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Symmetric and asymmetric solutions of oscillatory MHD flow due to eccentrically rotating disks

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Abstract

Exact solutions are obtained for the unsteady MHD flow of a viscous, electrically conducting, homogeneous, incompressible fluid between two infinite parallel, insulated, porous disks rotating with angular velocity Ω about two non-coincident axes. The disks are subjected to non-torsional oscillations of different frequencies and a uniform magnetic field is applied normal to the disks. The asymmetric or symmetric solutions containing arbitrary constants reduce to a single unique solution when one prescribes the pressure gradient. Necessary and sufficient conditions for a solution to be symmetric and asymmetric are obtained. In some special cases both the symmetric and asymmetric solutions for eccentrically rotating disks are evaluated numerically and discussed in detail.

Key words: Magnetohydrodynamics, non-coaxial rotating disks.

1. Introduction

Recently, asymmetric flow between two parallel rotating disks for both coaxial and non-coaxial axes of rotation has been studied by Lai *et al*[1]. The complete non-linear Navier-Stokes equations have been solved numerically by Galerkin's method with B-spline test functions and several interesting results were obtained. This work was motivated by the pioneering work of Berker[2,3], who has established the presence of one-parameter family of asymmetric and symmetric solutions for the flow between coaxial and non-coaxial rotating disks. Parter and Rajagopal[4] have investigated the flow between two disks rotating about a common axis, or about different axes and have established the existence of new asymmetric solutions for the full Navier-Stokes equations which are not isolated from classical solutions obtained by von Kármán[5] and Batchelor[6]. By the same numerical method of their earlier paper, Lai *et al*[7] have studied the asymmetric flow of an incompressible viscous fluid above a single rotating disk and generalised von Kármán solution to include non-axisymmetric solutions. The problem corresponding to a single rotating disk in the presence of uniform suction in a streaming flow has been solved by Szeri *et al*[8].

The importance of unsteady flows due to a single rotating disk or two parallel rotating disks is well known. Thornley[9] has presented an exact solution for the flow of an incompressible viscous fluid above a single disk or confined between two infinite disks with one of the disks performing non-torsional oscillations in its own plane in a rotating frame of reference rotating with uniform angular velocity Ω. Superposing the non-torsional oscillations of disks given by Thornley[9], on the results of Berker[3], Ramachandra Rao and Kasiviswanathan[10] have investigated the unsteady flow confined between two non-coaxially rotating disks. Recently, Kasiviswanathan and Ramachandra Rao[11] have studied the flow due to eccentrically rotating porous disk and a fluid at infinity.

Gopinath and Debnath[12] and Debnath[13] have extended the results of Thornley[9] to include the effects due to the presence of a transverse magnetic field. A general study of the unsteady hydrodynamic and hydromagnetic boundary layer flows including the effects of the pressure gradient and uniform suction or blowing has been made by Debnath[14]. An exact solution for the MHD flow of a viscous, incompressible, electrically conducting fluid between two infinite, parallel, insulated disks rotating with same angular velocity about two non-coincident axes under the application of a uniform transverse magnetic field has been obtained by Mohanty[15]. Ramachandra Rao and Raghupathi Rao[16] have studied the steady MHD flow between two disks rotating with different angular velocities about non-coincident parallel axes. This analysis has been further extended by Raghupathi Rao[17] to the case of torsionally oscillating eccentric disks.

Asymmetric and symmetric solutions for the unsteady MHD flow between two infinite, parallel, porous disks rotating with angular velocity Ω about two non-coincident axes are obtained in this paper. The disks are subjected to non-torsional oscillations of different frequencies and a uniform magnetic field is applied normal to the disks. Exact solutions are obtained by introducing an arbitrary boundary condition in the middle plane between the two disks. The arbitrariness of the boundary condition can be removed by prescribing the pressure gradient. The criteria for a solution to be symmetric or asymmetric are presented. Numerical results are discussed in some special cases.

2. Mathematical formulation

Consider the unsteady flow of a conducting, homogeneous, incompressible, viscous fluid between two infinite, insulated, parallel, non-torsionally oscillating, porous, rotating disks with different frequencies, ω_1 and ω_2, which are rotating with an angular velocity Ω about two non-coincident axes. A uniform magnetic field B_0 is applied perpendicular to the disks. Let the upper disk rotate about the point $P_1(x_1, y_1, h)$ and the lower disk about $P_2(-x_1, -y_1, -h)$ and let O the middle point of P_1P_2 be taken as the origin. The Ox and Oy axes which are perpendicular to each other, are chosen perpendicular to z-axis lying in the middle plane given by $z = 0$. The equations governing the flow under the usual MHD approximations[18] are

$$\rho\left(\frac{\partial \bar{V}}{\partial t}+(\bar{V}.\ \nabla)\bar{V}\right)=-\nabla p+\mu\nabla^2\bar{V}+\bar{J}\times\bar{B},\ \nabla.\bar{V}=0, \tag{1}$$

$$\nabla \times \bar{E} = -\partial \bar{B}/\partial t,\ \nabla \cdot \bar{B} = 0, \nabla \times \bar{B} = \mu' \bar{J},\ \bar{J} = \sigma(\bar{E} + \bar{V} \times \bar{B}), \tag{2}$$

where $\bar{V} = (u, v, w)$ is the velocity vector in the Cartesian coordinate system, ρ is the density, p is the pressure, μ is the coefficient of viscosity, $\bar{B}$ is the magnetic field, $\bar{E}$ is the electric field, $\bar{J}$ is the current density, μ' is the magnetic permeability and σ is the electrical conductivity. The velocity components satisfying the constraint of incompressibility for this type of flow are of the form similar to those of Rajagopal[19] and they are given by

$$u = -\Omega[y - g(z, t)],\ v = \Omega[x - f(z, t)],\ w = -W_0, \tag{3}$$

where W_0 is the uniform suction velocity. We observe that for the flow given in (3) the streamlines are concentric circles in planes z = constant for a given time t with centre at the stagnation point (f, g). In each plane the fluid rotates about a stagnation point and the locus of these stagnation points is a space curve Γ.

