

## Short Communication

# On the Clifford linearization of Laplacian

KIYOHARU NÔNO

Department of Mathematics, Fukuoka University of Education, Munakata, Fukuoka, 811-41, Japan.

AND

YOSHIKAZU INENAGA

Kyushu High School attached to Kyushu Industry University, Higashiku, Fukuoka, 813, Japan.

Received on July 29, 1986; Revised on May 6, 1987; Re-revised on June 8, 1987.

### Abstract

In this paper, we shall give a characterization of the differential operator  $D = \alpha_0 \partial / \partial x_0 + \alpha_1 \partial / \partial x_1 + \dots + \alpha_m \partial / \partial x_m$  with coefficients  $\alpha_i$  ( $i = 0, 1, \dots, m$ ) of Clifford number for which any solution of the differential equation  $Df=0$  is always a solution of Laplace's equation  $\Delta f=0$ , where  $\Delta = \partial^2 / \partial x_0^2 + \partial^2 / \partial x_1^2 + \dots + \partial^2 / \partial x_m^2$ .

**Key words:** Clifford algebra, linearization, Laplacian.

## 1. Introduction

The Clifford algebra was constructed by W. K. Clifford in 1878 as a generalization of the quaternion algebra. It was studied in the 1930s in connection with the theory of spinors<sup>1-4</sup> and has been studied in mathematics and physics<sup>5-30</sup>.

The generalized Clifford algebra which is a generalization of the ordinary Clifford algebra was constructed as a generalization of the theory of spinors and had extensively been studied by many authors<sup>10-18</sup>. In 1971, T. Nôno<sup>17-18</sup> had also generalized the concept of linearization of wave equation for the more general differential equation as another generalization of the theory of spinors.

Many authors have studied higher dimensional function theories<sup>19-26</sup> as an extension of classical complex function theory. This study has also been applied within a number of areas of theoretical physics (for example, refs 27-29). Most of the function theories<sup>19-26</sup> associated with the solutions of first order elliptic differential equations are special linearizations of the Laplace's equation.

We gave<sup>30</sup> the characterization of the differential operator  $D = \alpha_0 \partial / \partial x_0 + \alpha_1 \partial / \partial x_1 + \alpha_2 \partial / \partial x_2 + \alpha_3 \partial / \partial x_3$  with quaternionic coefficients  $\alpha_i$  ( $i=0,1,2,3$ ) for which any solution of the differential equation  $Du = 0$  is always a solution of the Laplace's equation  $(\partial^2 / \partial x_0^2 + \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \partial^2 / \partial x_3^2)u = 0$ , where  $u$  is a quaternionic function.

In this paper, we shall give a generalization of the result in ref. 30 for Laplacian of  $(m+1)$ -real variables ( $m \geq 1$ ).

## 2. Preliminaries

Let  $A_n$  be the Clifford algebra over an  $n$ -dimensional vector space with orthogonal basis  $\{e_1, e_2, \dots, e_n\}$ . It is well known that  $A_n$  is a real  $2^n$ -dimensional associative, but the non-commutative algebra and its basis  $\{e_1, e_2, \dots, e_n, \dots, e_1 e_2 \dots e_n\}$  satisfy the following:

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad (i, j = 1, 2, \dots, n),$$

$e_0$  is identity of  $A_n$ ,

where  $\delta_{ij}$  is the Kronecker's delta. Then, every element  $z$  in  $A_n$  is of the form

$$z = e_0 x_0 + e_1 x_1 + \dots + e_n x_n + e_1 e_2 \dots e_n x_{12\dots n},$$

where  $x_0, x_1, \dots, x_{12\dots n}$  are real numbers. For two Clifford numbers  $z = e_0 x_0 + e_1 x_1 + \dots + e_n x_n + e_1 e_2 \dots e_n x_{12\dots n}$  and  $w = e_0 y_0 + e_1 y_1 + \dots + e_n y_n + e_1 e_2 \dots e_n y_{12\dots n}$ , the inner product  $(z, w)$  is defined by the following:

$$(z, w) = x_0 y_0 + x_1 y_1 + \dots + x_n y_n + \dots + x_{12\dots n} y_{12\dots n},$$

and the norm  $|z|$  of  $z$  is given by

$$|z| = \sqrt{(z, \bar{z})}.$$

In this paper, let  $m$  be an arbitrary integer such that  $1 \leq m \leq n$ . The subspace of  $A_n$  spanned by the elements  $e_0, e_1, \dots, e_m$  is identified with  $\mathbf{R}^{m+1}$ . For each element  $z = e_0 x_0 + e_1 x_1 + \dots + e_m x_m$  in  $\mathbf{R}^{m+1}$ , the conjugate number  $z^*$  and the inverse  $z^{-1}$  are given by the following:

$$z^* = e_0 x_0 - \sum_{i=1}^m e_i x_i, \quad z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0).$$

