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A translation plane of order 25 with small translation complement

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Abstract

Various translation planes of order 25 have so far been constructed and their translation complements determined. All these planes except the two flag transitive planes of order 25 of Foulser and one of the exceptional planes of Walker are such that at least one of their matrix representation sets admit non-trivial nuclei. In this paper we construct a new translation plane of order 25, all of whose matrix representation sets admit only trivial nucleus. Further the translation complement modulo the subgroup of scalar collineations is a dihedral group of order 24 and is the smallest when compared with all the planes reported so far. The translation complement of this plane divides the set of ideal points into 4 orbits of lengths 4, 4, 6 and 12.

Key words: Translation plane, orbit of ideal points.

1. Introduction

A one spread set $\mathscr{C}$ over GF(5) is a collection of 25 matrices including the zero and the identity matrices such that the difference of any two distinct matrices of $\mathscr{C}$ is nonsingular[1].

The reader is referred to references 1–3 for the preliminary results on spread sets and the associated collineations in the corresponding translation planes.

For a fixed j and k we define $\Gamma_{j,k}$ by

$$\Gamma_{j,k} = \{(M-M_j)^{-1} - (M_k-M_j)^{-1} \mid M \,\varepsilon\, \mathscr{C} \, U \, \{(\infty)\}$$

with the usual operations of inverse whenever M_0 and M_{25} appear in the above expression. It is known[3] that there exists a collineation of π mapping (25) onto (j) and (0) onto (k) if and only if there are $A, B \,\varepsilon\, GL\,(2,5)$ such that $A^{-1}\Gamma_{j,k}\,B = \mathscr{C}\,U\,\{(\infty)\}$. In particular if $I, 2I, 3I, 4I \,\varepsilon\, \mathscr{C}$, then there must be $M \,\varepsilon\, \Gamma_{j,k}$ such that $M, 2M, 3M, 4M \,\varepsilon\, \Gamma_{j,k}$. Let this property be called 'inherited property'.

2. Construction of a translation plane π of order 25

Let $M_0 = (\begin{smallmatrix}0&0\\0&0\end{smallmatrix})$; M_5 [illegible]; M_1 [illegible] and M_{4} [illegible] consider the mapping $\alpha : M \to 3$ $(\begin{smallmatrix}1&0\\0&1\end{smallmatrix})$ $[(M+3I)^{-1}+I]$ $(\begin{smallmatrix}1&0\\0&1\end{smallmatrix})$. The matrices M_i, $0 \leqslant i \leqslant 25$ are defined recursively as $\alpha M_0 = M_1$, $\alpha M_1 = M_2$, $\alpha M_2 = M_3$, $\alpha M_3 = M_4$, $\alpha M_4 = M_1$. $M_{5+i} = \alpha^i M_5$ $0 \leqslant i \leqslant 11$ $M_{17+i} = \alpha^i M_{17}$, $0 \leqslant i \leqslant 3$, $M_{21+i} = \alpha^i M_{21}$, $0 \leqslant i \leqslant 3$. It may be observed that $\alpha^{12} M_5 = M_5$, $\alpha^4 M_{17} = M_{17}$ and $\alpha^4 M_{21} = M_{21}$. We list in Table I these 25 matrices along with their characteristic polynomials. The entry a, b under the heading CP indicates that $\lambda^2 + a\lambda + b$ is the characteristic polynomial of the corresponding matrix.

Table 1

Spread set and collineations

i	$\Gamma_{[illegible]}$	CP	$\Gamma_{[illegible]}$	$\Gamma_{[illegible]}$	$\Gamma_{[illegible]}$	$\Gamma_{[illegible]}$	$\Gamma_{[illegible]}$	$\Gamma_{5,11}$
0	00	–	33	41	33	20	12	33
	00		34	11	43	03	14	31
1	40	2,1	23	31	11	03	30	40
	04		33	10	32	32	04	01
2	20	1,4	03	01	41	44	01	11
	02		31	13	30	43	32	10
3	30	4,4	13	21	01	24	31	43
	03		32	14	31	41	30	32
4	10	3,1	43	01	31	33	10	01
	01		30	12	34	30	02	14
5	22	2,3	00	13	43	41	41	–
	21		00	32	04	12	24	
6	03	2,4	31	44	24	11	32	23
	23		02	34	01	01	21	43
7	21	3,4	04	12	42	31	21	13
	10		44	21	43	13	11	30
8	41	3,3	24	32	12	21	33	20
	43		22	04	21	24	13	22
9	43	1,1	21	34	14	30	23	21
	30		14	41	13	04	34	11
10	01	4,1	34	42	22	43	11	02
	41		20	02	24	23	43	34

