepsilon < 1$ the form of (2.6) is

$$K^*\varphi_1 + \frac{\partial\varphi_1}{\partial y} = s\,(K^*\varphi_2 + \frac{\partial\varphi_2}{\partial y}) \text{ on } y = 0$$

where $K^* = K(1-K\varepsilon)^{-1}$.

This is merely a modification of the usual condition at the surface of separation for a two-fluid medium corresponding to $\varepsilon = 0$[3]. For $K\varepsilon < 1$, propagation of time-harmonic progressive wave is possible. However, for $K\varepsilon \geq 1$, the form of (2.6) is different and does not allow progressive waves. As noted by Rhodes-Robinson[4,6], these facts ensure that propagation of time-harmonic waves is possible if and only if $K\varepsilon < 1$.

Let $\bar{f}(p)$ denote the Laplace transform of $f(t)$ defined as

$$\bar{f}(p) = \int_0^\infty \exp(-pt)\, f(t)\, dt \quad (Re\ p > 0).$$

Then $\bar{\varphi}_i$ $(i = 1, 2)$ satisfies the boundary value problem described by

$$\nabla^2\bar{\varphi}_1 = 0,\ y > 0,$$

$$\nabla^2\bar{\varphi}_2 = 0,\ y < 0\ , \tag{2.7}$$

except at a point of singularity, the inertial surface of separation conditions

$$p^2\bar{\varphi}_1 - (g + \varepsilon p^2)\frac{\partial\bar{\varphi}_1}{\partial y} = s\left\{p^2\bar{\varphi}_2 - (g+\varepsilon p^2)\frac{\partial\bar{\varphi}_2}{\partial y}\right\} \text{ on } y = 0, \tag{2.8}$$

$$\frac{\partial\bar{\varphi}_1}{\partial y} = \frac{\partial\bar{\varphi}_2}{\partial y} \text{ on } y = 0, \tag{2.9}$$

and the conditions

$$\left.\begin{aligned}&\frac{\partial\bar{\varphi}_1}{\partial y} = 0 \text{ on } y = h,\\ &\nabla\bar{\varphi}_2 \to 0 \text{ as } y \to -\infty.\end{aligned}\right\} \tag{2.10}$$

$\bar{\varphi}_i$ $(i = 1,2)$ can be obtained by a technique, somewhat similar to that used by Gorgui and Kassem[3]. Laplace inversion will then give φ_i's.

3. Line singularities

We choose the y-axis passing through a singularity submerged in any one of the two fluids and is situated at either of the points $(0,\eta)$ or $(0,-\eta)$ $(\eta > 0)$ according as the singularity is in the lower or upper fluid respectively. We consider singularities which are symmetric in x only.

i) Wave source submerged in lower fluid

Here $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are the solutions of the boundary-value problem stated by (2.7) to (2.10) with

$$\bar{\varphi}_1 \sim \bar{m}\,(p)\ \ln\, r \ \text{ as } r = \{\, x^2 + (y-\eta)^2\,\}^{1/2} \to 0.$$

Let

$$\bar{\varphi}_1 = \bar{m}\,[\ln\, r + C_1\ \ln\, r'\,] + \int_0^\infty \{A_1(k)\ \cosh\ k(h-y)$$

$$+ B_1(k)\ \sinh\ ky\,\}\cos\, kx\ dk,$$

$$\bar{\varphi}_2 = \bar{m}\ C_2\ \ln\, r + \int_0^\infty A_2(k)\ \exp\,(ky)\ \cos\ kx\ dk$$

where $r' = \{\,x^2 + (y+\eta)^2\}^{1/2}$.