The initial and boundary conditions for the flow are

$$F = f(z, t) + i\ g(z, t) = 0,\ \partial F/\partial z = 0 \text{ for } t = 0, \tag{4}$$

$$F = a_1 e^{i\omega_1 t} + a_3 e^{-i\omega_1 t} + x_1 + iy_1 \text{ on } z = h \text{ for } t > 0, \tag{5}$$

$$F = a_2 e^{i\omega_2 t} + a_4 e^{-i\omega_2 t} - (x_1 + iy_1) \text{ on } z = -h \text{ for } t > 0, \tag{6}$$

where ω_1 and ω_2 are frequencies of the non-torsional oscillations of the upper and lower disks respectively, a_1, a_2, a_3, a_4 are complex constants which give the amplitudes of the oscillations and the last terms in (5) and (6) are due to the disks rotating about the axes through $(\pm x_1, \pm y_1)$. We assume that the magnetic Reynolds number is small which implies the induced magnetic field is negligible and this enables one to replace $\bar{B}$ by the applied magnetic field B_0. The third equation in (2) is ignored completely but its consequence $\nabla . \bar{J} = 0$ is retained (*see* Mohanty[14]). Since the disks are insulated $J_z = \sigma E_z = 0$. From the first equation of (2) we get $(\partial E_x/\partial z) = (\partial E_y/\partial z) = 0$, which means that E_x and E_y are functions of x, y and t only. Integrating J_x and J_y between $-h$ and h, since the current across the cross-section is zero, we get

$$\int_{-h}^{h} J_x \mathrm{d}z = 0,\quad \int_{-h}^{h} J_y \mathrm{d}z = 0. \tag{7}$$

Using (7) we obtain E_x and E_y as

$$E_x = \Omega B_0(P - x), E_y = \Omega B_0(Q - y), \tag{8}$$

where

$$P = \frac{1}{2h}\int_{-h}^{h} f(z, t)\ \mathrm{d}z,\ Q = \frac{1}{2h}\int_{-h}^{h} g(z, t)\ \mathrm{d}z. \tag{9}$$

Now the expressions for J_x and J_y are obtained by using (8) in the last equation of (2).

Using the expressions for J_x, J_y and velocity components (3) in (1) and eliminating pressure by differentiating with respect to z, as the pressure p is independent of z from the third component of (1), we get

$$\mu g_{zzz}+W_0\rho g_{zz}-\rho g_{zt}-\rho\Omega f_z-\rho B_0^2 g_z = 0, \tag{10}$$

$$\mu f_{zzz}+W_0\rho f_{zz}-\rho f_{zt}+\rho\Omega g_z-\rho B_0^2 f_z = 0, \tag{11}$$

where the suffixes denote the partial differentiation with respect to the corresponding variable. For pseudo-plane motions of first kind (Berker[2]) the velocity field is given by (3) in the absence of W_0. The non-linear governing equations (1) and (2) reduce to linear equations (10) and (11) for the flows given by (3). Now combining (10) and (11), we get

$$\mu F_{zzz}+\rho W_0 F_{zz}-\rho F_{zt}-(i\rho\Omega+\sigma B_0^2)F_z = 0, \tag{12}$$

where $F = f+ig$.

3. Symmetric and asymmetric solutions

Now the problem is to solve (12) subjected to the initial and boundary conditions given in (4)–(6). Taking the Laplace transform of (12) and using (4), we get

$$\mu\bar{F}_{zzz}+\rho W_0\bar{F}_{zz}-(i\rho\Omega+\rho B_0^2+\rho s)\bar{F}_z = 0, \tag{13}$$

where

$$\bar{F}(z,s) = \int_0^\infty F(z,t)\,e^{-st}\,dt. \tag{14}$$

The transformed boundary conditions corresponding to (5) and (6) are given by

$$\bar{F} = \frac{a_1}{s-i\omega_1}+\frac{a_3}{s+i\omega_1}+\frac{x_1+iy_1}{s} \quad \text{on } z = h, \tag{15}$$

$$\bar{F} = \frac{a_2}{s-i\omega_2}+\frac{a_4}{s+i\omega_2}-\frac{x_1+iy_1}{s} \quad \text{on } z = -h. \tag{16}$$

As the ordinary differential equation given in equation (13) is of third order, one needs three boundary conditions to determine the solution completely. But we have only two boundary conditions as given in (15) and (16). For completeness, we assume the third transformed boundary condition arbitrarily on the plane $z = 0$ by

$$\bar{F} = \frac{a_{01}}{s-i\omega_1}+\frac{a_{02}}{s-i\omega_2}+\frac{a_{03}}{s+i\omega_1}+\frac{a_{04}}{s+i\omega_2}+\frac{x_p+iy_p}{s} \quad \text{on } z = 0, \tag{17}$$

where a_{01}, a_{02}, a_{03}, a_{04} and x_p+iy_p are arbitrary complex constants. But afterwards, we shall give a method of evaluating these constants when the pressure gradient is prescribed. The solution of (13) satisfying (15), (16) and (17) is given by

$$\bar{F} = \sum_{r=1}^{4} \frac{a_{or}}{s-i\omega_r} + \frac{x_p+iy_p}{s} + \left[\left(\sum_{r=1}^{4} \frac{a_r-2a_{or}}{s-i\omega_r}\right) - \frac{2(x_p+iy_p)}{s}\right]$$

$$\cdot\left[\frac{\cosh \alpha h\ (e^{-\alpha z}\cosh \beta z-1)}{2(\cosh \beta h-\cosh \alpha h)} + \frac{\cosh \beta h \sinh \alpha h\ e^{-\alpha z} \sinh \beta z}{2 \sinh \beta h\ (\cosh \beta h-\cosh \alpha h)}\right]$$

$$+\left[\left(\sum_{r=1}^{4} \frac{(-1)^{r+1}a_r}{s-i\omega_r}\right) + \frac{2(x_1+iy_1)}{s}\right]\left[\frac{\sinh \alpha h\ (e^{-\alpha z} \cosh \beta z-1)}{2(\cosh \beta h-\cosh \alpha h)}\right.$$

$$\left. + \frac{(\cosh \beta h \cosh \alpha h-1)\ e^{-\alpha z} \sinh \beta z}{2 \sinh \beta h\ (\cosh \beta h-\cosh \alpha h)}\right], \tag{18}$$

where

$$\omega_3 = -\omega_1,\ \omega_4 = -\omega_2,\ \alpha = \rho W_0/2\mu,$$

$$\beta = \frac{1}{2\mu}[(\rho W_0)^2+4\mu(i\rho\Omega+\sigma B_0^2+\rho s)]^{1/2}. \tag{19}$$