11	23	2,3	01	14	44	23	22	00
	31		10	42	14	31	42	00
12	02	2,4	30	43	23	10	42	24
	33		12	44	11	11	33	33
13	24	3,4	02	10	40	12	44	42
	40		24	01	23	21	12	20
14	44	3,3	22	30	10	22	14	22
	13		42	24	41	14	31	02
15	42	1,1	20	33	13	40	13	34
	20		04	31	03	02	22	44
16	04	4,1	32	40	20	42	34	03
	11		40	22	44	33	44	24
17	14	0,3	42	00	30	–	24	10
	44		23	00	22		03	03
18	13	0,3	41	04	34	04	20	41
	24		03	30	02	10	23	42
19	11	0,2	44	02	32	00	40	30
	14		43	20	42	00	41	04
20	12	0,3	40	03	33	01	43	44
	34		13	40	12	40	01	12
21	34	0,3	12	20	00	32	–	12
	22		01	33	00	42		41
22	33	0,3	11	24	04	14	03	31
	12		41	23	40	34	10	23
23	31	0,3	14	22	02	13	00	32
	32		11	43	10	44	00	13
24	32	0,3	10	23	03	34	02	14
	42		21	03	20	22	40	21
25	–		–	–	–	02	04	04
						20	20	40

2.1 Theorem: $\mathscr{C}$ is a 1-spread set over GF(5)

Proof: Since $\mathscr{C}$ has 25 matrices including the zero and the identity matrices, it suffices to show that $|X-Y| \neq 0$, $X, Y \in \mathscr{C}$ and $X \neq Y$. If either X or Y is the zero matrix then the result follows from the fact that the nonzero matrices of $\mathscr{C}$ are nonsingular. This is also true when X and Y are scalar matrices. If X and Y are such that X is a scalar matrix and $Y = M_i$, $5 \leq i \leq 24$ then $(X-Y)$ is nonsingular since all the matrices in $\mathscr{C}$ other than the scalar matrices have irreducible characteristic polynomials. Suppose X and Y are nonscalar matrices so that $X = M_i$, $Y = M_j$, $i \neq j$. Then $\alpha(M_i) = M_{i+1}$, $\alpha(M_j) = M_{j+i}$. Now $|M_{i+1} - M_{j+i}| = 3\ |\begin{pmatrix}1&0\\0&3\end{pmatrix}|\ |(M+3I)^{-1}|\ |(M_i - M_i)|\ |(M_j+3I)^{-1}|\ |\begin{pmatrix}1&0\\0&2\end{pmatrix}|$. Thus $(M_{i+1} - M_{j+1})$ is nonsingular if and only if $(M_i - M_i)$ is nonsingular. An inspection of $\Gamma_{25,5}$; $\Gamma_{25,17}$ and $\Gamma_{25,21}$ in Table I reveals that $M_5 - M_j$, $j > 5$; $M_{17} - M_j$, $j > 17$; $M_{21} - M_j$, $j > 21$ are honsingular. The theorem now follows from repeated application of the fact that $M_{i+1} - M_{j+1}$ is nonsingular if and only if $M_i - M_i$ is nonsingular. Hence the theorem.

We observe that the spread set $\mathscr{C}$ is not closed under matrix addition and therefore coordinatizes a non-desargusian translation plane π. Further $\mathscr{C}$ has 6 matrices with determinant 1, 12 matrices with determinant 3 and 6 matrices with determinant 4. That is the determinant structure of $\mathscr{C}$ is 1–6, 2–0, 3–12, 4–6. No matrix M in $\mathscr{C}$ other than I is such that M, $2M$, $3M$ and $4M \in \mathscr{C}$.

3. Some collineations of π

3.1 The mapping α defined in Section 2 induces a collineation on π whose action on the set of ideal points is seen to be

$\alpha = (0, 1, 2, 25, 3, 4)\ (5, \ldots, 16)\ (17, \ldots 20)\ (21, \ldots 24)$. Let β be the mapping defined by $\beta : M \to \begin{pmatrix}1&0\\0&4\end{pmatrix}^{-1} M \begin{pmatrix}1&0\\0&4\end{pmatrix}$. Then β maps the matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ onto $\begin{pmatrix}a&4b\\4c&d\end{pmatrix}$. An examination of Table I reveals that for each $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathscr{C}$, $\beta\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathscr{C}$. Thus β induces a collineation on π whose action on the ideal points is given by β : (0) (25) (1) (2) (3) (4) (5,11) (6,12) (7,13), (8,14) (9,15) (10,16) (17,19) (18,20) (21,23) (22,24).

