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On the spectral resolution of a differential operator III

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Abstract

The present paper is concerned with the Reisz means of: (i) the differentiated resolution matrix, and (ii) the differentiated generalized Fourier integrals. The resolution matrix $H(x, y, \lambda)$ is one generating the resolution of the identity of the self-adjoint differential operator

$$M = \begin{pmatrix} -D^2 + p(x) & r(x) \\ r(x) & -D^2 + q(x) \end{pmatrix}, \quad D \equiv \mathrm{d}/\mathrm{d}x. \tag{A}$$

Key words: Resolution matrix, Riemann matrices, Cauchy-type problem, Generalized orthogonal relations, ϕ, θ-Fourier transforms, Levitan-Tauberian theorem, finite function.

1. Introduction

Suppose that

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$$

is a real-valued matrix defined on $(-\infty, \infty)$ and is summable on every finite interval $(a, b) \subset (-\infty, \infty)$. Further, suppose that $Q(x)$ has derivative $Q'(x)$ which is also summable on $(a, b) \subset (-\infty, \infty)$.

Consider the differential system

$$MU = \lambda U \tag{1·1}$$

where M is given by (A), $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and λ is a complex parameter.

The boundary conditions considered are

$$[U, \phi_l]_a = 0 = [U, \phi_j]_b, l = 1, 2; \; j = 3, 4, \text{ with } [\phi_1, \phi_2] = 0 = [\phi_3, \phi_4],$$

where ϕ_l, ϕ_j are the boundary condition vectors of (1.1), *i.e.* vectors which together with their first derivatives take prescribed constant values independent of λ at $x = a$ and $x = b$, respectively; $[U, V]_\alpha$ is the value of the bilinear concomitant

$$[U,V] = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} + \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}, \quad U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad V = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \quad \text{at } x = \alpha.$$

The Fourier case is obtained when $p = q = r = 0$ in (1.1). It is well known[1-3] that the system (1.1) along with the boundary conditions prescribed as above, gives rise to a self-adjoint eigenvalue problem both in the finite as well as in the singular cases $[0,\infty)$ and $(-\infty,\infty)$.

Let $\phi_r(x,\lambda) = \begin{pmatrix} u_r \\ v_r \end{pmatrix}$, $r = 1,2$, be the solutions of (1.1) satisfying at $x=0$ the conditions $(u_j, v_j, u_j', v_j')|_{x=0} = \varepsilon_j, j = 1,2$, where ε_j is the jth unit vector in R^4.

Further, let

$$\theta_r(x,\lambda) = \begin{pmatrix} x_r \\ y_r \end{pmatrix}, \; r = 1,2,$$

be two other solutions of (1.1) connected with ϕ_r by the relations

$$[\phi_r, \theta_k] = \delta_{rk}; \; [\theta_1, \theta_2] = 0, \; r,k = 1,2.$$

Then ϕ_r, θ_k are linearly independent.

Let $H(x,y,\lambda)$, λ real, be the matrix

$$\begin{aligned} H(x,y,\lambda) &= \lim_{\nu\to 0} \int_0^\lambda \operatorname{im} G(x,y,\sigma+i\nu)\, d\sigma, \; \lambda > 0 \\ &= 0, \qquad \lambda = 0 \qquad\qquad (1.2) \\ &= -\lim_{\nu\to 0} \int_\lambda^0 \operatorname{im} G(x,y,\sigma+i\nu)\, d\sigma, \; \lambda < 0 \end{aligned}$$

where $G(\cdot)$ is the Green's matrix associated with the system (1.1). The explicit form of $H(x,y,\lambda)$ involving the matrices

$$\phi = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}, \quad \theta = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

and the matrices ξ, η, ζ occur in authors' paper[3] (p. 158).

If $K(x,y,\lambda) = H(x,y,\lambda-0) - H(x,y,-\infty)$, and

$$E(\lambda): f(x) \to 1/\pi \int_{-\infty}^{\infty} K(x,y,\lambda)\, f(y)\, \mathrm{d}y, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \varepsilon\, L_2(-\infty,\infty),$$

then it has been proved that[3] (p. 160), for real λ,

$$\int_{-\infty}^{\infty} (\tilde{f},g)\, \mathrm{d}x = \int_{-\infty}^{\infty} \lambda \left\{ \mathrm{d} \int_{-\infty}^{\infty} (E(\lambda) f, g) \right\} \mathrm{d}x, \quad \tilde{f} = Mf, g\, \varepsilon\, L_2(-\infty,\infty).$$

This shows that the self-adjoint extension T 'generated' by M and determined by the prescribed set of boundary conditions, is connected with the spectral resolution $E(\lambda)$ 'generated' by $H(x,y,\lambda)$ by the relation

$$T = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}E(\lambda).$$

The matrix $H(x,y,\lambda)$ is the spectral resolution or the resolution matrix associated with the differential system (1.1).

It is noted that discussion of the system of differential equations of type (1.1) is motivated by the fact that the Schröedinger equation for a deuteron (in its ground state) leads to such a system if tensor interaction forces are taken into account.

In the previous papers[3,4], we considered a number of properties of the matrix $H(x,y,\lambda)$ including some asymptotic formulae. An equiconvergence theorem involving the generalized Fourier integral of a vector-valued function $f(x)\ \varepsilon\ L_2(-\infty,\ \infty)$, as also the spectral representation theorem, and the generalized Parseval identity have been derived.

The object of the present paper is to study certain theorems on the asymptotic behaviour of the Riesz means of the derivatives of $H(x,y,\lambda)$ and to obtain a theorem on the Riesz summability of expansions of the differentiated generalized Fourier integral and of the Fourier integral of vector-valued functions of class $L_2(-\infty,\ \infty)$, associated with the system (1.1). The results can be extended to hold for similar problems involving higher order derivatives of $H(x,y,\lambda)$.

Introduction of abstract Hilbert space in mathematics leads to tremendous developments of abstract spectral theory of self-adjoint operators in the Hilbert space. But this theory sometimes fails to answer certain questions or it cannot give complete answers to certain specific problems, and a separate theory becomes necessary. Spectral theory of differential operators has thus received considerable attention in the present day analysis and a complete volume dealing with such theory by Levitan and Sargsyan[6] is an illustration in support of the point. Various methods are also being tried to develop the general theory. For example, in recent papers[7,8] Langer and Textorius define the notion of a spectral function for a symmetric linear relation with a directing mapping by means of a Parseval-Bessel inequality and apply their results to pairs of formally symmetric differential expressions and Hermitian differential systems. The papers contain useful references in connection with certain recent developments in the subject. Their method follows the directing functionals developed by M. G. Krein and others of his school.

In the following, we utilize the ideas initiated by Levitan and Sargsyan[5,6] in solving problems involving derivatives of spectral functions for (i) the Sturm-Liouville operator (ii) higher order linear differential operators and (iii) the Dirac operator. In their investigations they use Fourier cosine transforms and the asymptotic formulae for the derivatives of the eigenfunctions along with solutions of Cauchy problems for a one-dimensional wave equation and a similar problem for a one-dimensional Dirac system; boundary conditions being of type $u|_{t=0} \neq 0$, $\partial u/\partial t|_{t=0} = 0$. The formulation of their problem is such that the Fourier sine transform theory cannot be applied. We solve similar spectral problems arising in the system (1.1), where we use Fourier sine transform theory, the Cauchy type problem for a second order partial differential system satisfied by a vector-valued function u satisfying $u|_{t=0} \neq 0$, $\partial u/\partial t|_{t=0} \neq 0$ and certain results involving the derivatives of the resolution matrix $H(x, y, \lambda)$ not considered by them. Fourier cosine transform is inapplicable in our investigation. Since the analysis runs parallel in some stages to those of Levitan and Sargsyan, we therefore indicate steps only, emphasizing only those parts where we differ considerably.

2. Certain inequalities involving the Riemann matrices

We consider in conjunction with (1.1) the Cauchy type problem

$$\partial^2 U/\partial t^2 = \partial^2 U/\partial x^2 - Q(x)\,U \tag{2.1}$$

with

$$\begin{aligned} U(x,t)|_{t=0} &= f(x) \\ \partial U(x,t)\,\partial t|_{t=0} &= h(x) \end{aligned} \tag{2.1a}$$

where $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $h(x) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \neq 0.$

Then the solution of (2.1) is

$$U(x,t) = \tfrac{1}{2}[f(x+t) + f(x-t) + g(x+t) - g(x-t)] +$$

$$+\frac{1}{2}\int_{x-t}^{x+t} \{W(x,t,s)\,f(s) - T(x,t,s)\,g(s)\}\,\mathrm{d}s \tag{2.2}$$

where $g(x) = \int^x h(t)\,\mathrm{d}t$, an indefinite integral of $h(x)$, and $W(x,t,s)$ and $T(x,t,s)$ are the Riemann matrices associated with the system (2.1).

$$W(x,t,s) = \sum_{r=1}^{\infty} (-1)^r W_r(x,t,s) \tag{2.3}$$

where

$$W_n(x,t,s) = \frac{1}{2} \int_{\Omega\tau_y} Q(y)\, W_{n-1}(y,\tau,s)\, \mathrm{d}\tau \mathrm{d}y,\ n > 1,$$

$\Omega_{\tau y}$ being the domain

$$\Omega_{\tau y}:\ \{0 \le \tau \le t;\ x-t+\tau \le y \le x+t-\tau\};\ x-t \le s \le x+t,$$

$$W_1(x,t,s) = \frac{1}{2} \int_{\frac{1}{2}(s+x-t)}^{\frac{1}{2}(s+x+t)} Q(\sigma)\, \mathrm{d}\sigma. \tag{2.4}$$

Also

$$T(x,t,s) = \sum_{r=1}^{\infty} (-1)^r\, T_r(x,t,s) \tag{2.5}$$

where

$$T_n(x,t,s) = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} Q(y) T_{n-1}(y,\tau,s)\, \mathrm{d}\tau \mathrm{d}y. \tag{2.6}$$

$$T_1(x,t,s) = \frac{1}{2} \left[\int_{s}^{\frac{1}{2}(s+x+t)} + \int_{s}^{\frac{1}{2}(s+x-t)} \right] Q(\sigma)\, \mathrm{d}\sigma. \tag{2.6a}$$

$W_n(x,t,s)$, $T_n(x,t,s)$ satisfy the inequalities

$$|W_n(x,t,s)| \leqslant t^{n-1}/2^n(n-1)! \left(\int_{x-t}^{x+t} |Q(\sigma)|\, \mathrm{d}\sigma \right)^n$$

and

$$|T_n(x,t,s)| \leqslant t^{n-1}/2^{n-1}(n-1)! \left(\int_{x-t}^{x+t} |Q(\sigma)|\, \mathrm{d}\sigma \right)^n$$

for all integral values of $n \geq 1$ (*see* Chakravarty and Roy Paladhi[9], pp. 17–22). The series (2.3) and (2.5) are therefore uniformly convergent for x, t, s, lying in some fixed interval, by the M-test.