For elements  $z$  and  $w$  in  $\mathbf{R}^{m+1}$ , we obtain

$$|zw| = |z| |w|. \quad (2.1)$$

Let  $\partial/\partial x_i$  ( $i = 0, 1, \dots, m$ ) be the usual real differential operators. We consider the following Clifford differential operator:

$$D = \sum_{i=1}^m \alpha_i \frac{\partial}{\partial x_i}, \quad (2.2)$$

where  $\alpha_i = \sum_{j=0}^m e_j a_{ij}$  ( $i = 0, 1, \dots, m$ ) are elements in  $\mathbf{R}^{m+1}$ . For the above differential

operator  $D$ , the matrix  $A = (a_{ij})$  is said to be the *matrix* of  $D$ . Also, for the differential operator  $D$  of (2.2) we define the conjugate differential operator  $D^*$  by

$$D^* = \sum_{i=0}^m \alpha_i^* \frac{\partial}{\partial x_i},$$

where  $\alpha_i^*$  is the conjugate number of  $\alpha_i$  ( $i = 0, 1, \dots, m$ ).

Let  $G$  be a subset of  $\mathbf{R}^{m+1}$ . We consider a function  $f(z) = e_0 f_0(z) + e_1 f_1(z) + \dots + e_n f_n(z) + \dots + e_1 e_2 \dots e_n f_{12\dots n}(z)$  defined in  $G$ , where  $z = (x_0, x_1, \dots, x_m) \in G$ . We say that a differential operator  $D$  of (2.2) is a *linearization* of the Laplacian  $\Delta = \partial^2/\partial x_0^2 + \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_m^2$  if and only if any solution of a differential equation

$$Df = 0 \quad (2.3)$$

is always a solution of Laplace's equation

$$\Delta f = 0. \quad (2.4)$$

*Remark:* If  $\alpha_j = 0$  for some  $j$ , then the differential operator  $D$  is not a linearization of Laplacian  $\Delta$ , since  $Df = 0$  but  $\Delta f \neq 0$  for a function  $f(x_0, x_1, \dots, x_m) = x_j^2$ .

### 3. Main theorem

Let  $O(m+1)$  be the  $(m+1)$ -orthogonal group. We need the following two lemmas which are easily proved.

*Lemma 1:* Let  $D$  be a differential operator of (2.2),  $A = (a_{ij})$  be the matrix of  $D$ . Then, the following conditions (1), (2) and (3) are all equivalent.

- (1)  $A \in O(m+1)$ .
- (2)  $\alpha_i^* \alpha_j + \alpha_j^* \alpha_i = 2\delta_{ij}$  ( $i, j = 0, 1, \dots, m$ ).
- (3)  $(\alpha_i, \alpha_j) = \delta_{ij}$  ( $i, j = 0, 1, \dots, m$ ).

*Lemma 2:* The functions

$f_{ij}(x_0, x_1, \dots, x_m) = (\alpha_i^{-1} \alpha_j)^2 x_i^2 - 2(\alpha_i^{-1} \alpha_j) x_i x_j + x_j^2$ , ( $i, j = 0, 1, \dots, m$ ) are solutions of the differential equation (2.3), where  $\alpha_i$  ( $i = 0, 1, \dots, m$ ) are coefficients of  $D$ .

*Theorem:* Let  $D$  be a differential operator of (2.2) and  $A$  be the matrix of  $D$ . Then, the following conditions (1), (2) and (3) are all equivalent.

- (1)  $D$  is a linearization of Laplacian  $\Delta$ .
- (2)  $A = \lambda O$ ,  $O \in O(m+1)$ , ( $\lambda > 0$ ).
- (3)  $D^*D = \lambda^2 \Delta$ , ( $\lambda > 0$ ).