Let r be the mapping defined by r: $M \to \begin{pmatrix}0&1\\3&0\end{pmatrix}^{-1} 4M\begin{pmatrix}0&1\\3&0\end{pmatrix}$. Then for each $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathscr{C}$, $r\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}4d&3c\\2b&4a\end{pmatrix} \in \mathscr{C}$. Hence r induces a collination whose action is given by

r = (0) (25) (1,4) (2,3) (5,8) (6,7) (9,16) (10,15) (11,14) (12,13) (17,20) (18,19) (21,23) (22) (24).

Let δ be the mapping defined by $\delta : M \to \begin{pmatrix}0&1\\2&0\end{pmatrix}^{-1} 4M\begin{pmatrix}0&1\\2&0\end{pmatrix}$. Then for $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathscr{C}$, $\delta\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}4d&2c\\3b&4a\end{pmatrix} \in \mathscr{C}$. Hence δ induces a collineation whose action on the set of ideal points is given by

δ : (0) (25) (1,4) (2,3) (5,14) (6,13) (7,12) (8,17) (9,10) (15,16) (17,18) (19,20) (21) (22,24) (23).

3.2 *Conjugation collineations*

A mapping of the form $M \to A^{-1}MA$; $A \varepsilon GL(2,5)$ such that for each $M \varepsilon \mathscr{C}$, $A^{-1}MA \varepsilon \mathscr{C}$ induces a collineation on π called a conjugation collineation. Since there are only two matrices in $\mathscr{C}$, $P = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ in $\mathscr{C}$ with the same characteristic polynomial $\lambda^2 + 2\lambda + 3$, the group of conjugation collineations must leave $\{P, Q\}$ invariant. But β is a conjugation collineation mapping P onto Q. Thus the group of conjugation collineations is actually transitive on $\{P, Q\}$.

We now determine all the conjugation collineations that fix P. But any conjugation collineation fixing P must also fix Q. Thus the group of all conjugation collineations fixing P consists of the common elements of the normalizer of P and the normalizer of Q. Since the characteristic polynomials of P and Q are irreducible over $GF(5)$, their normalizers are the non-zero elements of the fields of matrices generated by $\{I, P\}$ and $\{I, Q\}$, but these fields, though isomorphic, have distinct matrix representations having only I, $2I$, $3I$ and $4I$ as common element. Conjugation of matrices in $\mathscr{C}$ by I, $2I$, $3I$ and $4I$ obviously induce collineation. These collineations are also called scalar collineations. Let the group of scalar collineations be denoted by S. Then S is of order 4.

Thus the group of conjugation collineation fixing P is the group S. Then G_1 is transitive on $\{P, Q\}$ and the subgroup consisting of all conjugation collineations fixing P is S. A coset decomposition of G_1 by S gives $G_1 = SUS\beta$ and $|G_1| = 4 \times 2 = 8$.

3.3 *Collineations fixing (0) and (25)*

Any collineation fixing (0) and (25) is induced by a mapping of the type $\phi : M \to A^{-1}MB, A, B \varepsilon GL(2,5)$ such that for each $M \varepsilon \mathscr{C}$, $A^{-1}MB \varepsilon \mathscr{C}$. Since $\mathscr{C}$ has no matrix M other than I, $2I$, $3I$, $4I$ such that $M, 2M, 3M, 4M \varepsilon \mathscr{C}$, ϕ must map I, $2I$, $3I$ and $4I$ among themselves. This implies that $|A^{-1}B| = kI$, $k = 1, 2, 3, 4$. The mapping then takes the form $\phi : M \to A^{-1}(kM)A$, $k = 1, 2, 3, 4$. If $k = 2$ the characteristic polynomial structures of $\mathscr{C}$ and $2\mathscr{C}$ do not tally. Therefore $M \to A^{-1}(2M)A$ does not induce a collineation on π. A similar argument shows that $M \to A^{-1}(3M)A$ also does not induce a collineation on π.

Thus the set of all collineations that fix (0) and (25) are given by mapping of the type $M \to A^{-1}MA$ or $M \to A^{-1}(4M)A$. An inspection of Table I reveals that M_5, M_{11}, $4M_{14}$ and $4M_8$ are the only matrices with the same characteristic polynomial $\lambda^2 + 2\lambda + 3$, and $4M_5$, $4M_{11}$, M_{14} and M_8 are the only matrices with the same characteristic polynomial $\lambda^2 + 3\lambda + 3$. This implies that the group of collineation fixing (0) and (25) is invariant on the set of ideal point consisting of $\{(5), (11), (8), (14)\}$. But $<\beta, r, \delta>$ which is a subgroup of G_2 is actually transitive on this set. Therefore G_2 is transitive on $\{(5), (8), (11), (14)\}$.