The inverse transform is given by

$$F(z,\ t) = \beta = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{F}(z,\ s)\ e^{st}\ ds, \tag{20}$$

where $\gamma > 0$ is a real number in the domain $\bar{F}(z,\ s)$. The integrand in (20) has simple poles at $s = 0,\ i\omega_1,\ i\omega_2,\ i\omega_3,\ i\omega_4, -(\mu/\rho)[(n^2\pi^2/h^2)-\alpha^2-(i\Omega\rho/\mu)-\sigma B_0^2/\mu)]$ for $n = 1,2,\ldots,$ and double poles at $(-4n^2\pi^2\mu/h^2\rho-\sigma B_0^2/\rho-i\Omega\pm 4in\pi\alpha\mu/h\rho)$ for $n = 1,2,\ldots.$ The residues at the poles $s = 0,\ i\omega_r,\ r = 1,2,3,4$ give the steady-state oscillatory solutions whereas the residues at the other poles give the transient part of the solution which vanishes at $t\to\infty$. The full transient solution is not presented for briefness; however, it is presented for a special case in which $\omega_1 = \omega_2 = \omega$ and $W_0 = 0$. The general steady-state oscillatory solution is given by

$$F(z,\ t) = x_p+iy_p+\sum_{r=1}^{4} a_{\text{or}}e^{i\omega_r t}+\sum_{r=1}^{4} (a_r-2a_{\text{or}})$$

$$\cdot\left[\frac{\cosh \alpha h\ (e^{-\alpha z} \cosh \beta_r z-1)}{2(\cosh \beta_r h-\cosh \alpha h)} + \frac{\cosh \beta_r h \sinh \alpha h\ e^{-\alpha z} \sinh \beta_r z}{2 \sinh \beta_r h\ (\cosh \beta_r h- \cosh \alpha h)}\right]$$

$$\cdot\ e^{i\omega_r t}-(x_p+iy_p)\left[\frac{\cosh \alpha h\ (e^{-\alpha z} \cosh \beta_5 z-1)}{(\cosh \beta_5 h-\cosh \alpha h)}\right.$$

$$\left.+\frac{\cosh \beta_5 h \sinh \alpha h\ e^{-\alpha z} \sinh \beta_5 z}{\sinh \beta_5 h\ (\cosh \beta_5 h-\cosh \alpha h)}\right]+\sum_{r=1}^{4} (-1)^{r+1} a_r\ e^{i\omega_r t}$$

$$\cdot\left[\frac{\sinh \alpha h\,(e^{-\alpha z}\cosh \beta_r z-1)}{2(\cosh \beta_r h-\cosh \alpha h)}+\frac{(\cosh \beta_r h\cosh \alpha h-1)\,e^{-\alpha z}\sinh \beta_r z}{2\sinh \beta_r h\,(\cosh \beta_r h-\cosh \alpha h)}\right]$$

$$+(x_1+iy_1)\left[\frac{\sinh \alpha h\,(e^{-\alpha z}\cosh \beta_5 h-1)}{(\cosh \beta_5 h-\cosh \alpha h)}\right.$$

$$\left.+\frac{(\cosh \beta_5 h\cosh \alpha h-1)\,e^{-\alpha z}\sinh \beta_5 z}{\sinh \beta_5 h\,(\cosh \beta_5 h-\cosh \alpha h)}\right], \tag{21}$$

where

$$\beta_r=\frac{1}{2\mu}\,[(\rho W_0)^2+4\mu(i\rho\Omega+\sigma B_0^2)+i\rho\omega_r]^{1/2},$$

$$r=1,2,3,4,5 \text{ and } \omega_5=0. \tag{22}$$

The solutions in (21) reduces to that given by Ramachandra Rao and Kasiviswanathan[10] when $\omega_1=\omega_2=\omega$ and $B_0=W_0=0$ and the steady solutions presented by Berker[3] are recovered from (21) when $B_0=W_0=0$ in the absence of forced oscillations. Further, the problem in which $B_0\neq 0$, $W_0=0$, $\omega_1=\omega_2=\omega$ has been studied in detail by Kasiviswanathan and Ramachandra Rao[20]. For the case when $\omega_1=\omega_2=\omega$, the solution (21) reduces to

$$F(z,t)=\frac{1}{2}\left[A\left\{\frac{\cosh \alpha h\,(e^{-\alpha z}\cosh \beta_1 z-1)}{(\cosh \beta_1 h-\cosh \alpha h)}\right.\right.$$

$$\left.+\frac{\cosh \beta_1 h\sinh \alpha h\,e^{-\alpha z}\sinh \beta_1 z}{\sinh \beta_1 h\,(\cosh \beta_1 h-\cosh \alpha h)}\right\}+B\left\{\frac{\sinh \alpha h\,(e^{-\alpha z}\cosh \beta_1 z-1)}{(\cosh \beta_1 h-\cosh \alpha h)}\right.$$

$$\left.\left.+\frac{(\cosh \beta_1 h\cosh \alpha h-1)e^{-\alpha z}\sinh \beta_1 z}{\sinh \beta_1 h\,(\cosh \beta_1 h-\cosh \alpha h)}\right\}+2(a_{01}+a_{02})\right]e^{i\omega t}$$

$$+\frac{1}{2}\left[C\left\{\frac{\cosh \alpha h\,(e^{-\alpha z}\cosh \beta_3 z-1)}{(\cosh \beta_3 h-\cosh \alpha h)}\right.\right.$$

$$\left.+\frac{\cosh \beta_3 h\sinh \alpha h\,e^{-\alpha z}\sinh \beta_3 z}{\sinh \beta_3 h\,(\cosh \beta_3 h-\cosh \alpha h)}\right\}$$

