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Approximate solution of an integral equation of interface crack

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Abstract

A simple and straight forward analysis is presented for obtaining an approximate solution of a singular integral equation arising in an interface crack problem.

Key words: Interface crack, integral equation, bounded solution, controlling factor.

1. Introduction

Studies on interface crack problems in the linearised theory of elasticity have attracted the attention of numerous workers. Comninou[1] derived the following basic integral equation for an interface crack problem:

$$(1-\gamma^2x^2)^{1/2}\, T\phi - \beta^2 T\,[(1-\gamma^2x^2)^{1/2}\,\phi] = (1-\gamma^2x^2)^{1/2}, \quad (\,|\,x\,| < 1) \tag{1}$$

with $\beta^2 < 1,\ \gamma = 1-\epsilon\ (\epsilon \ll 1,\ \text{unknown})$, and

$$T\phi = \frac{1}{\pi}\int_{-1}^{1} \frac{\phi\,(y)\mathrm{d}y}{y-x} \tag{2}$$

Atkinson and Leppington[2] have attacked this integral equation for its asymptotic solution for very small values of the unknown parameter ε, for the determination of which the extra condition required is that ϕ should be bounded at the two end points $x = \pm 1$.

Comninou obtained the bounded solution of equation (1) by using numerical methods, as developed by Erdogan and Gupta[3]. Atkinson and Leppington obtained an asymptotic solution of the integral equation (1) by employing a matching technique that involves simultaneous use of the theory of singular integral equations[4] and the Wiener-Hopf theory[5].

Recently, Gautesen and Dunders[6,7] have derived exact solutions of integral equations of the type (1) by employing an eigenfunction expansion technique of a special type.

In the present paper we have shown that by using well-known ideas involving bounded solutions of singular integral equations of the first kind, as explained in Muskhelishvili's book[4] (Section 89), the integral equation (1) can be reduced to a homogeneous Fredholm integral equation from which the determination of the unknown parameter ϵ ($\ll 1$) and the function ϕ can be completed by means of straight forward manipulations after evaluating the kernel of the resulting Fredholm equation approximately, for very small values of ϵ, for a class of problems for which β^2 is a small quantity as compared to $\beta^2 \ln \epsilon$.

A simple approximate formula is derived connecting the two parameters β^2 and ϵ of the problem and an estimate is obtained for the unknown parameter ϵ, in the case when $\beta^2 = 0.23561$ ($\beta = 0.4854$), as considered by the previous workers. The range of values of β for which the present approach is expected to produce results of a desired accuracy is shown to depend on a 'controlling factor' τ introduced into our analysis.

2. The analysis

If we rewrite the integral equation (1) in the form

$$(1-\beta^2)T\phi = 1 + \frac{\beta^2}{\pi}\int_{-1}^{1} K(y,x)\,\phi(y)\mathrm{d}y, \tag{3}$$

with

$$K(y,x) = \frac{1}{(y-x)}\left[\left(\frac{1-\gamma^2 y^2}{1-\gamma^2 x^2}\right)^{1/2} - 1\right], \tag{4}$$

and assume, for the time being, that the right hand side of equation (3) is known, we find from Muskhelishvili[4] (Section 89) that the unique bounded solution $\phi(x)$ of equation (3) is given by

$$\phi(x) = -\frac{\beta^2}{\pi^2(1-\beta^2)}\,(1-x^2)^{1/2}\int_{-1}^{1}\frac{(1-u^2)^{-1/2}}{(u-x)}\,\mathrm{d}u\int_{-1}^{1}K(y,u)\,\phi(y)\mathrm{d}y, \tag{5}$$

under the subsidiary condition that

$$\int_{-1}^{1}(1-\gamma^2x^2)^{1/2}\phi(x)\;K_1(x)\;\mathrm{d}x = \pi^2/\beta^2, \tag{6}$$

where

$$K_1(x) = \int_{-1}^{1}\frac{[1-u^2)(1-\gamma^2u^2)]^{-1/2}}{(u-x)}\;\mathrm{d}u. \tag{7}$$

We thus observe that the problem of determining the bounded solution of equation (3) becomes equivalent to that of determining the non-trivial solution of the homogeneous Fredholm integral (5), which can also be expressed in the form

$$\phi(x) - \frac{\beta^2}{\pi^2 (1-\beta^2)} \int_{-1}^{1} \bar{K}(x, y)\phi(y)\mathrm{d}y = 0, \tag{8}$$

where

$$\bar{K}(x,y) = (1-x^2)^{1/2} (1-\gamma^2 y^2)^{1/2} \left[\frac{K_1(x)-K_1(y)}{x-y}\right], \tag{9}$$

with K_1 as given by equation (7).