We now determine all collineations of G_2 that fix M_5. Obviously a mapping of the type $M \to A^{-1}(4M)A$ cannot fix M_5. Thus the subgroup of all collineations of G_2 that fix M_5 must be generated by the mappings of the form $M \to A^{-1}MA$. Since a conjugation collineation fixing M_5 fixes M_{11} also, the subgroup of G_2 fixing M_5 is S. A coset decomposition of G_2 by S gives $G_2 = \cup S. \Theta x$, where Θxs are chosen such that M_5 is mapped onto each of M_5, M_{11}, M_8 and M_{14}. Further Θxs may be taken from $<\beta, r, \delta>$. Thus $G_2 = <\beta, r, \delta, S>$ and $|G_2| = 16$.

3.4 Orbit structure of ideal points under $G = <\alpha, \beta, r, \delta, S>$

Let $G = <\alpha, \beta, r, \delta, S>$. Then G divides the set of ideal points into 4 orbits O_1, O_2, O_3 and O_4 of lengths 6, 12, 4, 4 given by

$$O_1 = \{(0), (1), (2), (3), (4), (25)\}; O_2 = \{(5), (6), \ldots (16)\}$$
$$O_3 = \{(17), (18), (19), (20)\}; \quad O_4 = \{(21), (22), (23), (24)\}.$$

4. Translation complement of π

In this section we show that G is in fact the translation complement of π. The scarcity of collineations in π makes it extremely difficult to prove the non-existence of certain types of collineations in π.

4.1 Definition: Two ideal points (i) and (j) are said to be companions under a collineation group H, if every collineation from H that fixes (i) fixes (j) and *vice versa*. That is a collineation from H either fixes both (i) and (j) or moves both (i) and (j).

The significance of companions is that any collineation must map companions onto companions. Companion of an ideal point under a collineation group may fail to be companion under a different collineation group. But a companion under a subgroup H is a possible companion under a bigger group of which H is a subgroup.

4.2 Lemma: The ideal points (0) and (25) are companions under the translation complement.

Proof: It suffices to prove that there is no collineation of π fixing one of (0) or (25) and moving the other. Since α^3 is a collineation flipping (0) and (25) it is enough if we consider the situation when (25) is fixed and (0) is moved onto an ideal point other than (25). Since the orbit structure of G_2 on the set of ideal points is $\{(0)\}$, $\{(25)\}$ $\{(1), (4)\}$ $\{(2), (3)\}$, $\{(5), (11), (14), (8)\}$, $\{(6), (7), (12), (13)\}$, $\{(15), (9), (16), (10)\}$ $\{(17), (20), (18), (19)\}$, $\{(21), (23)\}$, $\{(24), (22)\}$, we consider the possibilities of a collineation of π fixing (25) and moving (0) onto one ideal point in each of the orbits. The 'inherited property' condition requires that $\Gamma_{25,i}$, where (i) is one ideal point in each of the orbits, cannot have matrices whose determinants are 1, 2, 3 and 4. This is because $\mathscr{C}$ has matrices with determinants 1, 3 and 4 only. Further $\Gamma_{25,i}$ should have a matrix M such that $M, 2M, 3M, 4M \in \Gamma_{25,i}$. When $i = 5, 17, 21$ an inspection of Table I reveals that there are matrices in $\Gamma_{25,5}$, $\Gamma_{25,17}$ and $\Gamma_{25,21}$ with determinants 1, 2, 3 and 4 contradicting the 'inherited property' condition. When $i = 1$, $\Gamma_{25,1}$ has 4 matrices $\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}3 & 2\\2 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 3\\2 & 4\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2 & 0\\0 & 2\end{smallmatrix}\right)$ corresponding to $\left(\begin{smallmatrix}0 & 0\\0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2 & 2\\2 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 3\\2 & 3\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$ in $\Gamma_{25,0}$, whose determinants are 1, 2, 3 and 4, contradicting the 'inherited property' condition. For $i = 2, 9, 12$ and 22 a similar computation shows that $\Gamma_{25,2}$, $\Gamma_{25,9}$, $\Gamma_{25,12}$ and $\Gamma_{25,22}$ have matrices with determinants 1, 2, 3 and 4 which contradicts the 'inherited property' condition. The working in each case is straightforward and is omitted to save space. Hence the lemma.