We choose C_1, C_2, $A_1(k)$, $A_2(k)$, and $B_1(k)$ such that the conditions (2.8), (2.9) and (2.10) are satisfied and the integrals converge. Using the representations

$$\frac{\partial}{\partial y}(\ln\ r) = \begin{cases} \displaystyle\int_0^\infty \exp\{-k(y-\eta)\}\ \cos\ kx\ \mathrm{d}k,\ y > \eta, \\ \displaystyle -\int_0^\infty \exp\{-k(\eta-y)\}\ \cos\ kx\ \mathrm{d}k,\ y < \eta, \end{cases}$$

and $\displaystyle\frac{\partial}{\partial y}(\ln\ r') = \int_0^\infty \exp\{-k(y+\eta)\}\ \cos\ kx\ \mathrm{d}k,\ y > -\eta,$

the above stated conditions are satisfied if

$$kA_1\ \sinh\ kh + A_2k = \bar{m}\,[(C_1+C_2-1)\ \exp(-k\eta) - \exp(-kh)\,\{\exp(k\eta)$$

$$+ C_1\ \exp\,(-k\eta)\}\ \mathrm{sech}\,kh] = \bar{m}\ F(k),\ \text{say}, \tag{3.1}$$

$$1 + C_1 - sC_2 = 0, \tag{3.2}$$

$$A_1\,\{\,p^2\ \cosh\ kh + (g + \varepsilon p^2)k\ \sinh\ kh\} - A_2\,\{\,sp^2 - s(g + \varepsilon p^2)k\}$$
$$- (g + \varepsilon p^2)k\ B_1 = 2\bar{m}\,(g + \varepsilon p^2)C_1\ \exp(-k\eta), \tag{3.3}$$

$$B_1 k = -\bar{m}\ \exp(-kh)\{\exp(k\eta) + C_1\ \exp(-k\eta)\}\text{sech } kh. \tag{3.4}$$

The integrals will be convergent if $F(k) = 0$ at $k = 0$ which gives $C_2 = 2$ so that

$$C_1 = 2s - 1; \tag{3.5}$$

hence

$$\triangle A_1 = \bar{m}\left[\ (g + \varepsilon p^2)\{-2\ \exp(-k\eta) + F(k)\} + \frac{sF(k)p^2}{k(1-s)}\right], \tag{3.6}$$

$$\triangle A_2 = \bar{m}\left[\ 2(g + \varepsilon p^2)\ \exp(-k\eta)\ \sinh\ kh + \frac{F(k)p^2}{k(1-s)}\ \cosh\ kh\right], \tag{3.7}$$

$$kB_1(k) = -2\bar{m}\ \exp(-kh)\ (\sinh\ k\eta + s\ \exp(-k\eta))\ \text{sech}\ kh, \tag{3.8}$$

where

$$\triangle(k, p) = \frac{1}{1-s}\{p^2(\cosh\ kh + s\ \sinh\ kh)\} + (g + \varepsilon p^2)k\ \sinh\ kh. \tag{3.9}$$

Thus $\bar{\varphi}_1$, $\bar{\varphi}_2$ are obtained after rearrangement as

$$\bar{\varphi}_1 = \bar{m}\left[\ X_1(x,y) + \int_0^\infty \{-2E(k)\ \exp(-k\eta) + F(k)\ \cosh\ kh\} \right.$$
$$\left. \frac{\cosh\ k(h-y)}{k\ D(k)\ \sinh\ kh}\cos\ kx\ \frac{\mu^2}{\mu^2 + p^2}\ \mathrm{d}k\ \right],$$

$$\bar{\varphi}_2 = \bar{m}\left[\ X_2(x,y) + 2\int_0^\infty \frac{\exp(ky)\ \cosh\ k(h-\eta)}{k\ D(k)}\ \cos\ kx\ \frac{\mu^2}{\mu^2 + p^2}\mathrm{d}k\right],$$

