The solution of the nonlinear heat transfer problems having mixed and nonlinear boundary conditions using optimization principles

ANAND M. SHARAN AND BHIMAVARAPU S. REDDY

Faculty of Engineering, Memorial University of Newfoundland, St. John's, Newfoundland, A1B 3X5, Canada.

Abstract

In this work, the equation for obtaining the transient temperature distribution within a solid are derived using the nonlinear finite element analysis. The mixed boundary conditions in this formulation can be made up of four different forms which are: (a) the specified heat flux, (b) the convective heat flux, (c) the radiative heat flux and (d) the specified surface temperature distribution. The nonlinear set of equations are solved using a method based on optimization principles as well as three other numerical techniques. The results show that the methods based on the optimization principles can be successfully used in solving such problems.

Key words: Nonlinear finite elements analysis, heat transfer, optimization principles.

1. Introduction

There are a large number of industrial processes where one has to solve the transient nonlinear finite element-modelled heat transfer equations[1]. The nonlinearities in these problems arise due to two reasons; the first reason is the variation of the material properties with the temperature and the second one is due to the radiative heat flux where higher powers of temperatures are involved. The first type of nonlinearity can be taken care by evaluating the elemental matrices at each time increment; thus these matrices are updated at each time-step. The second type of the nonlinearity can be analysed by formulating the problem using the variational principles or the method of weighted residuals[2,3].

Irrespective of the method used, one arrives at a set of nonlinear partial differential equations which must be solved by an iterative, time-marching scheme. Some of the commonly known techniques which have been used to solve such problems are: (a) the iteration method, (b) the Newton-Raphson method, (c) the Gauss-Seidel iteration method. In reference 4, there is an excellent discussion on some of the routines for solving the nonlinear problems which includes the gear predictor-corrector routine.

The specific contribution of the present investigation is to obtain the solution of such heat transfer problems using the methods based on the optimization principles. These methods have been successfully utilized in obtaining the minima or the maxima of the functions, and have proved as excellent and reliable tools for design problems over a

very long period of time. The application of these techniques in the fluid mechanics area can be seen in Bristeav *et al*'s works[5-8].

These optimization methods are of two types. They are: (a) the direct search methods, and (b) the gradient search methods. In the present work, the heat transfer problem is solved using the Davidon-Fletcher-Powell method (DFP method), which is a gradient method. This method has been selected because it has been accepted as one among a few reliable methods to obtain the optima, but not necessarily the most efficient method in terms of CPU time. The purpose of the present work is to demonstrate the applicability of the methods based on the optimization principles to the nonlinear heat transfer problems. Therefore this reliable method has been selected. The efficiency in terms of the CPU time is quite dependent upon the type of problem being solved; thus in the opinion of authors, to start with, one should choose the method based on reliability rather than on efficiency. These were the reasons for selecting the DFP method. Once the feasibility of solving nonlinear heat transfer problems can be established using a reliable method then one can go for the efficiency.

2. Mathematical formulation

The differential equation of the heat conduction process in solids can be written as[2]

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial T}{\partial z}\right)+Q = \rho c\frac{\partial T}{\partial t} \tag{1}$$

with the boundary conditions

$$T = T_B \text{ on } S_1, \text{ and} \tag{2}$$

$$K_x\frac{\partial T}{\partial x}l_x + K_y\frac{\partial T}{\partial y}l_y + K_z\frac{\partial T}{\partial z}l_z + q + h(T-T_\infty) + \sigma \, \varepsilon(T^4 - T_\infty^4) = 0 \text{ on } S_2. \tag{3}$$

The union of both the surfaces, S_1 and S_2, forms the complete boundary of the solid having a volume V. The functional formulation which is equivalent to eqns (1) and (2) can be written as[1]

$$\chi = \int_V \frac{1}{2}\left[K_x\left(\frac{\partial T}{\partial x}\right)^2 + K_y\left(\frac{\partial T}{\partial y}\right)^2 + K_z\left(\frac{\partial T}{\partial z}\right)^2 + 2\rho c T\frac{\partial T}{\partial t}\right]\mathrm{d}V$$

$$+ \int_S \left[qT + \frac{h}{2}(T-T_\infty)^2 + \sigma \, \varepsilon\left(\frac{T^5}{5} - T_\infty{}^4 T\right)\right]\mathrm{d}S. \tag{4}$$

The contribution of each of the elements can be added up and the resulting global matrices can be written as[1]

$$[C^G]\frac{\partial\{T^G\}}{\partial t} + [K^G]\{T^G\} = \{F_Q^G\} - \{F_q^G\} + \{F_c^G\} + \{F_r^G\} \tag{5}$$

where the expressions for the matrices $[C^e]$, $[K^e]$ and the vectors $\{F^e_Q\}$, $\{F^e_q\}$, $\{F^e_c\}$, $\{F^e_r\}$ are given in Appendix I; the superscripts G and e refer to the global and elemental matrices respectively.