Put

$$\int_{x-t}^{x+t} \{|Q'(\sigma)|+|Q(\sigma)|\}\, d\sigma = \chi(x,t).$$

Then $$\left|\frac{\partial}{\partial x} T_n(x,t,s)\right| \leq t^{n-1}/2^{n-1}(n-1)!\,\chi^n(x,t).$$

Differentiating (2.6a), it follows that

$|\partial/\partial x\, T_1(x,t,s)| \leq \chi\,(x,t)$,

where $x-t \leq \frac{1}{2}(s+x-t) \leq s \leq \frac{1}{2}(s+x+t) \leq x+t$.

Let $$|\partial/\partial x\, T_{n-1}(x,t,s)| \leq t^{n-2}/2^{n-2}(n-2)!\;\chi^{n-1}(x,t) \tag{2.7}$$

for a positive integer $n \geq 1$.

Differentiating (2.6) we obtain

$$\begin{aligned}
|\partial/\partial x\, T_n(x,t,s)| &= \frac{1}{2}\left|\int_0^t d\tau\{Q(x+t-\tau)T_{n-1}(x+t-\tau,\tau,s)-\right.\\
&\qquad \left. -Q(x-t+\tau)T_{n-1}(x-t+\tau,\tau,s)\}\right|\\
&= \frac{1}{2}\left|\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} d\{Q(y)\,T_{n-1}(y,\tau,s)\}\,dy\right|\\
&\leq \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} \{|Q'(y)|+|Q(y)|\}\;\{|\,T_{n-1}(y,\tau,s)|+\\
&\qquad +|\,T'_{n-1}(y,\tau,s)|\}\,dy \leq 1/2^{n-1}(n-2)! \int_0^t \tau^{n-2}\,d\tau\times\\
&\qquad \times \int_{x-t}^{x+t} \{|Q'(y)|+|Q(y)|\}\left[\int_{y-\tau}^{y+\tau} \{|Q'(u)|+|Q(u)|\}\,du\right]^{n-1} dy\\
&= t^{n-1}/2^{n-1}(n-1)!\;\chi^n(x,t).
\end{aligned}$$

Hence $|\partial/\partial x T_n(x,t,s)| \leqslant t^{n-1}/2^{n-1}(n-1)!\,\chi^n(x,t)$, (2.8)

valid for all integral values of $n \geqslant 1$.

Hence from (2.5) by uniform convergence we have the inequality, for $x-t \leqslant s \leqslant x+t$,

$$|\partial/\partial x\, T(x,t,s)| \leqslant \int_{x-t}^{x+t} \{|Q'(\sigma)|+|Q(\sigma)|\}\, d\sigma \times$$

$$\times \exp\left[\frac{1}{2}\, t \int_{x-t}^{x+t} \{|Q'(\sigma)|+|Q(\sigma)|\}\, d\sigma\right]. \tag{2.9}$$

Similarly,

$$|\partial/\partial x\, W(x,t,s)| \leqslant \frac{1}{2} \int_{x-t}^{x+t} \{|Q'(\sigma)|+|Q(\sigma)|\}\, d\sigma \times$$

$$\times \exp\left[\frac{1}{2}\, t \int_{x-t}^{x+t} \{|Q'(\sigma)|+|Q(\sigma)|\}\, d\sigma\right]. \tag{2.10a}$$

Let $C(\alpha,\beta,\gamma,\ldots.)$ denote various constants depending on the arguments shown. Then we have

Theorem 2.1: Let $Q(x)$ and $Q'(x)$ in (1.1) be summable in each finite interval. Then for every arbitrary, finite and fixed interval (x_0,x_1) and an arbitrary fixed positive number $t_0>0$ it is possible to determine a constant $C \equiv C(x_0,x_1,t_0)$ such that, for $t<t_0$, $|\partial/\partial x\, T(x,t,s)| < C$ holds.

The theorem is an immediate consequence of the inequality (2.9).

Further, we have

Theorem 2.2: If $Q'(x)$ satisfy the relation

$$\int_{x-t}^{x+t} |Q'(\sigma)|\, d\sigma \leqslant C\, t^{a+1},\ a>0, \tag{2.10}$$

then $\quad |\partial/\partial x\, T(x,t,s)| \leqslant C\, t^{a+1}$,

where $\quad C = C(x_0,x_1), \quad t \leqslant t_0$ and $a \equiv a(x_0,x_1)$.

It is easy to deduce from (2.10) by the mean value theorem that

$$\int_{x-t}^{x+1} |Q(\sigma)|\, d\sigma \leq C\, t^{a+1},\ a>0, \tag{2.11}$$

The theorem then follows from the inequality (2.9).

Similar theorems also hold for $\partial/\partial x\, W(x,t,s)$.

$$\text{Put} \quad K(x,t,s) = \Omega(x,t,s) - A\Omega(x,t,x+t) - B\Omega(x,t,x-t) \tag{2.12}$$

where the constants A, B are defined as follows

$$A = 1, \text{ if } s\,\varepsilon(0, x+\varepsilon) \text{ and } B = 1, \text{ if } s\ \varepsilon\ (0, x-\varepsilon)$$
$$= 0 \quad \text{otherwise}; \qquad\qquad = 0 \quad \text{otherwise};$$

ε being an arbitrary fixed positive number; and $\Omega(x,t,s) = \int^s T(x,t,y)\mathrm{d}y$, the indefinite integral of $T(x,t,s)$.

Since

$$T(x,t,x\pm t) = \frac{1}{2}\int_0^t Q(x\pm s)\mathrm{d}s$$

(*see* Chakravarty and Roy Paladhi[9], formula (3.11)) it follows from (2.11) that

$$T(x,t,x\pm t) \leqslant C\,t^{a+1}$$

Hence if $Q'(x)$ satisfy (2.10), we obtain from theorem 2.2,

$$|\partial/\partial x\, K(x,t,s)| \leqslant C\,t^{a+1},\ a > 0. \tag{2.13}$$

$$|\partial^2/\partial x\,\partial s\, K(x,t,s)| \leqslant C\,t^{a+1},\ a > 0. \tag{2.14}$$

3. Generalized orthogonal relations for derivatives of the resolution matrix

Let $M(x,\lambda) = \begin{pmatrix} p-\lambda & -r \\ -r & q-\lambda \end{pmatrix}$, so that $\operatorname{re} M(x,\lambda) = \begin{pmatrix} p-\mu & -r \\ -r & q-\mu \end{pmatrix}$

and $\operatorname{im} M(x,\lambda) = -\nu I$, where I is the unit 2×2 matrix and $\lambda = \mu + i\nu$.

If $G(x,y,\lambda)$ be the Green's matrix in the singular case $(-\infty,\infty)$ for the system (1.1), it follows from the Titchmarsh formula[10] (p. 34)

$$(\xi-x)^2 h(x) = \int_x^\xi (\xi-y)^2(y-x)h''(y)\mathrm{d}y - \int_x^\xi (6y-2x-4\xi)h(y)\mathrm{d}y$$

that

$$(\xi-x)^2 G(x,y,\lambda) = \int_x^\xi (\xi-u)^2 (u-x) M(u,\lambda) G(u,y,\lambda)\,du + \\ + \int_x^\xi (2x+4\xi-6u)\; G(u,y,\lambda)du \tag{3.1}$$

where ξ is an arbitrary but fixed number. It may be noted that for the uniqueness of the Green's matrix in the singular case quite a number of stringent conditions are to be imposed on $Q(x)$ (*see* Chakravarty[2]). When necessary, such conditions are assumed to be satisfied in what follows.

Taking imaginary parts on both sides of (3.1) and then differentiating the result with respect to x, we obtain

$$(\xi-x)^2\, \partial/\partial x \operatorname{im} G(x,y,\lambda) = 2(\xi-x)\operatorname{im} G(x,y,\lambda) + \\ + \int_x^\xi (2-(\xi-u)^2 \operatorname{re} M(u,\lambda)) \operatorname{im} G(u,y,\lambda)du - \\ - \int_x^\xi (\xi-u)^2 \operatorname{im} M(u,\lambda) \operatorname{re} G(u,y,\lambda)du. \tag{3.2}$$

Putting $\lambda = \mu + i\nu$, $\nu > 0$, it follows from (3.2) and the lemmas 2 and 3 as given in Chakravarty and Roy Paladhi[3] (p. 153) that the integral with respect to μ over any interval of each term on the right hand side of (3.2) is bounded as ν tends to zero uniformly for x in any compact interval which does not contain y. Therefore

$$\lim_{\nu\to 0} \int_0^\mu \partial/\partial x \operatorname{im} G(x,y,\sigma+i\nu)d\sigma \equiv \partial/\partial x\, H(x,y,\mu) \tag{3.3}$$

is bounded in x, where $H(x,y,\mu)$ is the resolution matrix.