where

$$X_1(x,y) = \ln r + (2s-1)\ \ln r' + \int_0^\infty \frac{(1-s)}{D(k)}\left[\ \varepsilon\{-2\ \exp(-k\eta) + F(k)\}\right.$$
$$\left. + \frac{s}{1-s}\frac{F(k)}{k}\right]\cosh\ k(h-y)\ \cos\ kx\ \mathrm{d}k$$
$$-2\int_0^\infty \frac{\exp(-kh)}{k}\ (\sinh\ k\eta + s\ \exp(-k\eta))\ \frac{\sinh\ ky}{\cosh\ kh}\ \cos\ kx\ \mathrm{d}k,$$

$$X_2(x,y) = 2\ \ln\ r + \int_0^\infty \frac{1-s}{D(k)}\ \exp(ky)\{\ 2\varepsilon\ \exp(-k\eta)\ \sinh\ kh$$
$$+ \frac{F(k)\ \cosh\ kh}{k(1-s)}\}\ \cos\ kx\ \mathrm{d}k,$$

$$E(k) = \cosh kh + s \sinh kh,$$

$$D(k) = \cosh kh + \{ s(1-k\varepsilon) + k\varepsilon \} \sinh kh,$$

$$\mu^2 = \frac{gk(1-s) \sinh kh}{D(k)}.$$

Laplace inversion then gives

$$\varphi_1 = m(t)\ X_1 - 2\int_0^\infty \mu \frac{\cosh k(h-\eta) \cosh k(h-y)}{k\ D(k) \sinh kh} \cos kx \int_0^t m(\tau) \sin \mu(t-\tau)\, d\tau\, dk,$$

$$\varphi_2 = m(t)\ X_2 + 2\int_0^\infty \mu \frac{\exp(ky)}{k\ D(k)} \cosh k(h-\eta) \cos kx \int_0^t m(\tau) \sin \mu(t-\tau)\, d\tau\, dk. \quad (3.10)$$

(3.10) is the general result for the potential functions due to a line source of time-dependent strength $m(t)$ submerged in the lower fluid. To obtain some qualitative aspects of these results we consider the following three particular forms of the source strength $m(t)$:

(a) For an impulsive source, $m(t) = \delta(t)$ so that the corresponding potentials are

$$\varphi_1^{(o)} = \delta(t)X_1 - 2\int_0^\infty \mu \frac{\cosh k(h-\eta) \cosh k(h-y)}{k\ D(k) \sinh kh} \cos kx \sin \mu t\, dk,$$

$$\varphi_2^{(o)} = \delta(t)X_2 + 2\int_0^\infty \mu \frac{\exp(ky)}{k\ D(k)} \cosh k(h-\eta) \cos kx \sin \mu t\, dk. \quad (3.11)$$

We note that both $\varphi_1^{(o)}(t)$ and $\varphi_2^{(o)}(t)$ die out as $t \to \infty$. This result is only expected, because, as the source acts only instantaneously at $t = 0$, its effect will not be felt anywhere in the fluid region after a long lapse of time. The same conclusion can be arrived at for the shape of the inertial surface as well as the nonhydrostatic part of the pressure distribution at any point of the two fluids.

(b) For the classical wave source of constant strength we take $m(t) = 1$, so that the potentials become

$$\varphi_1^{(1)} = X_1 - 2\int_0^{\infty} \frac{\cosh k(h-\eta)\,\cosh k(h-y)}{k\,D(k)\,\sinh kh} \cos kx\,(1-\cos \mu t)\,\mathrm{d}k,$$

$$\varphi_2^{(1)} = X_2 + 2\int_0^{\infty} \frac{\exp(ky)}{k\,D(k)} \cosh k(h-\eta)\,\cos kx\,(1-\cos \mu t)\,\mathrm{d}k. \tag{3.12}$$

These potentials exist for finite time only. As $t \to \infty$, $\varphi_i^{(1)}(t)$ does not possess a finite limit although $\nabla \varphi_i^{(1)}$ has a finite limit. This was also observed by Mandal and Kundu[5] for anone-fluid medium.