$$+D\left\{\frac{\sinh \alpha h\,(e^{-\alpha z}\cosh \beta_3 z-1)}{(\cosh \beta_3 h-\cosh \alpha h)}\right.$$

$$\left.\left.+\frac{(\cosh \beta_3 h\cosh \alpha h-1)e^{-\alpha z}\sinh \beta_3 z}{\sinh \beta_3 h\,(\cosh \beta_3 h-\cosh \alpha h)}\right\}+2(a_{03}+a_{04})\right]e^{-i\omega t}$$

$$+(x_1+iy_1)\left[\frac{\sinh \alpha h\,(e^{-\alpha z}\cosh \beta_5 z-1)}{(\cosh \beta_5 h-\cosh \alpha h)}\right.$$

$$+\frac{(\cosh\beta_5 h\cosh\alpha h-1)\,e^{-\alpha z}\sinh\beta_5 z}{\sinh\beta_5 h\,(\cosh\beta_5 h-\cosh\alpha h)}\Bigg]$$

$$-(x_p+iy_p)\left[\frac{\cosh\alpha h\,(e^{-\alpha z}\cosh\beta_5 z-1)}{(\cosh\beta_5 h-\cosh\alpha h)}\right.$$

$$\left.+\frac{\cosh\beta_5 h\sinh\alpha h\,e^{-\alpha z}\sinh\beta_5 z}{\sinh\beta_5 h\,(\cosh\beta_5 h-\cosh\alpha h)}-1\right], \tag{23}$$

where

$$A=a_1+a_2-2(a_{01}+a_{02}),\quad B=a_1-a_2,$$

$$C=a_3+a_4-2(a_{03}+a_{04}),\quad D=a_3-a_4.$$

The complete solution including the transient part for a special case in the absence of suction and $\omega_1=\omega_2=\omega$ is given by

$$F(z,t)=\frac{1}{2}\left[A\frac{\cosh m_1 z-1}{\cosh m_1 h-1}+\frac{B\sinh m_1 z}{\sinh m_1 h}+2(a_{01}+a_{02})\right]e^{i\omega t}$$

$$+\frac{1}{2}\left[C\frac{\cosh m_2 z-1}{\cosh m_2 h-1}+D\frac{\sinh m_2 z}{\sinh m_2 h}+2(a_{03}+a_{04})\right]e^{-i\omega t}$$

$$+\left[(x_1+iy_1)\frac{\sinh m_3 z}{\sinh m_3 h}+(x_p+iy_p)\left(1-\frac{\cosh m_3 z-1}{\cosh m_3 h-1}\right)\right]$$

$$+\sum_{n=1}^{\infty}\{Q_1(n,t)\,[\cos(2\pi nz/h)-1]+Q_2(n)(z/h)\,\sin(2\pi nz/h)\}\cdot e^{-D_3\nu t/h^2}$$

$$+\sum_{n=1}^{\infty}Q_3(n)\,\sin(\pi nz/h)\,e^{-D_6\nu t/h^2}, \tag{24}$$

where

$$\begin{aligned}
Q_1(n,t) &= A(32t\nu\pi^2n^2D_1-4N_1h^2)/h^2D_1^2\\
&\quad+B(32t\nu\pi^2n^2D_2-4N_2h^2)/h^2D_2^2\\
&\quad-2(x_p+iy_p)(32t\nu\pi^2n^2D_3-4N_3h^2)/h^2D_3^2,\\
Q_2(n) &= 8\pi n\,(A/D_1+B/D_2-2(x_p+iy_p)/D_3),\\
Q_3(n) &= 2\pi\,(-1)^n.n(C/D_4+D/D_5+2(x_1+iy_1)/D_6),\\
D_1 &= i\,(\Omega+\omega)h^2/\nu+\sigma B_0^2h^2/\mu+4\pi^2n^2,\\
D_2 &= i\,(\Omega-\omega)h^2/\nu+\sigma B_0^2h^2/\mu+4\pi^2n^2,\\
D_3 &= i\,\Omega h^2/\nu+\sigma B_0^2h^2/\mu+4\pi^2n^2,\\
D_4 &= i(\Omega+\omega)h^2/\nu+\sigma B_0^2h^2/\mu+\pi^2n^2,
\end{aligned}$$

$$
\begin{aligned}
D_5 &= i(\Omega-\omega)h^2/\nu+\sigma B_0^2 h^2/\mu+\pi^2 n^2,\\
D_6 &= i\Omega h^2/\nu+\sigma B_0^2 h^2/\mu+\pi^2 n^2,\\
N_1 &= i(\Omega+\omega)h^2/\nu+\sigma B_0^2 h^2/\mu+\pi^2 n^2,\\
N_2 &= i(\Omega-\omega)h^2/\nu+\sigma B_0^2 h^2/\mu-4\pi^2 n^2,\\
N_3 &= i\Omega h^2/\nu+\sigma B_0^2 h^2/\mu-4\pi^2 n^2,\\
m_1 &= \{\, i(\Omega+\omega)/\nu+\sigma B_0^2/\mu\}^{1/2},\\
m_2 &= \{\, i(\Omega-\omega)/\nu+\sigma B_0^2/\mu\}^{1/2},\\
m_3 &= \{\, (i\Omega/\nu+\sigma B_0^2/\mu\}^{1/2} \text{ and } \nu = \mu/\sigma.
\end{aligned} \tag{25}
$$

It is interesting to observe that the transient part in (24) given in terms of infinite series in the last two terms vanish at $z = 0, \pm h$ and thus it consists of the eigenfunctions of the corresponding problem. The solution given in (21) contains five complex arbitrary constants a_{01}, a_{02}, a_{03}, a_{04}, x_p+iy_p whereas (24) contains three complex arbitrary constants $a_{01}+a_{02}$, $a_{03}+a_{04}$ and x_p+iy_p. Each of these arbitrary constants gives one parameter family of solutions.

