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Short Communication

Separation and limit-classification of special fourth-order differential expression

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Abstract

Two examples have been presented which show that there exist some p, q for which $M[y] = -(py')' + qy$ is separated and $M^2[y]$ is in the limit-2 case at infinity and there exist some p, q for which $M[y]$ is separated but $M^2[y]$ is in the limit-3 case at infinity.

Key words: Separation, fourth-order differential expression.

1. *Def. 1.* The differential expression $M[f] = -(pf^{(1)})^{(1)} + qf$, where both p and q are real-valued and $p^{(1)}$ is absolutely continuous on $[0, X]$ for all $X > 0$, is said to be in the *limit-point* case at infinity if only one solution of $M[f] = \lambda f$ belongs to $L^2(0, \infty)$ for all λ with $\operatorname{im}\lambda \neq 0$ [$L^2(0, \infty)$ denotes the set of all complex-valued functions f such that $|f|^2$ is Lebesque integrable on $(0, \infty)$][1].

Def. 2. Let $\Delta(p, q) \subset L^2(0, \infty)$ be defined as follows: $f \in \Delta(p, q)$ if (i) $f \in L^2(0, \infty)$, (ii) $f^{(1)}$ is absolutely continuous locally on $[0, \infty)$ and (iii) $M[f] \in L^2(0, \infty)$ is said to be *separated* if

$$f \in \Delta(p, q) \Longrightarrow (pf^{(1)})^{(1)} \text{ and } qf \in L^2(0, \infty)^{2,3}. \tag{1.1}$$

It is known [Ch. III] that if $M[\cdot]$ is separated then $M[\cdot]$ is in the limit-point case at infinity but not *vice versa*. That is, there exists $M[\cdot]$ which is in the limit-point case at infinity but which is not separated [vide §7,5].

We further know[5,6] that if $M[\cdot]$ is in the limit-point case, then $M^2[\cdot]$, where, with suitable p, q,

$$M^2[f] = (p^2 f^{(2)})^{(2)} - ((2pq - pp^{(2)}) f^{(1)})^{(1)} + (q^2 - (pq^{(1)})^{(1)}) f$$

may be in the limit-2 or in the limit-3 case at infinity *i.e.*, $M^2[f] = \lambda f$ has exactly two or three linearly independent solutions respectively belonging to $L^2(0, \infty)$ for any λ with $\operatorname{im}\lambda \neq 0$.

As 'to be separated' is a stronger condition on $M[\cdot]$, than 'to be in the limit-point case' one may hope that a separated $M[\cdot]$ will generate an $M^2[\cdot]$ which can be definitely qualified as to be in the limit-2 or in the limit-3 case at infinity. The idea of this present note is to exhibit that the actual situation is different. That is there exist some p, q for which $M[\cdot]$ is separated and $M^2[\cdot]$ is in the limit-2 case at infinity and some p, q for which $M[\cdot]$ is separated but $M^2[\cdot]$ is in the limit-3 case at infinity. The examples presented here are the following

Ex. 1. $p(x) = x^3$, $q(x) = 3x$, $x \in [1, \infty)$,
Ex. 2. $p(x) = x^n$, $q(x) = nx^{n-2}$ for $n \geq 4$ for $x \in [1, \infty)$.

In both the examples, $M[\cdot]$ is separated while in Example 1, $M^2[\cdot]$ is in the limit-2 case at infinity but in Example 2, $M^2[\cdot]$ is the limit-3 case at infinity. We discuss the above examples in the following section.

2. The fact that in both the above examples $M[\cdot]$ is separated follows[1] from separation theorem 1, since the coefficients p and q of $M[\cdot]$ satisfy the conditions of the said theorem.

In Example 1, the solutions of $M[y] = 0$ are x and $1/x^3$ which show that $M[\cdot]$ is in the limit-point case at infinity since $x \notin L^2(1, \infty)$ and $1/x^3 \in L^2(1, \infty)$ while the solutions of $M^2[y] = 0$ for the same p, q are $1, x, 1/x^3, 1/x^4$ which show that $M^2[\cdot]$ is in the limit-2 case at infinity since the solutions 1 and x are not in $L^2(1, \infty)$ but $1/x^3$ and $1/x^4$ are in $L^2(1, \infty)$.

In Example 2, the solutions of $M[y] = 0$ are x and x^{-n} $(n \geqslant 4)$ which show that $M[\cdot]$ is in the limit-point case at infinity since $x \notin L^2(1, \infty)$ and $x^{-n} \in L^2(1, \infty)$ for $n \geqslant 4$ while the solutions of $M^2[y] = 0$ for the same p, q are x, $1/x^n$, x^{-n+3} and x^{2-2n} of which the solutions $x \notin L^2(1, \infty)$ but the rest of the solutions are in $L^2(1, \infty)$ for $n \geqslant 4$ showing $M^2[\cdot]$ is in the limit-3 case at infinity.

References

1. KAUFFMAN, R. M., READ, T. T. AND ZETTL, A. — The deficiency index problem for powers of ordinary differential expressions, *Lecture Notes in Mathematics* 621, (Springer-Verlag edited by A. Dold and B. Eckmann), 1977.
2. EVERITT, W. N. AND GIERTZ, M. — Some properties of the domains of certain differential operators, *Proc. Lond. Math. Soc.* 1971, **23**(3), 301–324
3. DAS, J. AND DEY, J. — On the separation property of symmetric ordinary second-order differential expressions, *Questiones Math.*, 1976, **1**, 145–154.
4. DEY, J. — *Some separation problems of ordinary differential equation*, Ph.D. thesis, Department of Pure Mathematics, University of Calcutta, July 1979.
5. NAIMARK, M. A. — *Linear differential operators II*, Ungar, New York, 1968.
6. CHAUDHURI, J. AND EVERITT, W. N. — On the square of a formally self-adjoint differential expression, *J. Lond. Math. Soc.*, 1969, **1**(2), 661–673.