The vector $\{F^G_r\}$ is the global force vector due to the radiative heat transfer. This vector contains the terms of higher powers of the nodal temperatures. Since eqn. (5) represents a system of nonlinear first order differential equations, it must be solved by the nonlinear methods. As mentioned earlier, eqn. (5) can be solved using the DFP method and for comparison purposes by a few other commonly known methods such as (a) the iteration method, (b) the Newton-Raphson method, and (c) the Gauss-Seidel iteration method.

3. Methods of solution

3.1 Transformation of the simultaneous differential equations

There are two commonly known methods for solving the nonlinear set of transient temperature equations which is eqn. (5). It is a system of first order nonlinear differential equation. The first method of solving these types of equations is by using the finite element method defined in the time domain[2]. The second method[3] for solving these equations is by approximating the time derivative using a finite difference scheme. However, the number of computations involved in the first method are very large. On the other hand, the Crank-Nicolson central finite difference method[3,9] is unconditionally stable and can be easily used in the present investigation for obtaining the temperature distribution.

Using this central finite difference method, one can rewrite eqn. (5) in the following form:

$$([K^G] + \frac{2}{\Delta t}[C^G])\{T^G\}_{t+\Delta t/2} = \{A_1\} + \{F^G_Q\}_{t+\Delta t/2} - \{F^G_q\}_{t+\Delta t/2}$$

$$+ \{F^G_c\}_{t+\Delta t/2} + \{F^G_r\}_{t+\Delta t/2} \tag{6}$$

where $\{A_1\}$ is known because all the parameters at time t are known. Equation (6) contains vector $\{F^G_r\}_{t+\Delta t/2}$ and the unknown nodal temperatures whereas, $\{F^G_c\}_{t+\Delta t/2}$ contains T_∞ at $t+\Delta t/2$ which is known. In the present problem, there are no heat generation or specified heat flux terms so that $\{F^G_Q\}_{t+\Delta t/2}$ and $\{F^G_q\}_{t+\Delta t/2}$ are zero. Since the unknown nodal temperatures are present in the nonlinear form here, one has to solve this equation by an iteration technique.

3.2 Methods of solving the nonlinear system of algebraic equations

The method of solving the nonlinear system of algebraic equation such as the Gauss-Seidel iteration technique, or the Newton-Raphson technique, etc., is well known

and will not be discussed in detail here. Only a brief introduction to the application of the DFP method would be desirable and is mentioned here.

If we denote the global temperature vector at the ith iteration stage of the optimization as $\{T_o^i\}_{t+\Delta t/2}$ and substitute on the right hand side of eqn. (6) then we can compute the force vector, $\{F_r^G\}_{t+\Delta t/2}$. Then, this equation reduces to a system of linear equations which can be solved for the unknown temperature vector at time $t+\Delta t/2$. This calculated vector can be denoted as $\{T_c^i\}_{t+\Delta t/2}$. So we can define the objective function θ for the minimization as

$$\theta = \sum_{i=1}^{n} (T_c^i - T_o^i)^2_{t+\Delta t/2} \tag{7}$$

where n denotes the total number of nodes.

Therefore, the global temperature vector $\{T^G\}_{t+\Delta t/2}$ is obtained by minimizing the objective function θ. The iterative procedure of this method can be seen in Rao[10].