Similarly with $\partial/\partial y$ and $\partial^2/\partial x\,\partial y$.

In (3.2) let ν tend to zero. Then using the Schwarz inequality and the inequality

$$|a+b|^2 \leqslant 2(|a|^2+|b|^2), \tag{3.3a}$$

we obtain $|\xi-x|^4\, |\partial/\partial x\, H(x,y,\mu)|^2 \leqslant 8(\xi-x)^2 |H(x,y,\mu)|^2 +$

$$+2\int_x^\xi (2-(\xi-u)^2 \operatorname{re} M(u,\lambda))^2 du \int_x^\xi |H(u,y,\mu)|^2 du.$$

Since, as a function of $y, H(x,y,\mu)\,\varepsilon L_2(-\infty,\infty)$, it follows that considered as a function of $y, \partial/\partial x\, H(x,y,\mu)\,\varepsilon L_2(-\infty,\infty)$. In fact,

$$\int_{-\infty}^{\infty} |\partial/\partial x\, H(x,y,\mu)|^2\, dy < K(x,\mu).$$

Similarly, as a function of x, $\partial/\partial y\, H(x,y,\mu)\,\varepsilon L_2(-\infty,\infty)$, and as a function of x and also as a function of y, $\partial^2/\partial x\,\partial y\, H(x,y,\mu)\,\varepsilon L_2(-\infty,\infty)$.

We now establish the following lemmas required for our further discussions. As before, $K(x,\ldots)$ denotes various constants depending on the arguments shown.

Lemma 3.1: If $\lambda = \mu + i\nu$, $0 < \nu \leqslant 1$, $x \neq y$, then

$$\int_{\alpha}^{\beta} |\partial/\partial x\, \mathrm{im}\, G(x,y,\lambda)|\, d\mu < K(x,y,\alpha,\beta).$$

This immediately follows from (3.2) and the lemmas 2 and 3 in Chakravarty and Roy Paladhi[3] (p. 153).

Lemma 3.2: If $\lambda = \mu + i\nu$, $0 < \nu \leqslant 1$, $x \neq y$, then

$$\int_{\alpha}^{\beta} |\partial/\partial x\, G(x,y,\lambda)|\, d\mu < K(x,y,\alpha,\beta)\, \nu^{-\frac{1}{2}}.$$

Differentiating (3.1) with respect to x,

$$(\xi - x)^2\, \partial/\partial x\, G(x,y,\lambda) = -2(\xi - x)\, G(x,y,\lambda) -$$

$$- \int_{x}^{\xi} (\xi - u)^2 M(u,\lambda)\, G(u,y,\lambda)\, du + 2\int_{x}^{\xi} G(u,y,\lambda)\, du. \tag{3.4}$$

Hence

$$|\xi - x|^2 \int_{\alpha}^{\beta} |\partial/\partial x\, G(x,y,\lambda)|\, d\mu \leqslant 2|\xi - x| \int_{\alpha}^{\beta} |G(x,y,\lambda)|\, d\mu +$$

$$+ \int_{x}^{\xi} |\xi - u|^2\, du \int_{\alpha}^{\beta} |M(u,\lambda)|\, |G(u,y,\lambda)|\, d\mu + 2\int_{x}^{\xi} du \int_{\alpha}^{\beta} |G(u,y,\lambda)|\, d\mu.$$

Since $|M(u,\lambda)| \leqslant C$ for $0 < \nu \leqslant 1$ for all u in $x < u < \xi$, the lemma follows by using lemma 3 in Chakravarty and Roy Paladhi[3] (p. 153).

Lemma 3.3: If $0 < \nu \leqslant 1,\ x \neq y,$ then

$$\int_{\alpha}^{\beta} \mathrm{d}\mu \, \|\partial/\partial x \, G(x,y,\lambda)\|_{-\infty,\infty} < K(x,\alpha,\beta)\nu^{-1},$$

where $\|\partial/\partial x \, G(x,y,\lambda)\|_{-\infty,\infty}$ stands for $\displaystyle\int_{-\infty}^{\infty} |\partial/\partial x \, G(x,y,\lambda)|^2 \, \mathrm{d}y.$

It follows from (3.4), the inequality (3.3a) and the Schwarz inequality that

$$|\xi - x|^4 \int_{\alpha}^{\beta} \mathrm{d}\mu \, \|\partial/\partial x \, G(x,y,\lambda)\|_{-\infty,\infty} \leqslant 8|\xi - x|^2 \int_{\alpha}^{\beta} \mathrm{d}\mu \, \|G(x,y,\lambda)\|_{-\infty,\infty}$$

$$+2 \int_{x}^{\xi} |(\xi - u)^2 M(u,\lambda) - 2|^2 \, \mathrm{d}u \int_{\alpha}^{\beta} \mathrm{d}\mu \, \|G(x,y,\lambda)\|_{-\infty,\infty}.$$

The lemma follows by using lemma 5 in Chakravarty and Roy Paladhi[3] (p. 153) and the inequality $|M(u,\lambda)| \leqslant C,$ for $0 < \nu \leqslant 1,\ u \, \varepsilon (x,\xi).$

Lemma 3.4: If $0 < \nu \leqslant 1,\ x \neq y,$ then

$$\left\| \int_{\alpha}^{\beta} \partial/\partial x \, \mathrm{im} \, G(x,y,\lambda) \, \mathrm{d}\mu \right\|_{-\infty,\infty} < K(x,\alpha,\beta).$$

It follows from (3.2) and the inequality $|a+b+c|^2 \leqslant 3(|a|^2 + |b|^2 + |c|^2)$ that

$$|\xi - x|^4 \left| \int_{\alpha}^{\beta} \partial/\partial x \, \mathrm{im} \, G(x,y,\lambda) \, \mathrm{d}\mu \right|^2 \leqslant 12|\xi - x|^2 \left| \int_{\alpha}^{\beta} \mathrm{im} \, G(x,y,\lambda) \, \mathrm{d}\mu \right|^2 +$$

$$+12 \left| \int_{x}^{\xi} \mathrm{d}u \int_{\alpha}^{\beta} \mathrm{im} \, G(u,y,\lambda) \, \mathrm{d}\mu \right|^2 +$$

$$+3 \left| \int_{x}^{\xi} \left| \xi - u \right|^2 \mathrm{d}u \int_{\alpha}^{\beta} \mathrm{im} \, (M(u,\lambda) \, G(u,y,\lambda)) \, \mathrm{d}\mu \right|^2. \tag{3.5}$$

Now

$$\left| \int_\alpha^\beta \operatorname{im}(M(u,\lambda)\, G(u,y,\lambda))\, \mathrm{d}\mu - \int_\alpha^\beta \operatorname{re} M(u,\lambda) \operatorname{im} G(u,y,\lambda)\, \mathrm{d}\mu \right|$$

$$\leqslant \int_\alpha^\beta |\operatorname{im} M(u,\lambda)|\ |\operatorname{re} G(u,y,\lambda)| \mathrm{d}\mu$$

$$\leqslant \nu \int_\alpha^\beta |G(u,y,\lambda)| \mathrm{d}\mu.$$

Thus

$$\int_\alpha^\beta \operatorname{im}(M(u,\lambda)\, G(u,y,\lambda))\, \mathrm{d}\mu = \int_\alpha^\beta \operatorname{re} M(u,\lambda) \operatorname{im} G(u,y,\lambda)\, \mathrm{d}\mu +$$

$$+ 0\left(\nu \int_\alpha^\beta |G(u,y,\lambda)|\, \mathrm{d}\mu\right),$$

so that

$$\left(\int_\alpha^\beta \operatorname{im}(M(u,\lambda)\, G(u,y,\lambda))\, \mathrm{d}\mu\right)^2 \leqslant 2C^2 \left(\int_\alpha^\beta \operatorname{im} G(u,y,\lambda)\, \mathrm{d}\mu\right)^2 +$$

$$+ 0\left(\nu^2(\beta-\alpha) \int_\alpha^\beta |G(u,y,\lambda)|^2 \mathrm{d}\mu\right).$$

Hence, from (3.5) and the Schwarz inequality,

$$|\xi - x|^4 \left\| \int_\alpha^\beta \operatorname{im} G(u,y,\lambda)\, \mathrm{d}\mu \right\|_{\gamma,\delta} \leqslant 12\, |\xi - x|^2 \left\| \int_\alpha^\beta \operatorname{im} G(x,y,\lambda)\, \mathrm{d}\mu \right\|_{\gamma,\delta} +$$

$$+ 12\, |\xi - x| \left\| \int_\alpha^\beta \operatorname{im} G(x,y,\lambda)\, \mathrm{d}\mu \right\|_{\gamma,\delta} +$$

$$+ 6C^2/5\, (\xi - x)^5 \int_x^\xi \mathrm{d}u \left\| \int_\alpha^\beta \operatorname{im} G(u,y,\lambda)\, \mathrm{d}\mu \right\|_{\gamma,\delta} +$$

$$+ 0\left(\nu^2(\beta-\alpha) \int_\alpha^\beta \mathrm{d}\mu \int_\gamma^\delta |G(x,y,\lambda)|^2\, \mathrm{d}y\right).$$

Now, let γ tend to $-\infty$ and δ to ∞. Then the result follows by utilizing lemma 4 in Chakravarty and Roy Paladhi[3] (p. 153).