*Proof:* We first prove that (1)  $\rightarrow$  (2). Let  $f_{ij}$  ( $i \neq j$ ,  $i, j = 0, 1, \dots, m$ ) be the functions in Lemma 2. Since we have

$$\Delta f_{ij} = 2(\alpha_i^{-1} \alpha_j)^2 + 2,$$

from  $\Delta f_{ij} = 0$  it follows that

$$(\alpha_i^{-1} \alpha_j)^2 + 1 = 0 \quad (i \neq j, i, j = 0, 1, \dots, m). \quad (3.1)$$

From (3.1) and remark in Section 2, we obtain

$$\alpha_i^{-1} \alpha_j + \alpha_j^{-1} \alpha_i = 0 \quad (i \neq j, i, j = 0, 1, \dots, m).$$

From (3.1), we have  $|\alpha_i^{-1}| |\alpha_j| = |\alpha_j^{-1}| |\alpha_i|$  ( $i \neq j$ ,  $i, j = 0, 1, \dots, m$ ) which implies  $|\alpha_i| = |\alpha_j| > 0$  for all  $i, j = 0, 1, \dots, m$ . It now easily follows that

$$\alpha_i^* \alpha_j + \alpha_j^* \alpha_i = 0 \quad (i \neq j, i, j = 0, 1, \dots, m). \quad (3.2)$$

Put

$$\lambda = |\alpha_0| = |\alpha_1| = \dots = |\alpha_m| > 0. \quad (3.3)$$

Thus, (3.2), (3.3) and Lemma 1 show that  $A = \lambda O$ ,  $O \in O(m+1)$ , ( $\lambda > 0$ ).

Next, we prove that (2)  $\rightarrow$  (3). By operating  $D^*$  on the left side of  $D$ , from Lemma 1 and condition (2) we have

$$D^* D = \lambda^2 \Delta + \sum_{\substack{i, j=0 \\ i < j}}^m (\alpha_i^* \alpha_j + \alpha_j^* \alpha_i) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \lambda^2 \Delta.$$

At last, we prove that (3)  $\rightarrow$  (1). Let  $f$  be any solution of  $Df=0$ . From condition (3), it follows that  $0=D^* Df = \lambda^2 \Delta f$ . Thus,  $D$  is a linearization of Laplacian  $\Delta$ .

*Corollary:* Let  $D^*$  be the conjugate differential operator of  $D$  and  $A^*$  be the matrix of  $D^*$ . Then, the following conditions are all equivalent.

- (1)  $D$  is a linearization of Laplacian  $\Delta$ .
- (2)  $D^*$  is a linearization of Laplacian  $\Delta$ .
- (3)  $A = \lambda O$ ,  $O \in O(m+1)$ , ( $\lambda > 0$ ).
- (4)  $A^* = \lambda O$ ,  $O \in O(m+1)$ , ( $\lambda > 0$ ).
- (5)  $D^* D = \lambda^2 \Delta$ , ( $\lambda > 0$ ).
- (6)  $DD^* = \lambda^2 \Delta$ , ( $\lambda > 0$ ).

## References

1. BRAUER, R. AND WEYL, H. Spinors in  $n$ -dimensions, *Am. J. Math.*, 1935, 57, 425-449.
2. CHEVALLEY, C. *The algebraic theory of spinors*, Columbia Univ. Press, New York, 1954.
3. CARTAN, E. *Theory of spinors*, Dover, New York, 1966.