Any solution, the velocity field of which is symmetric with respect to a point 0 is called a symmetric solution with respect to that point 0. Thus, a symmetric solution satisfies the condition

$$\bar{V}(-x,-y,-z,t) = -\bar{V}(x,y,z,t), \tag{26}$$

and this in our problem implies

$$F(-z,t) = -F(z,t). \tag{27}$$

Here, the symmetric solution is different from the usual axisymmetric solution. Solutions which are not symmetric are called asymmetric solutions. The solutions given in (21), (23) and (24) do not satisfy condition (27) and therefore they are asymmetric solutions. The symmetry condition given in (27) cannot be satisfied by (21) for any choice of the arbitrary constants owing to the presence of suction or injection. Whereas the solution (23) in the absence of suction or injection satisfies (27) when $a_{01} = a_{02} = a_{03} = a_{04} = x_p = y_p = 0$, $a_1+a_2 = 0$ and $a_3+a_4 = 0$. Thus for a symmetric solution all the arbitrary constants become zero and hence it is unique. The symmetric solution corresponding to (24) in the absence of suction, as $t \to \infty$, is given by

$$
\begin{aligned}
F = (x_1+iy_1)\,&\frac{\sinh m_3 z}{\sinh m_3 h} + a_1 e^{i\omega t}\frac{\sinh m_1 z}{\sinh m_1 h}\\
&+ a_3 e^{-i\omega t}\frac{\sinh m_2 z}{\sinh m_2 h},
\end{aligned} \tag{28}
$$

and contains only odd functions in z.

4. Some important properties of the motions

In obtaining the solutions in the previous sections, equation (12) was derived by eliminating pressure by differentiating with respect to z from the equations of motion which has resulted in an increase in the order of the differential equation governing the motion. In order to determine the solutions, an artificial boundary condition (17) was introduced which leads to non-unique solutions. It is well-known that the solution is unique for a motion in which the pressure gradient is prescribed. Now, we examine, by knowing the pressure gradient whether it will be possible to prescribe the boundary conditions at $z = 0$ without any arbitrariness. Using (3) in (1), we get

$$\mu\Omega g_{zz} + \rho W_0 \Omega g_z - \rho\Omega g_t - \rho\Omega^2 f - \sigma\Omega B_0^2 g$$
$$= \frac{\partial p}{\partial x} - \sigma\Omega B_0^2 Q - \rho\Omega^2 x, \tag{29}$$
$$\mu\Omega f_{zz} + \rho W_0 \Omega f_z - \rho\Omega f_t + \rho\Omega^2 g - \sigma\Omega B_0^2 f$$
$$= -\frac{\partial p}{\partial y} - \sigma\Omega B_0^2 P + \rho\Omega^2 y. \tag{30}$$

Combining (29) and (30), we get

$$\nu F_{zz} + W_0 F_z - F_t - \left(i\Omega + \frac{\sigma B_0^2}{\rho}\right)F = i\nabla P', \tag{31}$$

where

$$\nabla P' = \frac{\partial P'}{\partial x} + i\frac{\partial P'}{\partial y} \text{ and } P' = \frac{p}{\rho\Omega} - \frac{\Omega}{2}(x^2+y^2) - \frac{\sigma B_0^2}{\rho}(Qx - Py). \tag{32}$$

Without any loss of generality, in what follows, we base all our discussions for the case $\omega_1 = \omega_2 = \omega$ and $t \to \infty$, as the essential qualitative features remain the same compared with the general case. The pressure can be determined from (29) and (30) but it is not presented here since all our discussions are based on the modified pressure gradient. Modified pressure gradient $\nabla P'$ is calculated from (31) making use of the expression for F given in (23) and it is given by

$$i\nabla P' = \left\{(x_p + iy_p)\left(\frac{\cosh\alpha h}{\cosh\beta_5 h - \cosh\alpha h} + 1\right)\right.$$
$$\left. - (x_1 + iy_1)\frac{\sinh\alpha h}{\cosh\beta_5 h - \cosh\alpha h}\right\}\nu(\alpha^2 - \beta_5^2) + \tfrac{1}{2}\left\{2(a_{01} + a_{02})\right.$$
$$\left. - \left(\frac{A\cosh\alpha h + B\sinh\alpha h}{\cosh\beta_1 h - \cosh\alpha h}\right)\right\}\nu(\alpha^2 - \beta_1^2)e^{i\omega t} + \tfrac{1}{2}\left\{2(a_{03} + a_{04})\right.$$

$$-\left(\frac{C \cosh \alpha h + D \sinh \alpha h}{\cosh \beta_3 h - \cosh \alpha h}\right)\Bigg\} \nu(\alpha^2 - \beta_3^2) e^{-i\omega t}. \tag{33}$$

In the absence of suction, (33) reduces to

$$\begin{aligned} i \nabla P' &= \frac{1}{2}\left\{ \frac{A}{\cosh \beta_1 h - 1} - 2 \ (a_{01} + a_{02})\right\} \nu \beta_1^2 e^{i\omega t} \\ &+ \frac{1}{2}\left\{ \frac{C}{\cosh \beta_3 h - 1} - 2 \ (a_{03} + a_{04})\right\} \nu \beta_3^2 e^{-i\omega t} \\ &- (x_p + iy_p)\left(\frac{1}{\cosh \beta_5 h - 1} + 1\right) \nu \beta_5^2. \end{aligned} \tag{34}$$

If we assume that the modified pressure gradient is prescribed and has the form

$$\nabla P' = p_0 + p_1 \ e^{i\omega t} + p_2 \ e^{-i\omega t}, \tag{35}$$

where p_0, p_1 and p_2 are complex constants, then by comparing (33) or (34) with (35) we can determine the arbitrary constants in (33) or (34) uniquely. Thus for a flow in which the modified pressure gradient is prescribed, we have a unique solution as the boundary condition on $z = 0$ is determined without any arbitrariness. Further, we prove the following theorems.