In what follows next, we assume that $\epsilon \ll 1$ and that $\epsilon \to 0$ corresponds to the fact that $\beta^2 \to 0$ also. In fact, in the interface crack problem of Comninou, it is observed that the case $\epsilon = 0$ corresponds to the case $\beta = 0$ and this, in turn, corresponds to the situation of a single crack in a complete elastic plate. In order to facilitate the approximate calculation of the solution $\phi(x)$ of equation (8), as well as the unknown parameter ϵ appearing there, we shall first express equation (8) in the symmetric form as given by

$$\phi^*(x) - \frac{\beta^2}{\pi^2 (1-\beta^2)} \int_{-1}^{1} K^*(x,y)\phi^*(y)\mathrm{d}y = 0, \tag{10}$$

with

$$\phi^*(x) = \left(\frac{1-\gamma^2 x^2}{1-x^2}\right)^{1/4} \phi(x), \tag{11}$$

and

$$K^*(x,y) = [(1-x^2)(1-y^2)(1-\gamma^2 x^2)(1-\gamma^2 y^2)]^{1/4} \left[\frac{K_1(x)-K_1(y)}{x-y}\right]. \tag{12}$$

We then note that for very small value of ϵ, the constant $\gamma = 1-\epsilon \approx 1$, (the symbol '$\approx$' means 'approximately equal to') and that the expression $(1-\gamma^2 u^2)$ can be expressed in the following two forms:

$$1-\gamma^2 u^2 \equiv (1-u^2) + (1-\gamma^2)u^2$$

$$\approx \begin{cases} 1-\gamma^2, & \text{when } u \approx \pm 1 \\ 1-u^2, & \text{when } u \not\approx \pm 1. \end{cases} \tag{13}$$

Using the results (13) in the relation (7) we therefore find that $K_1(x)$ has the following approximate form, for $\epsilon \ll 1$:

$$K_1(x) \approx \int_{-1}^{-1+\epsilon'} \frac{du/(u-x)}{[(1-u^2)(1-\gamma^2)]^{1/2}} + \int_{-1+\epsilon'}^{1-\epsilon'} \left[\frac{du/(u-x)}{(1-\gamma^2u^2)}\right]$$

$$+ \int_{1-\epsilon'}^{1} \frac{du/(u-x)}{[1-u^2)(1-\gamma^2)]^{1/2}}, \tag{14}$$

where ϵ' is a very small positive number with which the approximations (13) work. Evaluating the integrals in equation (14) by standard methods, we thus obtain that, for $\epsilon \ll 1$,

$$K_1(x) \approx \frac{1}{[(1-\gamma^2)\,(1-x^2)]^{1/2}} \ln \frac{\left|\left[\delta' + \frac{1-(1-x^2)^{1/2}}{x}\right]\left[\delta' - \frac{1+(1-x^2)^{1/2}}{x}\right]\right|}{\left|\left[\delta' + \frac{1+(1-x^2)^{1/2}}{x}\right]\left[\delta' - \frac{1-(1-x^2)^{1/2}}{x}\right]\right|}$$

$$+ \frac{x\gamma}{(\gamma^2x^2-1)}\left[\ln(\epsilon + \epsilon' - \epsilon\epsilon') - \ln(2-\epsilon - \epsilon' + \epsilon\epsilon')\right.$$

$$\left. - \frac{1}{x\gamma} \ln \left|\frac{1-x-\epsilon'}{1+x-\epsilon'}\right|\right], \tag{15}$$

with

$$\delta' = \frac{1-(1-\gamma'^2)^{1/2}}{\gamma'} \; ; \quad (\gamma' = 1-\epsilon')$$

i.e.,

$$K_1(x) \approx \frac{\gamma x}{(\gamma^2x^2-1)}[\ln \epsilon - \ln(2-\epsilon)] + \frac{1}{(\gamma^2x^2-1)} \ln \left|\frac{1+x}{1-x}\right| . \tag{16}$$

It now becomes apparent, by using the above approximation to $K_1(x)$ in equations (10) and (11), that for very small values of ϵ and β^2 together, the symmetric integral equation (10) can be approximated by the following new equation:

$$\phi^*(x) - \lambda \int_{-1}^{1} (\gamma^2xy+1)\left[\frac{(1-x^2)\,(1-y^2)}{(1-\gamma^2x^2)^3\,(1-\gamma^2y^2)^3}\right]^{1/4} \phi^*(y)\mathrm{d}y = 0, \tag{17}$$

where

$$\lambda = (-\beta^2 \ln \epsilon)/\pi^2, \tag{18}$$

assuming that $\beta^2 \ln \epsilon$ tends to a finite limit as both ϵ and β^2 tend to zero. Note that $\beta^2K_1(x) \approx \frac{-\pi^2\lambda x}{(\gamma^2x^2-1)}$, for small values of ϵ and β^2.