4. Numerical example

The transient heat transfer process of the body shown in fig. 1 was studied using the finite element method[1]. It is a ceramic body whose sides were maintained at 300°C and the bottom surface was insulated. The top surface exchanged heat with the surrounding fluid at 50°C. This heat transfer process takes place due to the convection and radiation mechanisms. Initially the whole body was at 300°C, and was analyzed as a two-dimensional problem. Because of the symmetry about the vertical axis, heat transfer analysis was carried out for the right half of the body only. Linear finite triangular
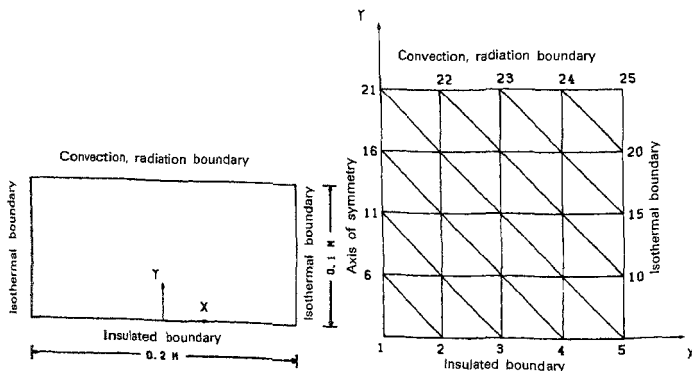

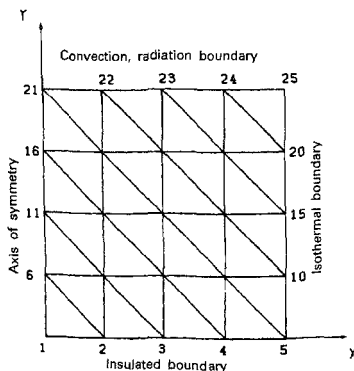FIG. 1. System configuration with various boundary conditions.

FIG. 2. Discretization of the system into finite elements.

elements were used in this study. The finite element discretization of the system is shown in fig. 2. The unsteady heat transfer process of this body is represented by eqn. (6). Equation (6) was solved for $\{T^G\}_{t+\Delta t/2}$ using all the methods mentioned earlier *i.e.* the iteration method, the Newton-Raphson method, the optimization method, and the Gauss-Seidel method.

In the selection of proper size of the elements and the time step Δt, these were simultaneously varied so that the oscillations in the nodal temperature values died out soon. It should be pointed out here that using the consistent matrix approach, these oscillations are always present but they die out with time because the finite element method is an unconditionally stable method. If one uses the lumped matrix approach then these oscillations may not occur but the model would be less accurate. One can reduce the amplitude of each of the frequencies which are present by a suitable combination of the element size and the time step, Δt. In order to compare the finite element results with the finite difference results, this problem was also solved using the finite difference method. The nodes in the finite difference method coincided with those of the finite element method.

5. Results and discussion

Figure 3 shows the temperature distribution of the solid using the iteration technique and the continuous decline of temperatures for all the nodes with time. The temperatures of various nodes oscillate in the beginning, and then die out after some time. At a given time, the oscillation amplitudes are higher for the top surface nodes compared to the nodes at the insulated boundary. The rate of decrease of the temperature at the top surface is very high, whereas the temperatures at the nodes which are on insulated boundary decrease very slowly. At any given time, node 21 is at the lowest temperature. This is because it lies on the symmetrical axis losing heat to the surrounding fluid by convection as well as radiation. Moreover, it is the farthest point from insulated and isothermal surfaces. Figure 4 shows the time-temperature plot of node 1. It is clear from this figure that the temperature distributions obtained by different numerical methods are quite close. As expected, the finite element results show oscillations in the initial time period, whereas finite difference results do not exhibit such a behaviour. However, one has to take care of the stability criteria in the finite difference method. To solve equation (6) by the Newton-Raphson method, one has to compute the Jacobian matrix. This computation requires more computer memory storage and involves more number of computations as compared to the iteration method. The results obtained by the Gauss-Seidel method were the same as those of the Newton-Raphson method. The CPU time required for ten time steps for the various methods are shown in Table I. It shows that the iteration method is the most efficient. It is simpler to use also. The minimum time, in case of the Gauss-Seidel method, is taken when $\omega = 1.0$, *i.e.*, when the situation is unmodified. Both, under and over relaxations lead to increased CPU times. The Newton-Raphson technique yields very good results but not better than the iteration method. The optimization method takes considerable amount of time. As pointed out earlier, the CPU times in the
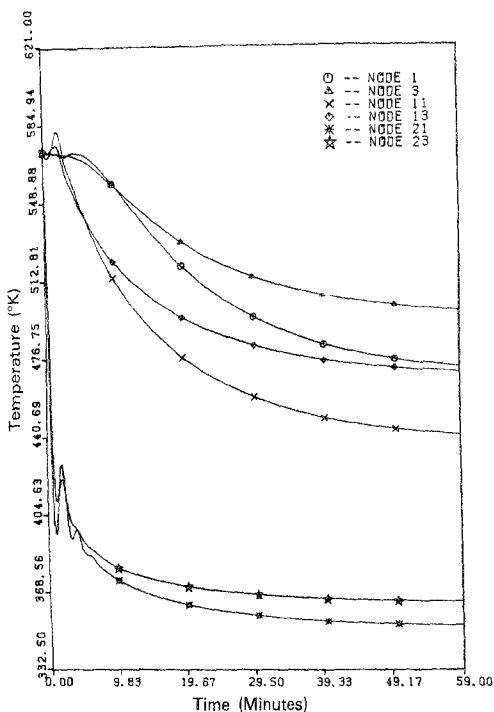
FIG. 3. The temperature–time plot of the ceramic at various nodes using variable metric method.