Theorem 1: Consider the unsteady MHD flow governed by (12) with $W_0 = 0$ and with the boundary conditions

$$F = a_1 e^{i\omega t} + a_3 e^{-i\omega t} + x_1 + iy_1, \ \text{on} \ z = h, \tag{36}$$

$$F = a_2 e^{i\omega t} + a_4 e^{-i\omega t} - (x_1 + iy_1), \ \text{on} \ z = -h, \tag{37}$$

$$F = (a_{01} + a_{02}) e^{i\omega t} + (a_{03} + a_{04}) e^{-i\omega t} + (x_p + iy_p), \ \text{on} \ z = 0. \tag{38}$$

A necessary condition for this flow to have a symmetric solution is that the modified pressure gradient in (34) must be zero. But this condition is not sufficient in general.

Proof: Let the solution be symmetric. The steady oscillatory solution given in (24) with $t \to \infty$ must satisfy the condition (27). This is true only when

$$\begin{aligned} &a_{01} = a_{02} = a_{03} = a_{04} = x_p = y_p = 0 \ \text{and} \\ &a_1 + a_2 = a_3 + a_4 = 0. \end{aligned} \tag{39}$$

Using (39) in the expression for the corresponding modified pressure gradient given in (34), we get $\nabla P' = 0$. Given $\nabla P' = 0$, the solution of (31) with right hand side zero satisfying the corresponding boundary conditions is obtained and is given by

$$F = \left\{\frac{a_1+a_2}{2}\frac{\cosh m_1 z}{\cosh m_1 h}+\frac{a_1-a_2}{2}\frac{\sinh m_1 z}{\sinh m_1 h}\right\}e^{i\omega t}$$

$$+\left\{\frac{a_3+a_4}{2}\frac{\cosh m_2 z}{\cosh m_2 h}+\frac{a_3-a_4}{2}\frac{\sinh m_2 z}{\sinh m_2 h}\right\}e^{-i\omega t}$$

$$+(x_1+iy_1)\frac{\sinh m_3 z}{\sinh m_3 h}. \tag{40}$$

The solution given in (40) does not satisfy the condition (27) for a symmetric solution. Thus $\nabla P' = 0$ is not a sufficient condition in general. Hence the theorem.

We observe that (40) will be a symmetric solution, if

$$\text{(i) } a_1+a_2 = 0,\ a_3+a_4 = 0, \tag{41}$$

or

$$\text{(ii) } a_1 = a_2 = a_3 = a_4 = 0. \tag{42}$$

In view of (41), $\nabla P' = 0$ will be a sufficient condition for an unsteady flow to have a symmetric solution if the amplitudes of the oscillations of the disks are equal and opposite in sign. Equation (42) implies that the motion is steady and we have the following theorem.

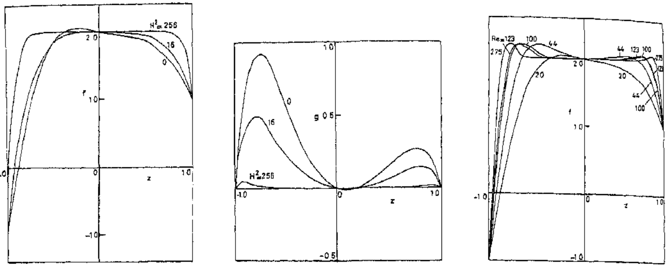
Theorem 2: The necessary and sufficient conditions for a steady MHD flow confined between two rotating disks to have a symmetric solution is that the modified pressure gradient is zero.

We observe that a steady MHD flow, confined between two rotating disks, has an asymmetric solution when the modified pressure gradient is different from zero and *vice versa*. The above results hold good even in the absence of magnetic field. Further we have:

Theorem 3: A sufficient condition for the solution of the flow given in Theorem 1 to be asymmetric is that $\nabla P' \neq 0$.

Proof: Solution in (40) is derived under assumption $\nabla P' = 0$ and it is not a symmetric solution even in the absence of suction. Hence $\nabla P' \neq 0$ is not a necessary condition for the solution to be asymmetric. Given $\nabla P' \neq 0$, we have to prove that the solution is asymmetric.

By comparing (34) and (35), we observe that the arbitrary constants in the boundary conditions at $z = 0$ can be determined uniquely and therefore they cannot satisfy the condition (39) for a symmetric solution. Hénce the theorem.



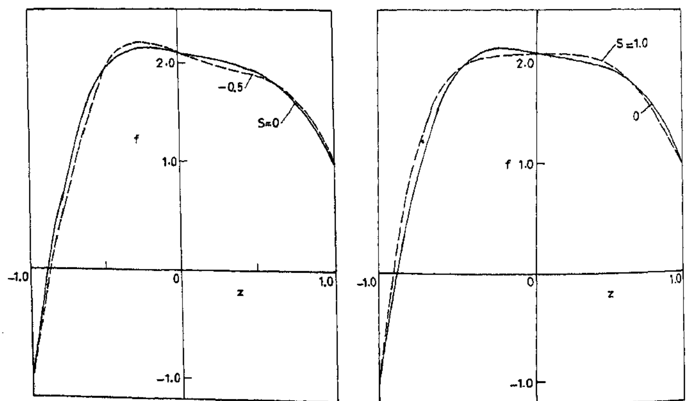
1 2 3

FIGS 1–3. 1. The variations of f when Re $= 20$, $S = 0$ for different values of H^2. 2. The variations of g when Re $= 20$, $S = 0$ for different values of H^2. 3. The variations of f when $S = 0$, $H^2 = 0$ for different values of Re.

It is clear from (33) and (35) that a symmetric solution is not possible for a flow in the presence of suction or injection.

5. Numerical discussion of the results

It is very difficult to clearly understand the effects of suction or injection, magnetic field and unsteadiness on the flow from the exact solutions presented in the third section as



(a) (b)

FIG. 4. The variations of f when Re $= 20$, $H^2 = 0$ for different values of S.

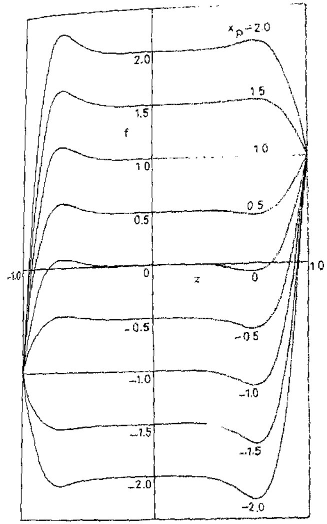


FIG. 5. The variations of f when Re = 100, S = 0, $H^2 = 0$ for different values of x_p.