Table II
Fixed points

Collineation	Fixed points	Collineation	Fixed points
α^4, α^8	(17)(18)(19)(20)(21)(22) (23)(24)	$\alpha^5 r$	(10)(16)(18)(20)
α^6	(0)(1)(2)(3)(4)(25)	$\alpha^6 r$	(0)(25)(21)(23)
α^{12}	all ideal points	$\alpha^7 r$	(15)(17)(19)(9)
r	(0)(25)(22)(24)	$\alpha^8 r$	(2)(4)(22)(24)
αr	(6)(12)(20)(18)	$\alpha^9 r$	(8)(14)(18)(20)
$\alpha^2 r$	(2)(4)(21)(23)	$\alpha^{10} r$	(1)(3)(21)(23)
$\alpha^3 r$	(5)(11)(17)(19)	$\alpha^{11} r$	(7)(13)(17)(19)
$\alpha^4 r$	(1)(3)(22)(24)		

4.3 Lemma: There is no collineation of π which maps (0) and (25) outside orbit O_1.

Proof: Since a collineation maps companions onto companions only, we have to determine companions in orbits O_2, O_3 and O_4 under the collineation group G, so that they are possible companions under the translation complement.

We list in Table II the fixed points under collineation group G.

An examination of Table II reveals that companions of (0), (5), (17), (21) under G are (25), (11), (19), (23) respectively. Since (0) and (25) are companions under the translation complement, the possible companions of (5), (17) and (21) under the translation complement are (11), (19) and (23) respectively. A reference to Table I once again reveals that in $\Gamma_{5,17}$, $\Gamma_{17,19}$ and $\Gamma_{21,23}$ there is no matrix M such that M, $2M$, $3M$ and $4M \varepsilon \Gamma_{5,17}, \Gamma_{17,19}, \Gamma_{21,23}$. *Hence there is no collineation that maps (0) and (25) onto* ideal points outside O_1.

4.4 Theorem: The group $G : <\alpha, \beta, r, \delta, S>$ is the translation complement of π. $|G| = 96$ and the quotient group G/S is a dihedral group of order 24. Further G divides the set of ideal points into four orbits of lengths 6, 12, 4 and 4.

Proof: From the fact that G accounts for all collineations that fix (0) and (25) and all the collineations that flip (0) and (25) it follows that the translation complement of π is G. There are no collineations of π which fixes either (0) or (25) and move the other. There are no collineations that move (0) and (25) simultaneously outside orbit O_1. The group G is transitive on O_1, containing the six ideal points (0), (1), (2), (3), (4) and (25). Since any collineation that fixes (25) fixes (0), the group of all collineations that fixes (25) is G_2 itself. A coset decomposition of G by G_2 gives $G = \cup G_2 x_i$, where x_is are those collineations that map (25) onto each of the six ideal points in O_1. Further x_is may be taken from $<\alpha>$. Thus $|G| = 16 \times 6 = 96$. Other parts of the theorem are obvious.

Lemma: Every matrix representation set of π has a trivial nucleus.

Proof: It is known that if every collineation of π either fixes more than two ideal points or fixes two ideal points and divides the remaining ideal points into orbits, at least two of which have unequal lengths, then every matrix representation set has only trivial nucleus. An examination of Table II reveals that π has either no fixed points or has four or more fixed points under any collineation of the translation complement. Hence the lemma.

5. Conclusion

The known translation planes of order 25 that have been reported so far are the following.

a) The fifteen planes reported by Davis[4]

b) The plane reported by Rao and Rao[5]

c) The plane reported by Rao and Satyanarayana[6]

d) The two exceptional planes reported by Walker[7]

e) The two flag transitive planes reported by Foulser[8].

The fifteen planes reported by Davis are such that at least one of the matrix representation sets admit nontrivial nuclei. One of the matrix representation sets of Rao and Rao and the plane of Rao and Satyanarayana admit nontrivial nuclei[9]. It has been shown by Rao and Satyanarayana[6] that one of Walker's planes is indeed a C-plane, one of whose matrix representation sets admit nontrivial nucleus. Thus the translation plane π, now under consideration is distinct from the 18 planes reported so far.

However the two flag transitive planes of Foulser and one of the exceptional planes of Walker are such that all the matrix representation sets admit a trivial nucleus. But the orbit structure of Walker's plane is 1, 25 and the flag transitive planes have only one orbit of length 26. Thus π is a new translation plane of order 25 with the smallest translation complement.

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It has been brought to the notice of the authors by a referee that this spread set finds a place in the unpublished thesis of Oakden[10]. The authors have arrived at the spread set independently and the exposition of this paper is in a different spirit altogether. However, the authors thank the referee for the information.

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