(c) For a time-harmonic wave source of circular frequency σ, we take $m(t) = \sin \sigma t$. Then the potentials become

$$\varphi_1^{(2)} = \sin \sigma t\, X_1 - 2\int_0^{\infty} \mu\, \frac{\cosh k(h-\eta)\,\cosh k(h-y)}{k\,D(k)\,\sinh kh} \cos kx$$

$$\frac{\mu \sin \sigma t - \sigma \sin \mu t}{\mu^2 - \sigma^2}\,\mathrm{d}k,$$

$$\varphi_2^{(2)} = \sin \sigma t\, X_2 + 2\int_0^{\infty} \mu\, \frac{\exp(ky)}{k\,D(k)} \cosh k(h-\eta)\,\cos kx$$

$$\frac{\mu \sin \sigma t - \sigma \sin \mu t}{\mu^2 - \sigma^2}\,\mathrm{d}k.$$

To study the behaviours of $\varphi_i^{(2)}$ $(i = 1, 2)$ as $t \to \infty$ we follow the technique of Rhodes-Robinson[4]. Two different situations arise according as the integrand vanishes or not in the range of integration $k > 0$. In fact $\mu^2 - \sigma^2$ or equivalently $\delta(k) = \{k(1-s)(1-K\varepsilon) - Ks\} \sinh kh - K \cosh kh$ has a positive zero when $0 \leqslant K\varepsilon < 1$ and none when $K\varepsilon \geqslant 1$.

Following Rhodes-Robinson, for $0 \leqslant K\varepsilon < 1$, we obtain as $t \to \infty$,

$$\varphi_1^{(2)} = \sin \sigma t \Bigg[\ln r + (2s-1) \ln r' + 2\int_0^{\infty} \frac{1}{k \cosh kh} \Bigg\{ s \exp(-k\eta)$$

$$- \frac{\{k(1-s)(1-K\varepsilon) - Ks\} \cosh k(h-\eta)}{\delta(k)} \Bigg\} \cosh k(h-y) \cos kx\,\mathrm{d}k$$

$$- 2\int_0^{\infty} \frac{\exp(-kh)}{k} (\sinh k\eta + s \exp(-k\eta)) \frac{\sinh ky}{\cosh kh} \cos kx\,\mathrm{d}x \Bigg]$$

$$+ 4\pi \frac{\cosh k_0^*(h-\eta)\,\cosh k_0^*(h-y)\,\cos k_0^* x}{\sinh 2k_0^* h + s \cosh 2k_0^* h + 2\,k_0^* h - s} \cos \sigma t, \tag{3.14}$$

$$\varphi_2^{(2)} = 2 \sin \sigma t \left[\ln r + \int_0^\infty \frac{\exp(ky)}{k} \left\{ \exp(-k\eta) + \frac{K \cosh k(h-\eta)}{\delta(k)} \right\} \cos kx \, dk \right] + 4\pi \frac{\cosh k_0^*(h-\eta) \cosh k_0^*(h-y) \sinh k_0^* h}{\sinh 2k_0^* h + s \cosh 2k_0^* h + 2 k_0^* h - s} \cos k_0^* x \cos \sigma t,$$

where

$$\delta(k) = \{ k(1-s)(1-K\varepsilon) - Ks \} \sinh kh - K \cosh kh \tag{3.15}$$

and k_0^* is the only positive real zero of $\delta(k)$. It can be shown that the forms given in (3.14) represent the outgoing wave as $|x| \to \infty$ (*cf.* Mandal and Kundu[5]). All these results coincide with those obtained by Gorgui and Kassem[3] by putting $\varepsilon = 0$, for an ordinary two-fluid medium.