Lemmas similar to those of 3.1–3.4 also hold when one replaces $\partial/\partial x$ by $\partial/\partial y$ or $\partial^2/\partial x\,\partial y$.

Let $G(a,b,x,y,\lambda)$ be the Green's matrix for the given boundary value problem of the interval (a,b). Then for non-real $\lambda, \lambda'(\lambda \neq \lambda')$ it is easy to deduce by using Green's theorem and subsequent differentiation under the sign of integration that

$$(\lambda-\lambda') \int_a^b \partial/\partial x\, G(a,b,t,x,\lambda)\, G^{r}(a,b,y,t,\lambda')\,\mathrm{d}t$$

$$= \partial/\partial x\, G(a,b,y,x,\lambda) - \partial/\partial x\, G(a,b,y,x,\lambda'). \tag{3.6}$$

If $G(x,y,\lambda)$ be the Green's matrix in the singular case $(-\infty,\infty)$, by the familiar extension procedure (*see* for example Chakravarty[2], p. 400) we have

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} G(a,b,y,x,\lambda) = G(y,x,\lambda) \tag{3.7}$$

uniformly for y,x,λ with $\operatorname{im}\lambda \gtrless 0$.

Considered as functions of x or y, since $G(a,b,y,x,\lambda)$ and $G(y,x,\lambda)$ satisfy the same differential system (1.1), we obtain from (3.4), for a fixed ξ,

$$(\xi-x)^2\,(\partial/\partial x\,\{G(a,b,y,x,\lambda)-G(y,x,\lambda)\})$$
$$= -2(\xi-x)\,(G(a,b,y,x,\lambda)-G(y,x,\lambda)) -$$

$$-\int_x^{\xi} (\xi-u)^2 M(u,\lambda)\,(G(a,b,y,u,\lambda)-G(y,u,\lambda))\mathrm{d}u +$$

$$+2\int_x^{\xi} (G(a,b,y,u,\lambda)-G(y,u,\lambda))\,\mathrm{d}u. \tag{3.8}$$

Hence, from (3.7) it follows that

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \partial/\partial x\, G(a,b,y,x,\lambda) = \partial/\partial x\, G(y,x,\lambda) \tag{3.9}$$

uniformly for y,x and λ (complex) with $\operatorname{im}\lambda \gtrless 0$.
Again from (3.8)

$$|\xi-x|^4\,||\partial/\partial x\,(G(a,b,y,x,\lambda)-G(y,x,\lambda))||_{a,b}$$
$$\leqslant 8\,|\xi-x|^2\,||G(a,b,y,x,\lambda)-G(y,x,\lambda)||_{a,b}+$$
$$+2\int_x^{\xi}|(\xi-u)^2M(u,\lambda)-2|^2\,\mathrm{d}u\,||G(a,b,y,x,\lambda)-G(y,x,\lambda)||_{a,b} \tag{3.10}$$

by the inequality (3.3a) and the Schwarz inequality. Since

$$\lim_{\substack{a\to-\infty\\ b\to\infty}}\ ||G(a,b,y,x,\lambda)-G(y,x,\lambda)||_{a,b}<\infty,$$

it follows from (3.10) that

$$\lim_{\substack{a\to-\infty\\ b\to\infty}}\ ||\partial/\partial x\,(G(a,b,y,x,\lambda)-G(y,x,\lambda))||_{a,b}<\infty.$$

Hence by the familiar extension procedure (*see* Titchmarsh[11], p. 58; Chakravarty[2] p. 410), we obtain from (3.6)

$$(\lambda-\lambda')\int_{-\infty}^{\infty}\partial/\partial x\,G(t,x,\lambda)\,G^T(y,t,\lambda')\mathrm{d}t=\partial/\partial x\,G(y,x,\lambda)-\partial/\partial x\,G(y,x,\lambda'). \tag{3.11}$$

A similar result holds if we replace $\partial/\partial x$ by $\partial/\partial y$. Also

$$(\lambda-\lambda')\int_{-\infty}^{\infty}\partial/\partial x\,G(t,x,\lambda)\ \partial/\partial y\ G^T(y,t,\lambda')\,\mathrm{d}t$$
$$=\partial^2/\partial x\,\partial y\,(G(y,x,\lambda)-G(y,x,\lambda')). \tag{3.12}$$

From (3.11) and the same formula obtained by replacing λ by its conjugate $\bar{\lambda}$, it follows that

$$\int_{-\infty}^{\infty}\mathrm{d}t\int_0^{\nu}\mathrm{im}\,\partial/\partial x\,G(t,x,\lambda)\,G^T(y,t,\lambda')\,\mathrm{d}\mu$$
$$=\int_0^{\nu}1/(\lambda-\lambda')\,(\mathrm{im}\ \partial/\partial x\,G(y,x,\lambda))\ \mathrm{d}\mu-$$
$$-\nu\int_0^{\nu}1/\{(\lambda-\lambda')\,(\bar{\lambda}-\lambda')\}\ \partial/\partial x\,G(y,x,\bar{\lambda})\mathrm{d}\mu+$$
$$+\nu\,\partial/\partial x\,G(y,x,\lambda')\int_0^{\nu}1/\{(\lambda-\lambda')\,(\bar{\lambda}-\lambda')\}\ \mathrm{d}\mu \tag{3.13}$$

where $\lambda = \mu + i\nu$, $\bar{\lambda} = \mu - i\nu$ and $\lambda' = \mu' + i\nu'$.

By closely following Titchmarsh[11], (pp. 41, 49 and 51) and by utilizing the lemmas 3.1–3.4 at relevant places, we obtain from (3.13)

$$\int_{-\infty}^{\infty} H'(t,x,\nu)G^T(y,t,\lambda')\,\mathrm{d}t = \int_0^{\nu} \mathrm{d}H'(y,x,\mu)/(\mu-\lambda')$$

$$= H'(y,x,\nu)/(\nu-\lambda') + \int_0^{\nu} H'(y,x,\mu)/(\mu-\lambda')^2\,\mathrm{d}\mu \tag{3.14}$$

where $H'(\cdot) = \partial/\partial x\, H(\cdot)$, the partial derivative of the resolution matrix $H(\cdot)$ with respect to x.

Integrating with respect to μ' over $(0, u)$, $u \leqslant \nu$, taking imaginary parts and then proceeding as in Titchmarsh[11] (p. 60) by utilizing the Cauchy singular integral, it follows from (3.14), after some reductions, that

$$\int_{-\infty}^{\infty} \partial/\partial x\; H(t,x,\Lambda)H^T(x,t,\Lambda')\,\mathrm{d}t = \pi\partial/\partial x\, H(y,x,\Lambda\cap\Lambda') \tag{3.15}$$

where $\Lambda = (\alpha,\beta)$, $\Lambda' = (\alpha',\beta')$ and $H(t,x,\Lambda) = H(t,x,\beta) - H(t,x,\alpha)$. In particular,

$$\int_{-\infty}^{\infty} \partial/\partial x\, H(t,x,\Lambda)\; H^T(x,t,\Lambda)\;\mathrm{d}t$$

$$= \pi\,\partial/\partial x\, H(y,x,\Lambda)|_{y=x} = \pi k(x,x,\Lambda), \text{ say.} \tag{3.16}$$

Similarly,

$$\int_{-\infty}^{\infty} \partial/\partial x\, H(t,x,\Lambda)\;\partial/\partial y\, H^T(y,t,\Lambda')\;\mathrm{d}t = \pi\partial^2/\partial x\,\partial y\; H(y,x,\Lambda\cap\Lambda'). \tag{3.17}$$

Finally, proceeding as before[4] (formula 1.14), we obtain

$$\partial/\partial x\, H(y,x,\Lambda) \leqslant \tfrac{1}{2}(g^T(x,x,\Lambda) + g(y,y,\Lambda)) \tag{3.18}$$

where

$$h = \begin{pmatrix} H_{11} & H_{22} \\ H_{11} & H_{22} \end{pmatrix},\; g = \partial h/\partial x,$$

and H_{ij} are the elements of the resolution matrix $H(\cdot)$;

$$\partial^2/\partial x\,\partial y\; H(y,x,\Lambda) \leqslant \tfrac{1}{2}(J^T(x,x,\Lambda) + J(y,y,\Lambda)) \tag{3.19}$$

where $J = \partial g/\partial y = \partial^2 h/\partial x\,\partial y$. The symbol '$\ll$' represents that the matrix on the right 'majorizes' the one on the left. A non-negative $n\times n$ matrix $B = (b_{ij})$ majorizes a complex $n\times n$ matrix $A = (a_{ij})$ (in symbol $A \ll B$), if $|a_{ij}| \leqslant b_{ij}$, $i,j = 1,2,\ldots,n$. (*see* Mirsky[12], p. 328).

From (3.17) it follows that $k(x,x,\Lambda)$ is symmetric and positive in the sense that the corresponding quadratic form is positive.

4. Some preliminary estimates

It easily follows from (2.2) that

$$\lambda^{-\frac{1}{2}} \sin\sqrt{\lambda}\,t\,\phi_j(x,\lambda) = \frac{1}{2}\int_{x-t}^{x+t} (I+\Omega(x,t,s))\phi_j(s,\lambda)\,\mathrm{d}s -$$

$$-\frac{1}{2}(\Omega(x,t,x+t)\int_0^{x+t}\phi_j(s,\lambda)\,\mathrm{d}s - \Omega(x,t,x-t)\int_0^{x-t}\phi_j(s,\lambda)\,\mathrm{d}s \tag{4.1}$$

where $\phi_r, \theta_r, \Omega(\cdot)$ are all defined as before (compare formula (2.2) in Chakravarty and Roy Paladhi[4]).