If we then notice that we want only the odd solution of equation (1) (or of equation (17)), we immediately find that the solution is given by

$$\phi^*(x) = \lambda c\gamma^2 x \left[\frac{1-x^2}{(1-\gamma^2x^2)^3}\right]^{1/4}, \tag{19}$$

in which the constant c can be determined by using the subsidiary condition (6) along with equation (11), and, we have that

$$c = \pi^2\beta^{-2}\gamma^{-2}\left[\int_{-1}^{1} \lambda x K_1(x) \frac{(1-x^2)^{1/2}}{(1-\gamma^2x^2)^{1/2}}\, dy\right]^{-1}. \tag{20}$$

The value of λ can be finally determined by substituting solution (19) in equation (17) and we obtain that

$$\lambda = \gamma^{-2}\left[\int_{-1}^{1} y^2 \frac{(1-y^2)^{1/2}}{(1-\gamma^2y^2)^{3/2}}\, dy\right]^{-1}. \tag{21}$$

Equations (19) and (11) together with relations (20) and (21) completely solve the integral equation (1) in an approximate manner.

3. Approximate determination of ϵ and c

In this section we shall obtain approximate values of the unknown parameter ϵ and the constant c by using the results (21), (18) and (20).

In order to achieve this goal, we shall adopt the following quick, but fruitful, though not a very rigorous, procedure:

We evaluate the integral in equation (21) approximately, by using the approximation (13) and writing the integral in equation (21) as

$$\int_{-1}^{1} \ldots dy = 2\int_{0}^{1} \ldots dy = 2\left[\int_{0}^{1-\epsilon''} \ldots dy + \int_{1-\epsilon''}^{1} \ldots dy\right]$$

where ϵ'' is another small positive number. We ultimately find that

$$\lambda = -\frac{\beta^2 \ln \epsilon}{\pi^2} \approx -[2-\ln 2+\ln \epsilon'']^{-1} = -[2-\ln 2+\ln \tau+\ln \epsilon]^{-1} \tag{22}$$

by introducing a 'controlling factor' τ as defined by $\epsilon'' = \tau\,\epsilon$ at this stage (Note that both ϵ and ϵ'' are very small quantities, and one of these can be controlled in our approach when the other is known).

We thus obtain, from equation (22), the result that

$$\epsilon = \exp\left[-1-\frac{1}{2}\ln\tau+\frac{1}{2}\ln 2-\left(\left(1+\frac{1}{2}\ln\tau-\frac{1}{2}\ln 2\right)^2+\frac{\pi^2}{\beta^2}\right)^{1/2}\right]. \tag{23}$$

We notice that a judicious decision about the choice of the controlling factor τ can be taken in order to produce results of a desired accuracy, and evaluate the unknown quantity ϵ, for a wide range of values of the physical parameter β. (Physically admissible values of β satisfy the inequalities $(0\leqslant\beta\leqslant\frac{1}{2})$. In a similar fashion, we also obtain, after approximating $\beta^2K_1(x)$ by the expression $(\beta^2\ln\epsilon)\ \frac{x}{(\gamma^2x^2-1)}$ (cf. equation (15)), that

$$c \approx \pi^2\left((2\beta^2\gamma^2\lambda\ \ln\epsilon)\left[\int_0^1 x^2\mathrm{d}x/(\gamma^2x^2-1)\right]\right)^{-1}$$

$$= -\left[2\lambda^2\left(1+\frac{1}{2\gamma}\ln\left(\epsilon/(2-\epsilon)\right)\right)\right]^{-1} \tag{24}$$

The determination of the numerical value of the constants c and ϵ for a given value of β is a straight forward job now when equation (23) is made use of after properly selecting the controlling factor τ.

We find that if τ is chosen to be unity, then we obtain, for $\beta = .4854$, $\epsilon = (0.7782) \times 10^{-3}$ which is, of course, a small number as expected, but higher than the values 10^{-4} (Comninou[1]), $(0.9484)\ 10^{-4}$ (Atkinson and Leppington[2]) and $(1.0517)\ 10^{-4}$ (Gautesen and Dunders[6]) respectively, reported earlier. However, if we must achieve the result $\epsilon = 10^{-4}$, for the problem considered, we notice from equation (23), that τ must be chosen to be equal to 28.6566, for $\beta = 0.4854$ and this is not an unnatural choice of the controlling factor τ introduced into our approach.

A further point that is to be noted with the present analysis is that the value 10^{-4} of the small quantity ϵ, as obtained by Comninou[1] can be achieved even with the particular choice $\tau = 1$, for $\beta = 0.3682$. The conclusion, therefore, is that in order to achieve the amount of accuracy desired (*i.e.* $\epsilon \leqslant 10^{-4}$) in the solution of the crack problem in which the integral equation (1) is basic, the amount of mathematical analysis required can be made much more simpler than those of Atkinson and Leppington[2], and of Gautesen and Dunders[6,7].

Analysing the controlling factor τ still remains an open problem and investigations are in progress in this direction.

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