When $K\varepsilon \geqslant 1$, there is no real zero of $\delta(k)$ for $k > 0$. Then by Riemann-Lebesgue lemma the integrals involving $\sin \mu t$ in (3.13) are wholly transient and after simplification we obtain

$$\varphi_1^{(2)} \sim \sin \sigma t \left[\ln r + (2s-1) \ln r' + \int_0^\infty \frac{1}{k \cosh kh} \left\{ s \exp(-k\eta) - \frac{\{ k(1-s)(K\varepsilon-1) + Ks \} \cosh k(h-\eta)}{\{ k(1-s)(K\varepsilon-1) + Ks \} \sinh kh + K \cosh kh} \right\} \cosh k(h-y) \cos kx \, dk \right], \tag{3.16}$$

$$\varphi_2^{(2)} \sim 2 \sin \sigma t \left[\ln r + \int_0^\infty \frac{\exp(ky)}{k} \left\{ \exp(-k\eta) - \frac{K \cosh k(h-\eta)}{\{ k(1-s)(K\varepsilon-1) + Ks \} \sinh kh + K \cosh kh} \right\} \cos kx \, dk \right].$$

In this case there is no outgoing progressive wave as $|x| \to \infty$.

Since the interface is horizontal initially, its form at any time t is $y = \zeta(x,t)$ where

$$\zeta(x,t) = \int_0^t \left(\frac{\delta \varphi_1}{\delta y} \right)_{y=0} dt \text{ or } \int_0^t \left(\frac{\delta \varphi_2}{\delta y} \right)_{y=0} dt.$$

From (3.14) it follows that for $0 \leqslant K\varepsilon < 1$, $\zeta(x,t)$ assumes the form of an outgoing wave for large $|x|$ and t. Again from (3.16) it follows that for $K\varepsilon \geqslant 1$, $\zeta(x,t)$ becomes small for large $|x|$ and t so that the interface remains almost undisturbed at a large distance from the source after a large time since the wave source starts operating.

The behaviour of the nonhydrostatic part of the pressure at any point of the two fluids can be studied from $\partial\varphi_1/\partial y$ or $\partial\varphi_2/\partial y$ according as the point is in the lower or upper fluid respectively.

ii) Wave source submerged in upper fluid

The boundary value problem for this case is similar to the previous one except that, now the singularity is at $(0, -\eta)$ in the upper fluid and we have $\bar{\varphi}_2 \sim \bar{m} \ln r'$ as $r' \to 0$. We can similarly obtain

$$\varphi_1 = m(t)\ Y_1 + 2s\int_0^\infty \mu\ \frac{\exp\ (-k\eta)}{k\ D(k)} \cosh\ k(h-\eta)$$

$$\cos\ kx \int_0^t m(\tau)\ \sin\ \mu(t-\tau)\ d\tau\ dk,$$

$$\varphi_2 = m(t)\ Y_2 - 2s\int_0^\infty \mu\ \frac{\exp\{k(y-\eta)\}}{k\ D(k)} \sinh\ kh$$

$$\cos\ kx \int_0^t m(\tau)\ \sin\ \mu(t-\tau)\ d\tau\ dk,$$

where

$$Y_1(x,y) = 2s\ \ln\ r - 2s\int_0^\infty \exp\ \{-k(h+\eta)\} \sinh\ ky\ \cos\ kx dk,$$

$$Y_2(x,y) = \ln(rr') + 2s\int_0^\infty \frac{\exp\ \{\ k(y-\eta)\ \}}{kD(k)}\ \sinh\ kh\ \cos\ kx\ dk,$$

and $D(k)$ is the same as in case (i).

Similar types of results as in (i) regarding the potentials, interface shape, etc., can be obtained for three different types of source strengths in this case also.

4. Conclusion

Potential functions due to a line source of time-dependent strength submerged in either of the fluids of a two-fluid medium have been obtained. The upper fluid is of infinite extent while the lower fluid is of finite depth. Known results in the absence of upper fluid can be deduced (*cf.* Mandal and Kundu[5]) by putting $s = 0$ in φ_1. Again if we put $\varepsilon = 0$ in the results corresponding to time-harmonic source strength, we recover the results for an ordinary two-fluid medium given by Gorgui and Kassem[3].

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