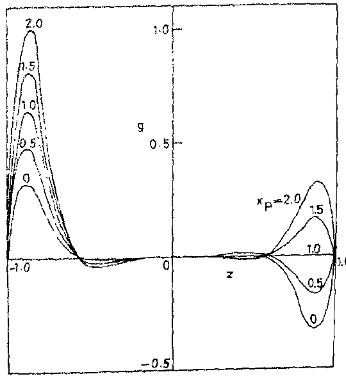


FIG. 6. The variations of g when Re = 100, S = 0, $H^2 = 0$ for different values of x_p.

the form remains the same in all cases. Some insight into the problem is obtained by evaluating the physically interesting quantities numerically in some special cases for various non-dimensional numbers like Re $= \Omega h^2/\nu$ (Reynolds number), $W = \omega h^2/\nu$ (Womersley number), $H^2 = \sigma B_0^2 h^2/\mu$ (Hartmann number) and $S = W_0 h/2\nu$ (suction parameter), governing the flow. In order to understand the effect of various parameters, we consider the numerical results for the flow of the fluid between two disks in different situations. In the unsteady case we will consider both disks oscillate with the same frequency. Solution (23) is non-dimensionalised and is computed for different values of the parameters for the special case in which the dimensionless constants are chosen as

$$a_1 = a_2 = a_3 = a_4 = 0;$$
$$a_{01} = a_{02} = a_{03} = a_{04} = 0;\ x_p = 2,\ y_p = 0;\ x_1 = 1,\ y_1 = 0. \tag{43}$$

The real and imaginary parts of (23) give us f and g and they correspond to a steady asymmetric solution for the choice of constants in (43).

Figures 1 and 2 show the variations of f and g respectively for the case in which Re = 20, S = 0 for different values of H^2. We observe that as Hartmann number

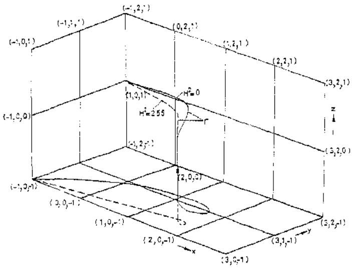


FIG. 7. Locus of stagnation points Γ for Re = 100, $S = 0$ for different values of H^2.

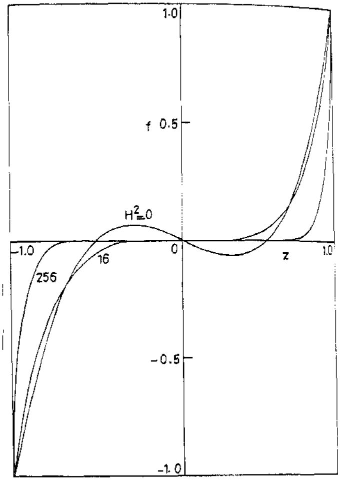


FIG. 8. Symmetric f with Re = 20, $S = 0$ for different values of H^2.

increases the variations of f and g are confined to the regions very near to the disks or the core region in which they do not vary increases with an increase in Hartmann number.

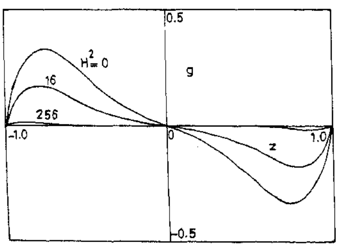


FIG. 9. Symmetric g with Re = 20, $S = 0$ for different values of H^2.

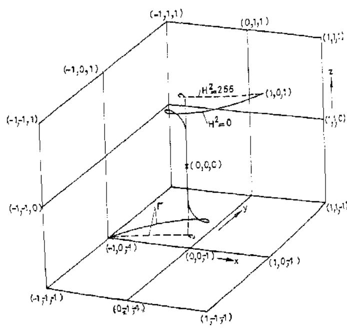


FIG. 10. Symmetric Γ with Re = 100, $S = 0$, for different values of H^2.

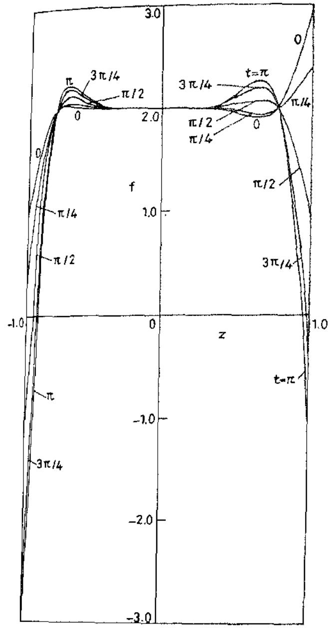


FIG. 11. Asymmetric f with Re = 100, S = 1.0, W = 5.0, H^2 = 0 for different times t.

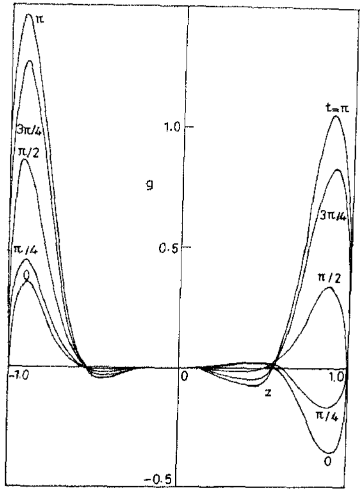


FIG. 12. Asymmetric g with Re = 100, S = 1.0, W = 5.0, H^2 = 0 for different times t.