Let $g_\varepsilon(t)$ be an odd function of t which vanishes outside $(-\varepsilon,\varepsilon)$ and satisfies certain smoothness conditions stated in sections 2 and 3 of the authors' paper[4]; $\psi_\varepsilon(\sqrt{\lambda})$ is the Fourier sine transform of $g_\varepsilon(t)$:

$$\psi_\varepsilon(\sqrt{\lambda}) = \int_0^\varepsilon g_\varepsilon(t)\sin\sqrt{\lambda}\,t\,\mathrm{d}t.$$

Differentiate (4.1) with respect to x, then multiply both sides of the differentiated result so obtained by $g_\varepsilon(t)$ and integrate with respect to t over $(0,\varepsilon)$. Then changing the order of integration we obtain after some easy manipulations

$$\lambda^{-\frac{1}{2}}\psi_\varepsilon(\sqrt{\lambda})\,\phi_j'(x,\lambda) = \frac{1}{2}\int_{x-\varepsilon}^{x+\varepsilon} Q(x,s,\varepsilon)\,\phi_j(s,\lambda)\,\mathrm{d}s \tag{4.2}$$

where

$$Q(x,s,\varepsilon) = g_\varepsilon(s-x)I + \int_{|x-s|}^{\varepsilon} K(x,t,s)g_\varepsilon(t)\,\mathrm{d}t, \tag{4.3}$$

$K(x,t,s)$ being defined by (2.12). Similarly,

$$\lambda^{-\frac{1}{2}}\psi_\varepsilon(\sqrt{\lambda})\,\theta_j'(x,\lambda)=\frac{1}{2}\int_{x-\varepsilon}^{x+\varepsilon}Q(x,s,\varepsilon)\,\theta_j(s,\lambda)\,\mathrm{d}s. \tag{4.4}$$

The relations (4.2) and (4.4) determine the ϕ and θ Fourier transforms, respectively, of the jth row vectors of $Q(x,s,\varepsilon)$, $j=1,2$, in $(x-\varepsilon,x+\varepsilon)$, vanishing outside the interval.

Then by the generalized Parseval relation[3] (p. 151) and the ϕ- and θ- Fourier transforms of row vectors of $Q(x,s,\varepsilon)$ and $Q(y,s,\varepsilon)$ we obtain[3] (p. 158) in view of the explicit representation of the resolution matrix $H(x,y,\lambda)$, the relation

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\lambda^{-1}\psi_\varepsilon^2(\sqrt{\lambda})\,\mathrm{d}_\lambda(\partial^2/\partial x\,\partial y\,H(x,y,\lambda))$$

$$=\frac{1}{4}\int_{\Delta_{xy}}Q(x,s,\varepsilon)\,Q^T(y,s,\varepsilon)\,\mathrm{d}s \tag{4.5}$$

where $\Delta_{xy}=(x-\varepsilon,\ x+\varepsilon)\cap(y-\varepsilon,y+\varepsilon)$.

In particular,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\lambda^{-1}\,\psi_\varepsilon^2(\sqrt{\lambda})\,\mathrm{d}_\lambda\,(\partial^2/\partial x^2 H(x,y,\lambda)|_{y=x}$$

$$=\frac{1}{4}\int_{x-\varepsilon}^{x+\varepsilon}Q(x,s,\varepsilon)\,Q^T(x,s,\varepsilon)\,\mathrm{d}s. \tag{4.6}$$

A similar consideration with the ϕ, θ-transforms of the different row vectors of $Q(x,s,\varepsilon)$ and the same of $P(y,s,\varepsilon)$:

$$\lambda^{-\frac{1}{2}}\psi_\varepsilon(\sqrt{\lambda})\begin{matrix}\phi_j(y,\lambda)\\ \theta_j(y,\lambda)\end{matrix}=\frac{1}{2}\int_{y-\varepsilon}^{y+\varepsilon}P(y,s,\varepsilon)\begin{matrix}\phi_j(s,\lambda)\\ \theta_j(s,\lambda)\end{matrix}\,\mathrm{d}s,$$

where

$$P(y,s,\varepsilon)=\int_{|y-s|}^{\varepsilon}(I+\Omega(y,t,s)-A\Omega(y,t,y+t)-B\Omega(y,t,y-t))g_\varepsilon(t)\,\mathrm{d}t$$

(*see* Chakravarty and Roy Paladhi[4], formulae (2.4) and (2.6)), and the generalized Parseval theorem leads to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \lambda^{-1}\,\psi_\varepsilon^2\,(\sqrt{\lambda})\,\mathrm{d}_\lambda\,(\partial/\partial x\,H(x,y,\lambda))$$

$$=\frac{1}{4}\int_{\Delta_{xy}} Q(x,x,\varepsilon)P^T(y,s,\varepsilon)\,\mathrm{d}s \tag{4.7}$$

with a similar result involving $\partial/\partial y\,H(x,y,\lambda)$.

In particular,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \lambda^{-1}\,\psi_\varepsilon^2\,(\sqrt{\lambda})\,\mathrm{d}_\lambda\,(\partial/\partial x\,H(x,y,\lambda))|_{y=x}$$

$$=\frac{1}{4}\int_{x-\varepsilon}^{x+\varepsilon} Q(x,s,\varepsilon)\,P^T(x,s,\varepsilon)\,\mathrm{d}s. \tag{4.8}$$

In our subsequent discussions we shall require certain lemmas involving derivatives of the resolution matrix $H(x,y,\lambda)$ similar to those obtained in Chakravarty and Roy Paladhi[4] (section 3). The methods of derivation of these lemmas do not materially differ from those in the previous paper with the exception of the use of the results involving the derivatives of $H(x,y,\lambda)$ obtained earlier. We therefore state without proof the final forms of the results in the form of a single lemma.

As before let (x_0,x_1) denote a fixed interval and $x, y\varepsilon(x_0,x_1)$; $C(\alpha,\beta,\dots)$ are various constants depending on the arguments shown.

Suppose that Q, Q' satisfy the relations (2.11) and (2.10) respectively.

If $\lambda=\mu^2$, we write $H(x,y,\lambda)=H_1(x,y,\mu)$; $H_1(\cdot)$ is continued to the negative half line as a matrix each element of which is an odd function of μ.

Let $Y(x,y,\lambda)=\partial/\partial x\,H(x,y,\lambda)$ or $\partial/\partial y\,H(x,y,\lambda)$ or $\partial^2/\partial x\,\partial y\,H(x,y,\lambda)$, the corresponding entities when $H(x,y,\lambda)$ is replaced by $H_1(x,y,\lambda)$ being represented by $Y_1(x,y,\mu)$.

Then we have,

Lemma 4.1: (A) $\displaystyle\int_{-\infty}^{0} \exp\,(\varepsilon_0\sqrt{|\lambda|})\,\mathrm{d}_\lambda\,Y(x,y,\lambda) \ll C(\varepsilon_0,x_0,x_1),\ \varepsilon_0,$

an arbitrary positive number.

In particular, $\int_{-\infty}^{0} \exp(\varepsilon\sqrt{|\lambda|})\, d_\lambda Y(x,y,\lambda)$ is finite for arbitrary finite ε, x, y and $Y(x,y,-\infty)$ is finite.

(B) $Y_1(x,y,\mu+\nu) - Y_1(x,y,\nu) \leqslant C(x_0,x_1)$, for a fixed ν;

that is $\sup_{-\infty<\mu<\infty} \bigvee_{\mu}^{\mu+\nu} Y_1(x,y,\mu) \leqslant C(x_0,x_1)$;

(C) $\int_{-\infty}^{0} |\lambda^{-\frac{1}{2}} \psi_\varepsilon(\sqrt{\lambda})|\, d_\lambda Y(x,y,\lambda) \leqslant C(x_0,x_1)$.

5. Asymptotic formulae involving derivatives of the resolution matrix

Put $g_\varepsilon(t,a) = g_\varepsilon(t)\cos at$, where a is an arbitrary positive number and

$$\psi_\varepsilon(\mu,a) = \int_0^\varepsilon g_\varepsilon(t,a)\sin \mu t\, dt. \tag{5.1}$$

Then $\psi_\varepsilon(\mu,a) = \frac{1}{2}(\psi_\varepsilon(\mu+a)+\psi_\varepsilon(\mu-a))$, where $\lambda=\mu^2$, and $\psi_\varepsilon(\mu,a)$ is an odd function of μ.

Let $f \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \varepsilon L_2(-\infty,\infty)$.