numerical methods are problem dependent. Therefore, this picture may change completely while solving other types of problems. In the present work only one out of a large number of optimization methods has been used to obtain the transient temperatures. It yielded correct temperatures but was not very efficient as compared to some other methods. However, one should consider other optimization methods which could be quite effective in obtaining the convergence. On the other hand, one can use some of the modifications to the DFP method[11,12] to economize on the CPU time. As a point of clarification it should be added here that the difference in the temperatures obtained at various nodes, with and without radiative terms, were quite significant even
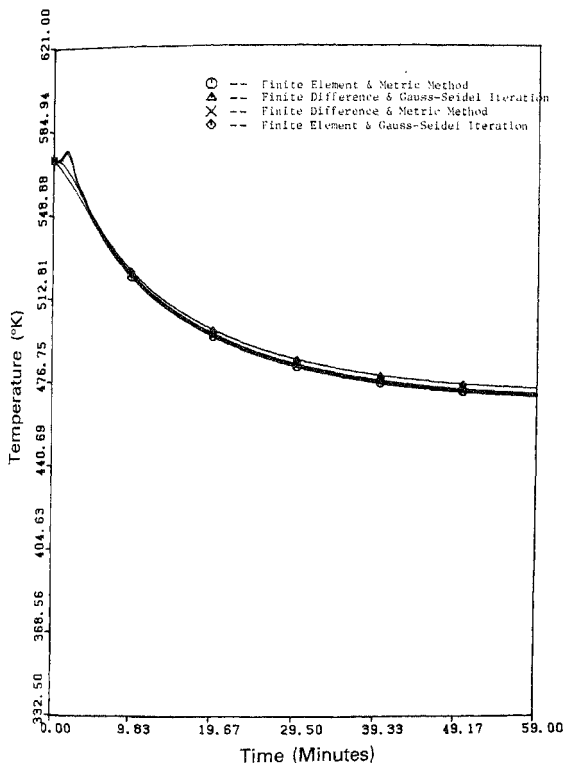
FIG. 4. The temperature–time plot at node 1.

though the solid was only at 300°C (not a very high temperature) initially. This difference will be more if the solid is cooled from higher temperatures.

6. Conclusions

In this paper, the equations for the transient nonlinear temperature distribution within a solid due to the convective and radiative heat flux from the surroundings were obtained using a combination of finite element and finite difference methods. The nonlinearities in this analysis were due to two reasons; the first was due to the variation of the material properties with the temperature and the second was due to the radiative heat flux. The

Table I
CPU time for ten Time Steps for various methods ($\triangle t = 60$ seconds)

Method	CPU time (Seconds)
Iteration	0.092
Newton-Raphson	0.140
Davidon-Fletcher-Powell	0.980
Gauss-Seidel $\omega = 0.2$	1.93
$\omega = 0.4$	0.97
$\omega = 0.6$	0.76
$\omega = 0.8$	0.27
$\omega = 1.0$	0.18
$\omega = 1.2$	0.60
$\omega = 1.4$	0.81

resulting nonlinear heat transfer equations were solved by (a) the iteration method, (b) the Newton-Raphson method, (c) the nonlinear optimization method, and (d) the Gauss-Seidel iteration method. Based on this study it can be concluded that the methods based on the optimization principles can be successfully used in solving nonlinear heat transfer problems.

