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Short communication

Note on pulsatile viscous flow in a tube of elliptical cross-section

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Abstract

Exact solution of the Navier–Stokes equations is obtained for a pulsatile flow of an incompressible viscous fluid in a tube of elliptical cross-section when the pressure gradient is prescribed by Fourier series in time.

Key words: Pulsatile flow, Mathieu functions.

1. Mathematical formulation

The importance of the study of pulsatile flows in channels and pipes is well known. Uchida[1] has obtained the exact solution of pulsating laminar flow superposed on the steady motion in a circular pipe. The flow is assumed to be unidirectional and has a non-vanishing mean. Here, we consider the same flow conditions as in Uchida[1] but in a pipe of elliptical cross section given by $x^2/a^2+y^2/b^2 = 1$ in Cartesian coordinate system. For this flow Navier–Stokes equations reduce to a single equation

$$u_t = -p_z/\rho+\nu(u_{xx}+u_{yy}), \tag{1}$$

where $u(x,y,t)$ is the velocity z-direction, ρ is the density, p is the pressure, ν is the kinematic viscosity and subscripts denote partial differentiation with respect to that variable. Following Uchida[1], we take

$$-p_z/\rho = p_0+\sum_{n=1}^{\infty} p_n \exp(int),\ u = u_0+\sum_{n=1}^{\infty} u_n \exp(int), \tag{2}$$

and substituting (2) in (1), we get

$$u_{nxx}+u_{nyy}+\alpha_n^2 u_n = p_n/\nu,\ \alpha_n^2 = -\mathrm{in}/\nu,\ n \geq 0. \tag{3}$$

The boundary conditions for the flow are given by

$$u_n = 0 \text{ on } x^2/a^2+y^2/b^2 = 1,\ n \geq 0, \tag{4}$$

as the velocity is zero on the boundary of the tube.

2. Exact solution

By introducing elliptic cylindrical coordinates (ξ, η, z) defined by

$$x = c \cosh \xi \quad \cos \eta, \; y = c \sinh \xi \quad \sin \eta, \quad z = z; \tag{5}$$

the equation (3) transforms into

$$u_{n\xi\xi} + u_{n\eta\eta} + \alpha_n^2 (c^2/2) \quad (\cosh 2\xi - \cos 2\eta)(u_n - p_n/\nu\alpha_n^2) = 0, \quad n \geq 0. \tag{6}$$

The corresponding boundary condition is

$$u_n = 0 \text{ on } \xi = \xi_0, \; n \geq 0. \tag{7}$$

The solution of (6) satisfying (7) is given in terms of Mathieu functions as McLachlan[2] and the final solution is

$$u = \frac{p_0 c^2}{8\nu \cosh 2\xi_0} (\cosh 2\xi - \cosh 2\xi_0)(\cosh 2\xi_0 - \cos 2\eta)$$

$$+ \sum_{n=1}^{\infty} \frac{p_n \exp(int)}{\nu\alpha_n^2} \left[1 - 2\pi \sum_{m=0}^{\infty} \frac{A_0^{(2m)} Ce_{2m}(\xi, \theta_n) ce_{2m}(\eta, \theta_n)}{L_{2m} Ce_{2m}(\xi_0, \theta_n)} \right] \tag{8}$$

where Ce_{2m} and ce_{2m} are modified Mathieu and Mathieu functions respectively, $A_0^{(2m)}$ is a known constant, $\theta_n = \alpha_n^2 c^2/4$ and $L_{2m} = \int_0^{2\pi} ce_{2m}^2(\eta, \theta_n) d\eta$. The orthogonal property of Mathieu functions ce_{2m} which are complete in interval $(0, 2\pi)$ is used to obtain the above result. The approximate series solution for the special case of (7) with $n = 1$ which is not complete has been presented recently by Mehrotra *et al*[3]. The complete exact solution for a pulsatile flow in an elliptic cylinder is given by (8).

The results corresponding to a circular cylinder are obtained by taking the eccentricity of the ellipse as zero ($a = b$ or $c \to 0$). As $c \to 0$, the systems of confocal ellipses become system of concentric circles. In this limit, we have from Mclachlan[2],

$$Ce_{2m}(\xi, \theta_n) \to J_{2m}(\alpha_n r), \; Ce_{2m}(\xi_0, \theta_n) \to J_{2m}(\alpha_n a_0)$$

$$Ce_{2m}(\eta, \theta_n) \to \cos_{2m}\theta, \; c_{e0}(\eta, \theta_n) \to A_0^{(0)} = (2)^{-1/2}, \tag{9}$$

$$A_0^{(2m)} \to 0, \; m \geq 1, \; L_{2m} \to \int_0^{2\pi} \cos^2 2m\theta \; d\theta = \pi$$

where $r^2 = x^2 + y^2$ and a_0 is the radius of the cylinder. Using (9) in (8), we obtain

$$u = -p_0(a_0^2 - r^2)/4 + \sum_{n=1}^{\infty} (p_n/\nu\alpha_n^2) \exp(int)[1 - J_0(\alpha_n r)/J_0(\alpha_n a_0)], \tag{10}$$

which coincides with the result given by Uchida[1].

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