Also let

$$Q(x,s,a,\varepsilon) = g_\varepsilon(s-x,a)I +$$

$$+ \int_{|x-s|}^{\varepsilon} \partial/\partial x(\Omega(x,t,s) - A\Omega(x,t,x+t) - B\Omega(x,t,x-t))g_\varepsilon(t,a)\, dt.$$

From the ϕ- and θ-Fourier transforms of f and the same for the column vectors of $Q(x,s,a,\varepsilon) = \begin{pmatrix} Q_{11} & Q_{21} \\ Q_{12} & Q_{22} \end{pmatrix}$ and of $Q(y,s,a,\varepsilon)$, we obtain, by the generalized

Parseval relation, after some reductions, that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\lambda^{-\frac{1}{2}}\psi_\varepsilon(\sqrt{\lambda},a)\int_{-\infty}^{\infty}\partial/\partial x\, H(x,s,\lambda)\, f(s)\,\mathrm{d}s$$

$$=\frac{1}{2}\int_{x-\varepsilon}^{x+\varepsilon} Q(x,s,a,\varepsilon)f(s)\,\mathrm{d}s. \tag{5.1a}$$

From this, in view of the arbitrariness of the vector f, we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\lambda^{-\frac{1}{2}}\,(\psi_\varepsilon(\sqrt{\lambda}+a)+\psi_\varepsilon(\sqrt{\lambda}-a))\,\mathrm{d}_\lambda\,(\partial/\partial x\, H(x,s,\lambda))$$

$$= Q(x,s,a,\varepsilon),\ \text{for } |x-s|\leqslant\varepsilon \tag{5.2}$$

$$=0,\ \text{for } |x-s|>\varepsilon.$$

Since the uniform boundedness of the integral on the left of (5.2) and the same integral when $\partial/\partial x\, H(\cdot)$ is replaced by $\partial^2/\partial x\,\partial y\, H(x,s,\lambda)$ follows from lemma 4.1 (C), it is possible to differentiate (5.2) with respect to s so as to obtain, if $s>x$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\overline{\lambda}^{\frac{1}{2}}\ \psi_\varepsilon(\sqrt{\lambda},a)\,\mathrm{d}_\lambda\,(\partial^2/\partial x\,\partial s\, H(x,s,\lambda))$$

$$=\partial/\partial s\, g_\varepsilon(s-x,a)I+K(x,s)g_\varepsilon(s-x,a)+$$

$$+\int_{s-x}^{\varepsilon} L(x,s,t)g_\varepsilon(t,a)\,\mathrm{d}t,\ \text{for } s-x\leqslant\varepsilon \tag{5.3}$$

$$=0,\ \text{for } s-x>\varepsilon$$

where $K(x,s)=-\partial/\partial x(\Omega(x,t,s)-A\Omega(x,t,x+t)-B\Omega(x,t,x-t))|_{t=s-x}$ (5.4)

and $L(x,s,t)=\partial^2/\partial x\,\partial s\,\Omega(x,t,s)$. (5.5)

A similar result holds for the case when $s<x$.

Let $\lambda=\mu^2$ and $H_1(x,y,\mu)$ have the same meaning as before. Then from (5.2) we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\mu^{-1}\psi_\varepsilon(\mu-a)\,\mathrm{d}_\mu(\partial/\partial x\, H_1(x,s,\mu))-Q(x,s,a,\varepsilon)\equiv I_1-Q(x,s,a,\varepsilon)$$

$$=-2/\pi\int_{-\infty}^{0}\int_{0}^{\varepsilon}g_\varepsilon(t,a)\sin\sqrt{\lambda}\,t/\sqrt{\lambda}\,\mathrm{d}t\,\mathrm{d}_\lambda(\partial/\partial x\, H(x,s,\lambda)),\ |x-s|\leqslant\varepsilon \tag{5.6}$$

and

$$I_1=-2/\pi\int_{-\infty}^{0}\int_{0}^{\varepsilon}g_\varepsilon(t,a)\sin\sqrt{\lambda}\,t/\sqrt{\lambda}\,\mathrm{d}t\,\mathrm{d}_\lambda(\partial/\partial x\, H(x,s,\lambda),\ |x-s|>\varepsilon.$$

Also from (5.3)

$$1/\pi\int_{-\infty}^{\infty}1/\mu\,\psi_\varepsilon(\mu-a)\,\mathrm{d}\mu\,(\partial^2/\partial x\,\partial s\, H_1(x,s,\mu))$$

$$=\partial/\partial s\, g_\varepsilon(s-x,a)\,I+K(x,s)g_\varepsilon(s-x,a)+\int_{|x-s|}^{\varepsilon}L(x,s,t)g_\varepsilon(t,a)\,\mathrm{d}t-$$

$$-2/\pi\int_{-\infty}^{0}\int_{0}^{\varepsilon}g_\varepsilon(t,a)\sin\sqrt{\lambda}\,t/\sqrt{\lambda}\,\mathrm{d}t\,\mathrm{d}_\lambda(\partial^2/\partial x\,\partial s\, H(x,s,\lambda)),\ |x-s|\leqslant\varepsilon \tag{5.7}$$

$$=-2/\pi\int_{-\infty}^{0}\int_{0}^{\varepsilon}g_\varepsilon(t,a)\sin\sqrt{\lambda}\,t/\sqrt{\lambda}\,\mathrm{d}t\,\mathrm{d}_\lambda(\partial^2/\partial x\,\partial s\, H(x,s,\lambda)),\ |x-s|>\varepsilon.$$

Let $$1/\mu\,\alpha_1(x,s,\mu)=\int_{|x-s|}^{1}\Omega^*(x,t,s)\sin\mu t\,\mathrm{d}t \tag{5.8}$$

where

$$\Omega^*(x,t,s)=\partial/\partial x(\Omega(x,t,s)-A\Omega(x,t,x+t)-B\Omega(x,t,x-t)). \tag{5.9}$$

Then by applying the Parseval theorem for the Fourier sine transform to each element of $1/\mu\,\alpha_1(x,s,\mu)$ and (5.1), we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}1/\mu\ \psi_\varepsilon(\mu-a)\,\alpha_1(x,s,\mu)\,\mathrm{d}\mu$$

$$=\int_{|x-s|}^{\varepsilon}\Omega^*(x,t,s)g_\varepsilon(t,a)\,\mathrm{d}t,\ |x-s|\leqslant\varepsilon$$

$$=0,\ |x-s|>\varepsilon.$$

Again, changing the order of integration,

$$\int_{-\infty}^{0}\int_{0}^{\varepsilon} g_\varepsilon(t,a)\sin\sqrt{\lambda}t/\sqrt{\lambda}\,\mathrm{d}t\,\mathrm{d}_\lambda\,(\partial/\partial x\,H(x,s,\lambda))$$

$$=\int_{0}^{\varepsilon} g_\varepsilon(t,a)\,h_1(x,s,t)\,\mathrm{d}t \tag{5.11}$$

where $h_1(x,s,t) = \displaystyle\int_{-\infty}^{0} \sin\sqrt{\lambda}t/\sqrt{\lambda}\,\mathrm{d}_\lambda\,(\partial/\partial x\,H(x,s,\lambda)).$ (5.12)

(5.12) exists uniformly for x, s, t, by lemma 4.1 (C).

Let $\mu^{-1}\beta_1(x,s,\mu) = \displaystyle\int_{0}^{1} \sin\mu t\,h_1(x,s,t)\,\mathrm{d}t.$ (5.13)

Applying the Parseval theorem for the Fourier sine transform to (5.1) and each element of $\beta_1(x,s,\mu)/\mu$, we obtain for $0<\varepsilon\leqslant 1$

$$\int_{-\infty}^{\infty}\psi_\varepsilon(\mu-a)/\mu\cdot\beta_1(x,s,\mu)\,\mathrm{d}\mu = \pi\int_{0}^{\varepsilon} g_\varepsilon(t,a)\,h_1(x,s,t)\,\mathrm{d}t. \tag{5.14}$$

Let $\Phi_1(x,s,\mu) = \partial/\partial x\,(H_1(x,s,\mu)-H_1^F(x,s,\mu))$,

where $H^F(x,s,\lambda)$ is the resolution matrix for the Fourier case and $H_1^F(x,s,\mu)$ that when λ is replaced by μ^2.

Then from (5.6), (5.10), (5.14) and a result obtained from (5.6) by replacing $H_1(x,s,\mu)$ by $H_1^F(x,s,\mu)$, we obtain

$$\int_{-\infty}^{\infty}\psi_\varepsilon(\mu-a)/\mu\,\mathrm{d}_\mu\,\Phi_1^*(x,s,\mu) = 0 \tag{5.15}$$

where

$$\Phi_1^*(x,s,\mu) = \Phi_1(x,s,\mu)-\int_{0}^{\mu}\alpha_1(x,s,\nu)\,\mathrm{d}\nu+2/\pi\int_{0}^{\mu}\beta_1(x,s,\nu)\,\mathrm{d}\nu. \tag{5.16}$$

Similarly, with

$$\Phi_2(x,s,\mu) = \partial^2/\partial x\,\partial s\,(H_1(x,s,\mu) - H_1^F(x,s,\mu))$$

$$\alpha_2(x,s,\mu)/\mu = \int_{|x-s|}^{1} L(x,s,t)\,\sin\,\mu t\,\mathrm{d}t \tag{5.17}$$

$$\beta_2(x,s,\mu)/\mu = \int_0^1 h_2(x,s,t)\,\sin\,\mu t\,\mathrm{d}t \tag{5.18}$$

$$h_2(x,s,t) = \int_{-\infty}^{0} \sin\,\sqrt{\lambda}\,t/\sqrt{\lambda}\,\mathrm{d}_\lambda\,(\partial^2/\partial x\,\partial s\,H(x,s,\lambda) \tag{5.19}$$

and a relation obtained by changing $H_1(\cdot)$ to $H_1^F(\cdot)$ in (5.7) we obtain

$$\int_{-\infty}^{\infty} \psi_\varepsilon(\mu - a)/\mu\,\mathrm{d}_\mu\,\Phi_2^*(x,s,\mu) = 0 \tag{5.20}$$

where

$$\Phi_2^*(x,s,\mu) = \Phi^*(x,s,\mu) - \int_0^\mu \alpha_2(x,s,\nu)\,\mathrm{d}\nu + 2/\pi \int_0^\mu \beta_2(x,s,\nu)\,\mathrm{d}\nu \tag{5.21}$$

with $\Phi^*(x,s,\mu) = \Phi_2(x,s,\mu) - \frac{1}{2}K(x,s)\,\partial/\partial x\,H_1^F(x,s,\mu).$ (5.22)

The following lemma holds.