Nomenclature

c	: specific heat
$[C]$	: capacitance matrix
$[D]$	: thermal material property matrix
$\{f^i\}$	: residual vector
$\{F_c\}$	: force vector due to convection
$\{F_q\}$	: force vector due to the specified heat flux
$\{F_Q\}$	: force vector due to heat generation within the body
$\{F_r\}$	: force vector due to radiation
h	: convection heat transfer coefficient
K_x, K_y, K_z	: thermal conductivities in the x, y and z directions respectively
$[K]$	: thermal conduction matrix
l_x, l_y, l_z	: direction cosines of the normal to the surface
n	: total number of nodes
$[N]$	: shape function matrix
q	: the specified heat flux
Q	: heat generated within the body
S_1	: surface experiencing heat flux
S_2	: surface experiencing convection and radiation heat transfer

t	: time
$\triangle t$	: time increment
$\{T\}$	: nodal temperature vector
T_∞	: fluid temperature
$\{T_0\}$	: optimization temperature vector
$\{T_c\}$	: surface nodal temperatures
ε	: emissivity of the body
θ	: objective function to be minimized for the solution of nonlinear heat transfer equations
ρ	: density of the material
σ	: Stefan-Boltzmann Constant
ω	: relaxation parameter
χ	: a functional

Acknowledgements

References

1. REDDY, B. S. AND SHARAN, A. M. — The transient heat transfer analysis of solids with radiative boundary condition using finite element analysis, *Int. Communs Heat Mass Transfer*, 1985, **12**, 169–178.

2. SEGERLIND, L. J. — *Applied finite element analysis*, John Wiley and Sons, Inc., New York, 1976, 212–222.

3. HUEBNER, K. AND THRONTON, E. A. — *The finite element method for engineers*, John Wiley and Sons, Inc., New York, 1982, 427–428; 431–435.

4. KRINKE, D. C. AND HUSTON, R. L. — An analysis of algorithms for solving differential equations, *Computers Struct.*, 1980, **11**, 69–74.

5. BRISTEAV, M. O. *et al* — Transonic flow simulations by finite element, and least square methods, *Finite Elem. Fluids*, 1982, **4**, 453–482.

6. BRISTEAV, M. O. *et al* — Application of optimal control and finite element methods to the calculation of transonic flows and incompressible viscous flows, *Numerical Methods in Applied Fluid Dynamics*, 1980, 203–312.

7. BRISTEAV, M. O. *et al* — On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite elements (I) Least square formulations and conjugate gradient solution of the continuous problems, *Computer Meth. Appl. Mech. Engng.*, 1979, **17/18**, 619–657.

8. BRISTEAV, M. O. *et al* — A finite element approximation of Navier-Stokes equations for incompressible viscous fluids, iterative methods of solution, *Appro-methods for Navier-Stokes problems*, R. Rautmann (ed.), *Lecture Notes in Math*, Vol. 771, Springer, Berlin, 1980, 78–128.

9. RAO, S. S.   *The finite element method in engineering*, Pergamon Press, Oxford, 1982, 493–494.

10. RAO, S. S.   *Optimization theory and applications*, Wiley Eastern Limited, New Delhi, 1978, 315–327.

11. REDDY, P. J. AND ASGHAR HUSAIN   Multivariable function optimization by quadratically convergent methods, *J. Computer Soc. India*, 1975, **5** (2), 37–42.

12. REDDY, P. J., ZIMMERMAN, H. J. AND ASGHAR HUSAIN   Numerical experiments with DFP method: A powerful minimization technique, *J. Computational Appl. Math*, 1975, **1**, 255–265.

Appendix I

The following are the expressions for the elemental matrices:

$$[C^e] = \int_{v^e} \rho c [N^e]^T [N^e] dV$$

$$[K^e] = \int_{v^e} [B^e]^T [D^e] [B^e] dV + \int_{S_2^e} h [N^e]^T [N^e] dS$$

$$\{F_Q^e\} = \int_{v^e} Q [N^e]^T dV$$

$$\{F_q^e\} = \int_{S_1^e} q [N^e]^T dS$$

$$\{F_c^e\} = \int_{S_2^e} h T_\infty [N^e]^T dS$$

$$\{F_r^e\} = \int_{S_2^e} \sigma \varepsilon T_\infty^4 [N^e]^T dS$$
$$\qquad - \int_{S_2^e} \sigma \varepsilon [N^e]^T ([N^e]\{T^e\})^4 dS.$$