4. RIESZ, M. *Clifford numbers and spinors*, Lecture Notes No. 38, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1958.
5. CHISHOLM, J. S. R. AND FARWELL, R. S. Spin gauge theory of electric magnetic spinors, *Proc. R. Soc. Lond.*, 1981, **A377**, 1-23.
6. HESTENES, D. Vectors, spinors and complex numbers in classical and quantum physics, *Am. J. Phys.*, 1971, **39**, 1013-1027.
7. LOUNESTO, P. Sur les idéaux à gauche des algèbres de Clifford et les produits scalaires des spineurs, *Ann. Inst. H. Poincaré, Sect. A*, 1980, **33**, 53-61.
8. MICALI, A. AND VILLAMATOR, O. E. Sur les algèbres de Clifford, *Ann. Ecole Normales Sup.*, 1968, 4<sup>e</sup> série 1. 271-304.
9. SALINGAROS, N. A. AND WENE, G. P. The Clifford algebra of differential forms, *Acta Appl. Math.*, 1985, **4**, 271-292.
10. MORINAGA, K. AND NÔNO, T. On the linearization of a form of higher degree and its representation, *J. Sci. Hiroshima Univ. (A)*, 1952, **16**, 13-41.
11. YAMAZAKI, K. On projective representations and ring extensions of finite groups, *J. Fac. Sci. Univ. Tokyo, Sec. I*, 1964, **10**, 147-195.
12. MORRIS, A. O. On generalized Clifford algebra, *Q. J. Math.*, 1967, **18**, 7-12.
13. RAMAKRISHNAN, A., VASUDEVAN, R., RANGANATHAN, N. R. AND CHANDRASEKARAN, P. S. On a generalization of the L-matrix hierarchy, *J. Math. Appl.*, 1968, **23**, 10-14.
14. RAMAKRISHNAN, A., SANTHANAM, T. S. AND CHANDRASEKARAN, P. S. On the representations of the generalized Clifford algebra, *J. Math. Phys. Sci.*, 1969, **3**, 307-313.
15. RAMAKRISHNAN, A., SANTHANAM, T. S. AND CHANDRASEKARAN, P. S. Helicity matrices for the generalized Clifford algebra, *J. Math. Anal. Appl.*, 1969, **26**, 275-278.
16. RAMAKRISHNAN, A., VASUDEVAN, R., CHANDRASEKARAN, P. S. AND RANGANATHAN, N. R. Kemmer algebra and generalized Clifford elements, *J. Math. Anal. Appl.*, 1969, **28**, 108-111.
17. NÔNO, T. On the linearization of a partial differential equation  $\nabla^m \psi = c^m \psi$ , *Bull. Fukuoka Univ. Ed.*, Part III, 1971, **21**, 35-41.
18. NÔNO, T. Generalized Clifford algebra and linearization of a partial differential equation:  $g^{i_1 i_2 \dots i_m} \nabla_{i_1} \nabla_{i_2} \dots \nabla_{i_m} \psi = C^m \psi$ , *Proc. Conf. on Clifford algebra, its generalization and applications*, 1971, 1-24, MATSCIENCE, Madras.
19. FUETER, R. Die funktionentheorie der differentialgleichungen  $\Delta u = 0$  und  $\Delta \Delta u = 0$  mit vier reellen variablen, *Comment. Math. Helv.*, 1935, **7**, 307-330.
20. STEIN, E. M. AND WEISS, G. Generalizations of the Cauchy-Riemann equations and representations of the rotation group, *Am. J. Math.*, 1968, **90**, 163-196.

21. DELANGHE, R. On regular-analytic functions with values in a Clifford algebra, *Math. Ann.*, 1970, **185**, 91-111.
22. LOUNESTO, P. Spinor valued regular functions in hypercomplex analysis. Doctoral thesis. 1979. Univ. of Technol., Helsinki.
23. SUDBERY, A. Quaternionic analysis, *Math. Proc. Camb. Phil. Soc.*, 1979, **85**, 199-225.
24. BRACKX, F., DELANGHE, R., SOMMEN, F. *Clifford analysis, Research Notes in Math.*, Pitman, London, 1982, **76**.
25. GOLDSCHMIDT, B. Regularity properties of generalized analytic vectors in  $R^n$ , *Math. Nachr.*, 1981, **103**, 245-254.
26. RYAN, J. Properties of isolated singularities of some functions taking values in real Clifford algebras, *Math. Proc. Camb. Phil. Soc.*, 1984, **95**, 277-298.
27. DEAVOURS, C. A. The quaternion calculus. *Am. Math. Mon.*, 1973, **80**, 995-1008.
28. IMAEDA, K. A new formulation of electromagnetism, *Nuovo Cimento*, 1976, **326**, 138-162.
29. GÜRSEY, F. AND TZE, H. C. Complex and quaternionic analyticity in chiral and gauge theories I, *Ann. Phys.*, 1980, **128**, 29-130.
30. NÔNO, K. On the quaternion linearization of Laplacian  $\Delta$ , *Bull. Fukuoka Univ. Ed.*, Part III, 1985, **35**, 5-10.