Lemma 5.1: Let $Q'(x)$ satisfy the relation (2.10). Then

i) $\displaystyle\int_0^\mu \alpha_1(x,s,\nu)\,\mathrm{d}\nu = o(1);\quad \int_0^\mu \alpha_2(x,s,\nu)\,\mathrm{d}\nu = o(1)$

ii) $\displaystyle\int_0^\mu \beta_1(x,s,\nu)\,\mathrm{d}\nu = -\tfrac{1}{2}\pi\partial/\partial x\,H_1(x,s,-\infty) + o(1)$

iii) $\displaystyle\int_0^\mu \beta_2(x,s,\nu)\,\mathrm{d}\nu = -\tfrac{1}{2}\pi\partial^2/\partial x\,\partial s\,H_1(x,s,-\infty) + o(1)$

as μ tends to ∞ uniformly in every finite domain which contains x and s, and $\alpha_j(\cdot)$, $\beta_j(\cdot)$ can be taken, without generality, to be positive.

The proofs are similar to those in Chakravarty and Roy Paladhi[4]. The difference, however, lies in utilization of the estimates of $\Omega^*(x,t,s)$ of (5.9) and $L(x,t,s)$ of (5.5), *i.e.*

$|\Omega^*(x,t,s)| \leq Ct^{a+1}$ and $|L(x,t,s)| \leq Ct^{a+1}, a > 0$, which can be easily deduced from theorem 2.2.

We now establish

Theorem 5.1: If $Q'(x)$ satisfy (2.10), then

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \, \mathrm{d}_\nu (\partial/\partial x\, H_1(x,s,\nu) - \partial/\partial x\, H_1^f(x,s,\nu)) +$$

$$+ \pi^{\frac{1}{2}} \; 2^{l-\frac{1}{2}} \; \Gamma(l+1)\mu \int_{|x-s|}^{1} \partial/\partial t\, \Omega^*(x,t,s) J_{l+\frac{1}{2}}(\mu t)/(\mu t)^{l+\frac{1}{2}} \, \mathrm{d}t = o(\mu),$$

as μ tends to infinity uniformly for x,s lying in a fixed interval (x_0, s_0), say. Here $J_\nu(\cdot)$ is the Bessel function of order ν and $\Omega^*(\cdot)$ is given by (5.9).

It is easy to show by using lemmas 4.1, 5.1 and the definition of $\psi_\varepsilon(\mu)/\mu$ that the conditions of the Levitan-Tauberian theorem (quoted in the appendix) are satisfied. Hence, from (5.15) and (5.20), for $l \geq 0$,

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \mathrm{d}_\nu \Phi_j^*(x,s,\nu) = o(\mu^{-l}), \; j = 1,2 \tag{5.23}$$

as μ tends to infinity uniformly for x,s contained in a fixed interval (x_0, s_0), say. From (5.23)

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \mathrm{d}_\nu (\partial/\partial x\, H_1(x,s,\nu) - \partial/\partial x\, H_1^F(x,s,\nu)) -$$

$$- \frac{1}{2} \int_0^{\mu} (1-\nu^2/\mu^2)^l \alpha_1(x,s,\nu)\,\mathrm{d}\nu + 2/\pi \int_0^{\mu} (1-\nu^2/\mu^2)^l \; \beta_1(x,s,\nu)\,\mathrm{d}\nu$$

$$= o(\mu^{-l}). \tag{5.24}$$

$$\text{Now } I_1 = \int_0^{\mu} (1-\nu^2/\mu^2)^l \ \alpha_1(x,s,\nu)\,\mathrm{d}\nu$$

$$= \int_{|x-s|}^{1} \Omega^*(x,t,s)\,\mathrm{d}t \int_0^{\mu} (1-\nu^2/\mu^2)^l \ \nu \sin \nu t\,\mathrm{d}\nu$$

by substituting for $\alpha_1(x,s,\nu)$ given by (5.10) and then changing the order of integration. Then, integrating by parts by utilizing the integral derived from differentiation with respect to t of

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \cos \nu t\,\mathrm{d}\nu = \pi^{\frac{1}{2}} \ \ \mu\, 2^{l-\frac{1}{2}}\,\Gamma(l+1) J_{l+\frac{1}{2}}(\mu t)/(\mu t)^{l+\frac{1}{2}}$$

(Watson[13], p. 48), we have, for large values of μ,

$$I_1 = \pi^{\frac{1}{2}} \ \ 2^{l-\frac{1}{2}}\,\Gamma(l+1)\mu \int_{|x-s|}^{1} \partial/\partial t\,\Omega^*(x,t,s)\, J_{l+\frac{1}{2}}(\mu t)/(\mu t)^{l+\frac{1}{2}}\,\mathrm{d}t. \tag{5.25}$$

Again, by integration by parts,

$$I_2 = \int_0^{\mu} (1-\nu^2/\mu^2)^l \beta_1(x,s,\nu)\,\mathrm{d}\nu$$

$$= 2l/\mu^2 \int_0^{\mu} (1-\nu^2/\mu^2)^{l-1}\,\nu\,\mathrm{d}\nu \int_0^{\nu} \beta_1(x,s,u)\,\mathrm{d}u$$

$$\ll C/\mu^2 \int_0^{\mu} (1-\nu^2/\mu^2)^{l-1}\,\nu\,\mathrm{d}\nu \ll C,$$

where

$$\int_0^{\nu} \beta_1(x,s,u)\,\mathrm{d}u \ll \int_0^{\mu} \beta_1(x,s,u\,\mathrm{d}u = O(1),$$

by lemma 5.1 (ii) as μ tends to infinity and C are different constant matrices independent of μ but functions of l, x_0, s_0; (x_0, s_0) an arbitrary but fixed interval. The theorem now follows from (5.23), (5.24) and (5.25).

In an exactly similar manner we have from (5.23), $(j = 2)$,

Theorem 5.2: If $Q'(x)$ satisfy (2.10), the asymptotic formula

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \, d_\nu \{(\partial^2/\partial x\, \partial s(H_1(x,s,\nu) - H_1^F(x,s,\nu)) -$$

$$-\tfrac{1}{2}K(x,s)\, \partial/\partial x\, H_1^F(x,s,\nu)\} +$$

$$+\pi^{\frac{1}{2}} 2^{l-\frac{1}{2}} \Gamma(l+1)\mu \int_{|x-s|}^{1} \partial^2/\partial x\, \partial t\, T(x,t,s)\, J_{l+\frac{1}{2}}(\mu t)/(\mu t)^{l+\frac{1}{2}} = o(\mu) \tag{5.26}$$

holds, as μ tends to infinity uniformly for x, s lying in a fixed interval; $K(x, s)$ is given by (5.4).

6. Summability of the differentiated Fourier expansion

Put $$S(x,\lambda) = \begin{pmatrix} s_{11}(x,\lambda) \\ s_{12}(x,\lambda) \end{pmatrix} = \int_{-\infty}^{\infty} H(x,s,\lambda) f(s)\, ds \tag{6.1}$$

to be called the generalized Fourier integral, where $f(x) \,\varepsilon\, L_2(-\infty, \infty)$ and

$$S^F(x,\lambda) = \int_{-\infty}^{\infty} H^F(x,s,\lambda) f(s)\, ds \tag{6.2}$$

is the Fourier integral corresponding to the system (1.1) with $p = q = r = 0$. When $\lambda = \mu^2$, put

$$H(x,y,\lambda) = H_1(x,y,\mu), H^F(x,y,\lambda) = H_1^F(x,y,\mu), S(x,\lambda) = S_1(x,\mu)$$

and $S^F(x,\lambda) = S_1^F(x,\mu)$; for every fixed x, y we assume, as before, that $H_1(x,y,\mu)$ is continued to the negative half-line as an odd function; with similar consideration for $H_1^F(x,y,\mu)$.

For $\mu < 0$, let $S_1(x,\mu) = -S_1(x,-\mu)$; $S_1^F(x,\mu) = -S_1^F(x,-\mu)$.

Let, $$E_1(\mu) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \phi^T(x,\mu) f(x)\, dx; \quad E_2(\mu) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \theta^T(x,\mu) f(x)\, dx$$

where ϕ, θ are the matrices which occur in the explicit representation for the resolution matrix $H(x, y, \lambda)$ (*see* Chakravarty and Roy Paladhi[3], p. 158). Then

$$S_1(x,\mu) = \int_{-\infty}^{\infty} H_1(x,s,\mu) f(s)\,\mathrm{d}s = \int_0^{\mu} (\{\phi(x,\mu)\,\mathrm{d}\xi(\mu) + \theta(x,\mu)\,\mathrm{d}\eta(\mu)\}\, E_1(\mu) +$$

$$+ \{\phi(x,\mu)\,\mathrm{d}\eta(\mu) + \theta(x,\mu)\,\mathrm{d}\zeta(\mu)\}\, E_2(\mu)),\ \mu > 0$$

with similar relations for $\mu < 0$.

$$\text{Hence } \partial/\partial x\, S_1(x,\mu) = \partial/\partial x \int_{-\infty}^{\infty} H_1(x,s,\mu) f(s)\,\mathrm{d}s$$

$$= \int_0^{\mu} (\{\phi'(x,\mu)\,\mathrm{d}\xi(\mu) + \theta'(x,\mu)\,\mathrm{d}\eta(\mu)\}\, E_1(\mu) +$$

$$+ \{\phi'(x,\mu)\,\mathrm{d}\eta(\mu) + \theta'(x,\mu)\,\mathrm{d}\zeta(\mu)\}\, E_2(\mu)) \tag{6.3}$$

for $\mu > 0$, and similarly for $\mu < 0$.
It evidently follows from (6.3) that

$$\partial/\partial x\, S_1(x,\mu) = \int_{-\infty}^{\infty} \partial/\partial x\, H_1(x,s,\mu) f(s)\,\mathrm{d}s \equiv S_2(x,\mu) \text{ say.} \tag{6.3a}$$