The variations of f for $S = 0$, $H^2 = 0$ for different values of Re are depicted in fig. 3 and it is seen that the core region in which f does not vary, increases with an increase in Re. Boundary layer-type behaviour is seen for large Re. The effect of suction or injection on the flow is seen from the variation of f, for Re = 20 and $H^2 = 0$ and for different values of S given in figs 4a and b. Figures 5 and 6 show the variations of f and g respectively for fixed $H^2 = 0$, $S = 0$, Re = 100, for x_p given in (43) taking different values. The curves f pass through the value of x_p at $z = 0$ whereas the g curves pass through zero at $z = 0$ for all x_p. This is due to the fact that Γ, space curve giving the locus of the stagnation points, passes through $(x_p, 0)$ at $z = 0$. The curves for $x_p = 0$ correspond to symmetric solution, and the rest represent asymmetric solutions. The three-dimensional picture of the space curve Γ, for Re = 100, $S = 0$ and for different values of H^2, is depicted in fig. 7. It is interesting to observe that the curve Γ with large H^2 has straight core region larger than the corresponding non-magnetic case confirming the observations made earlier for asymmetric solutions. In order to get the effect of magnetic field on the flow, the symmetric solutions f and g, *i. e.* with $x_p = 0$ in (43) for fixed Re = 20, $S = 0$ and for different values of H^2 are plotted in figs. 8 and 9. It is observed that the curves for f and g flatten in the core region as Hartmann number increases. Figure 10 depicts the locus of the stagnation points Γ for Re = 100, $S = 0$ and for different values of H^2. These three-dimensional curves confirm the fact that magnetic field increases the straight core region even for symmetric solutions.

Now we pass on to the study of unsteady flows. Here we choose the dimensionless constants as

$$a_1 = a_2 = a_3 = a_4 = 1;$$
$$a_{01} = a_{02} = a_{03} = a_{04} = 0;\ x_p = 2,\ y_p = 0;\ x_1 = 1,\ y_1 = 0. \tag{44}$$

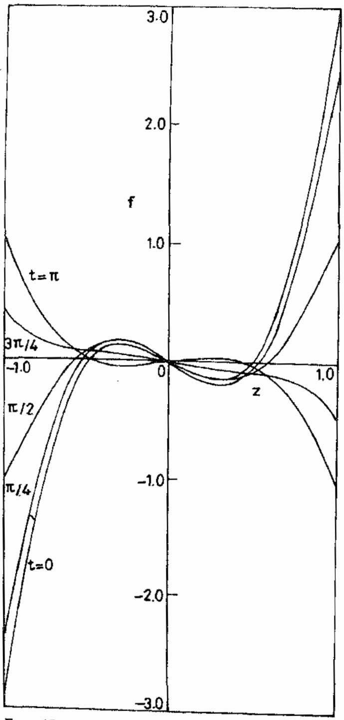


FIG. 13. Symmetric f with Re = 20, $W = 5.0$ $S = 0$, $H^2 = 0$ for different times t.

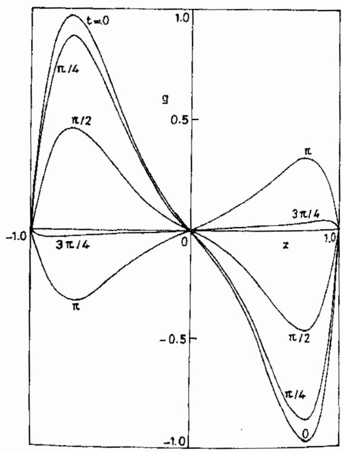


FIG. 14. Symmetric g with Re = 20, $W = 5.0$, $S = 0$, $H^2 = 0$ for different times t.

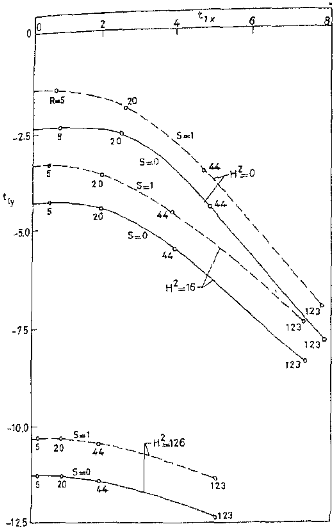


FIG. 15. The shear stress components t_{1x} and t_{1y} on the upper disk for different values of Re, S and H^2

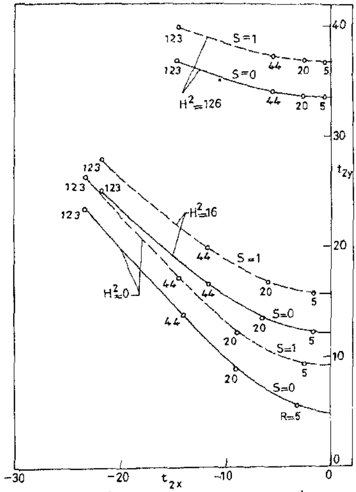


FIG. 16. The shear stress components t_{2x} and t_{2y} on the lower disk for different values of Re, S and H^2.

The flow due to the above choice of the constants is asymmetric and unsteady. Figures 11 and 12 respectively show the variations of f and g for Re $= 100$, $W = 5$, $S = 1.0$, $H^2 = 0$ and for different times t. We observe that the core region is not disturbed by the non-torsional oscillations of the disks. These graphs include all the effects due to suction, unsteadiness except the magnetic field. We have already seen the effect of magnetic field on the solutions. An unsteady motion is symmetric if the amplitudes of the oscillations of the disks are equal and opposite in sign. Figures 13 and 14 depict the variations of f and g respectively for Re $= 20$, $W = 5.0$, $H^2 = 0$, $a_1 = -a_2 = 1$, $a_3 = -a_4 = 1$ and $x_p = 0$. Here the disturbance due to oscillations extends to the entire core region as the Reynolds number for the flow is small.

The stress applied by the fluid at any point of the upper plate is given by t_1 and the components in x, y, z directions are given by

$$t_{1x} = -\mu\omega g_z(h, t), \quad t_{1y} = \mu\omega f_z(h, t), \quad t_{1z} = p. \tag{45}$$

Similarly the stress applied on the lower plate at any point can be calculated.

The non-dimensionalised shear stress components t_{1x} *vs* t_{1y} on the upper disk $z = 1$ for the case given in (43) for different values of Re, H^2 and S are shown in fig. 15. Figure 16 gives non-dimensionalised stress components t_{2x} *vs* t_{2y} on the lower disk for the same values of the parameters. It is clearly seen from figures how the suction and magnetic field affect the shear stress distribution on the disks.

Acknowledgement

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