Again, as before we have the relation (*see* formula (5.1a))

$$1/\pi \int_{-\infty}^{\infty} 1/\mu\, \psi_\varepsilon(\mu,a)\, (\phi'(x,\mu)\,\mathrm{d}\xi(\mu)\, E_1(\mu) + \phi'(x,\mu)\,\mathrm{d}\eta(\mu)\, E_2(\mu) +$$

$$+ \theta'(x,\mu)\,\mathrm{d}\eta(\mu)\, E_1(\mu) + \theta'(x,\mu)\,\mathrm{d}\zeta(\mu)\, E_2(\mu)) = \frac{1}{2} \int_{x-\varepsilon}^{x-\varepsilon} Q(x,s,a,\varepsilon) f(s)\,\mathrm{d}s. \tag{6.4}$$

From (5.1), (6.3) and (6.4)

$$\int_{-\infty}^{\infty} 1/\mu\, \psi_\varepsilon(\mu - a)\, \mathrm{d}_\mu(\partial/\partial x\, S_1(x,\mu)) = \pi \int_{x-\varepsilon}^{x+\varepsilon} Q(x,s,a,\varepsilon) f(s)\,\mathrm{d}s -$$

$$-2 \int_{-\infty}^{0} \left(\int_0^{\varepsilon} g_\varepsilon(t,a) \sin \mu t/\mu\,\mathrm{d}t\right) \mathrm{d}_\mu(\partial/\partial x\, S_1(x,\mu)). \tag{6.5}$$

Put $R(x,\mu) = \partial/\partial x(S_1(x,\mu) - S_1^F(x,\mu))$. (6.6)

Then from (6.5) and a similar result involving S_1^F in place of S_1, we obtain

$$\int_{-\infty}^{\infty} \psi_\varepsilon(\mu-a)/\mu \, \mathrm{d}_\mu R(x,\mu) = \pi \int_{x-\varepsilon}^{x+\varepsilon} g(x,s,a,\varepsilon) f(s)\,\mathrm{d}s -$$

$$-2 \int_{-\infty}^{0} \left(\int_0^{\varepsilon} g_\varepsilon(t,a) \sin \mu t/\mu \, \mathrm{d}t \right) \mathrm{d}_\mu (\partial/\partial x\, S_1(x,\mu)) \tag{6.7}$$

where

$$g(x,s,a,\varepsilon) = \int_{|x-s|}^{\varepsilon} K(x,t,s)\, g_\varepsilon(t,a)\,\mathrm{d}t,$$

$K(x,t,s)$ being given by (2.12). Hence using the same procedure as before (*see* section 5), we can ultimately obtain from (6.7)

$$\int_{-\infty}^{\infty} \psi_\varepsilon(\mu)/\mu \, \mathrm{d}_\mu R^*(x,\mu) = 0 \tag{6.8}$$

where $R^*(x,\mu) = R(x,\mu) - \frac{1}{2} \int_0^{\mu} A(x,\nu)\,\mathrm{d}\nu + 2/\pi \int_0^{\mu} B(x,\nu)\,\mathrm{d}\nu$ (6.9)

$$A(x,\nu) = \begin{pmatrix} A_1(x,\nu) \\ A_2(x,\nu) \end{pmatrix} = \int_0^1 \nu y(x,t) \sin \nu t \,\mathrm{d}t$$

$$y(x,t) = \int_{x-t}^{x+t} \partial/\partial x\, K(x,t,s) f(s)\,\mathrm{d}s$$

$$B(x,\nu) = \int_0^1 \nu z(x,t) \sin \nu t\,\mathrm{d}t$$

$$z(x,t) = \int_{-\infty}^{0} \sin \mu t/\mu \, \mathrm{d}_\mu S_2(x,\mu).$$

The Stieltjes integral exists in accordance with Radon's definition[14] (p. 307). By adopting the analysis of Chakravarty and Roy Paladhi[4], we obtain the following lemma.

Lemma 6.1: Let $Q'(x)$ satisfy the condition (2.10) and let $f(x) \varepsilon L_2(-\infty, \infty)$. Then for fixed ν, as μ tends to infinity uniformly in any finite interval containing x,

(A) $$\int_0^\mu A(x,u)\,\mathrm{d}u = o(1); \quad \bigvee_\mu^{\mu+\nu} A(x,u) = o(1);$$

(B) $$\frac{2}{\pi} \int_0^\mu B(x,u)\,\mathrm{d}u = -S_2(x,-\infty) + o(1): \quad \bigvee_\mu^{\mu+\nu} B(x,u) = o(1);$$

(C) $$\bigvee_\mu^{\mu+\nu} S_2(x,u) = o(1).$$

We next obtain

Theorem 6.1: If $f(x) \varepsilon L_2(-\infty, \infty)$ and $Q'(x)$ satisfy the condition (2.10), then

$$\int_0^\mu (1-\nu^2/\mu^2)^l \,\mathrm{d}_\nu (\partial/\partial x\, S_1(x,\nu) - \partial/\partial x\, S_1^F(x,\nu)) = o(\mu),$$

as μ tends to infinity uniformly for x in any finite interval.

It follows from lemma 6.1 and the relation (6.9) that $R^*(x,\mu)$ is of bounded variation over every finite interval containing μ where

$$\sup_{-\infty<\mu<\infty} \bigvee_\mu^{\mu+\nu} R^*(x,u) = o(1);$$

so that by the Levitan-Tauberian theorem we obtain from (6.8),

$$\int_0^\mu (1-\nu^2/\mu^2)^l \,\mathrm{d}_\nu R(x,\nu) = \frac{1}{2} \int_0^\mu (1-\nu^2/\mu^2)^l A(x,\nu)\,\mathrm{d}\nu -$$

$$-2/\pi \int_0^\mu (1-\nu^2/\mu^2)^l B(x,\nu)\,\mathrm{d}\nu = o(\mu^{-l}), \; l > 0, \tag{6.10}$$

as μ tends to infinity uniformly for x belonging to any finite interval. Now

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l A(x,\nu)\,\mathrm{d}\nu = \int_0^{\mu} (1-\nu^2/\mu^2)^l \mathrm{d}\nu \int_0^{\mu} A(x,u)\,\mathrm{d}u$$

$$= 2l \int_0^{\mu} (1-\nu^2/\mu^2)^{l-1} \;\nu/\mu^2\,\mathrm{d}\nu \int_0^{\nu} A(x,u)\,\mathrm{d}u, \text{ by integration by parts.}$$

$$= O\left(\int_0^{\mu} A(x,u)\,\mathrm{d}u\right) = o(1),$$

as μ tends to infinity uniformly for x in a finite interval, by lemma 6.1 (A). Similarly by lemma 6.1 (B),

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l B(x,\nu)\,\mathrm{d}\nu = O\left(\int_0^{\mu} B(x,u)\,\mathrm{d}u\right) = O(1),$$

as μ tends to infinity uniformly for x in a finite interval. Hence from (6.10)

$$\int_0^{\mu} (1-\nu^2/\mu^2)^l \mathrm{d}_\nu R(x,\nu) = o(\mu),$$

as μ tends to infinity uniformly for x in any finite interval. The theorem therefore follows.

Suppose that the eigenvalue problem is considered over the interval $[0,\infty)$ and the spectrum is assumed to be bounded below (conditions for which for the present problem remain to be decided). Then the term containing $B(x,\nu)$ contributes $o(1)$, as μ tends to infinity. Theorem 6.1 now reduces to

Theorem 6.2: If $l>0$ and $f(x)\,\varepsilon\,L_2[0,\infty)$, $Q'(x)$ satisfies the relation (2.10), then

$$\lim_{\mu\to\infty} \int_0^{\mu} (1-\nu^2/\mu^2)^l \mathrm{d}_\nu(\partial/\partial x(S_1(x,\nu)-S_1^F(x,\nu)) = 0,$$

uniformly for x belonging to any finite interval. Theorems 5.1 and 5.2 can also be modified similarly.

lgement

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below the Levitan-Tauberian theorem for Fourier integrals for convenience. :an and Sargsyan[5], p. 85).

Let $\sigma(\nu)$ satisfy

s of bounded variation over every finite interval;

ii) $\sup\limits_{-\infty<\mu<\infty} \overset{\mu+1}{\underset{\mu}{\mathrm{V}}} \sigma(\nu) = o(\mu^r),\ r \geq 0;$

iii) $\int\limits_{-\infty}^{\infty} h(\nu)\, \mathrm{d}\sigma(\nu) = 0$, where $h(\nu) = 1/2\pi \int\limits_{-\varepsilon}^{\varepsilon} g_\varepsilon(t) \exp(-i\nu t)\, \mathrm{d}t$

for every finite function $g_\varepsilon(t)$; *i.e.* $g_\varepsilon(t)$ is a function having bounded $r+2$th derivative but vanishing outside $(-\varepsilon, \varepsilon)$.

Then as μ tends to infinity,

iv) $\int\limits_{-\infty}^{\infty} (1-\nu^2/\mu^2)^s\, \mathrm{d}\sigma(\nu) = o(|\mu|^{r-s}), s \geq 0.$

The theorem remains true when o is replaced by O in (ii) and (iv); also one can consider the Fourier cosine or sine transform of the finite function according as it is even or odd.