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BOOK REVIEWS

Percy Alexander MacMahon, Collected papers, Vol. II (*Number Theory, invariants and applications*), edited and prefaced by George E. Andrews; introduction by Gian-Carlo Rota. The MIT Press, 28, Carleton Street, Cambridge, Mass. 02142, USA, 1986, pp. lxxv+952, $ 109.25. Indian orders to Affiliated East–West Press Pvt. Ltd., 6, Roselyn Gardens Apartments, 20/1A, Barnaby Road, Madras 600 010.

§1. The first volume of the Collected papers of the late Percy Alexander MacMahon (1854–1929) appeared in 1978 (as no. 13) in the series "Mathematicians of our time" of the MIT Press. Together with it, the present volume (no. 24 in the same series) completes the collection of all of MacMahon's (nearly) 120 mathematical papers, published during 1881–1928. However, for reasons of their mathematical interest being secondary, some articles of MacMahon, in the Minutes of the Proceedings of the Royal Artillery Institution, are not included in these volumes; but their complete list is given in the Bibliography (on pp. xxiv–xxv in volume II). Apart from these, MacMahon wrote three books: (1) the famous *Combinatory analysis* in two volumes[1], (2) *An introduction to combinatory analysis*[2], and (3) the recreational little book *New mathematical pastimes*[3]. The present volume also contains two obituaries[4,5]. On three occasions (in Chapters 13, 15 and 18), an entire paper by another mathematician is included by way of stressing links between MacMahon's investigations and related modern work.

§2. For a comprehensive review of the first volume, we refer the reader to P. R. Stein's article[6]. This review also includes a short sketch of the life and work of MacMahon. Both volumes are organised in a way to help the reader to appreciate the work of MacMahon and to recognise its part in later developments (through summaries and references, in each chapter). Every chapter, generally, is divided into sections comprising reprints of papers, an introductory discussion of their topic, their summaries and references to more recent work. The second volume, unlike the first, pertains to more diverse topics and so, we limit our more detailed account to a particular chapter (no. 16) whereas the descriptions of others would be mainly from the editor's summaries and comments. While the reader may (understandably) find the terminology and notation somewhat antiquated, it must be added that the papers of MacMahon are highly readable. Of the eight chapters, in volume II, the last one consists of (reprints of) the two obituaries (mentioned in §1); the second was presented before the Royal Astronomical Society, of which MacMahon was president in 1917–18. MacMahon has but a single paper in Astronomy (Chapter 17, pp. 437–438), which (we learn) attracted comments from Sir Arthur Eddington but, nevertheless, found application much later.

§3. Perhaps the best introductory papers are the surveys in Chapter 19. The four mathematical papers (of the eight) here, appearing in *Nature* and *Encyclopaedia*

Britannica (Edition 11) are both interesting and informative; particularly the expository 'Algebraic forms', giving an account of classical invariant theory, provides a good introduction to the papers in Chapter 18. The treatment of magic squares, in this collection, from *Nature*, would be interesting even to a general reader.

Over a third of the present volume concerns with invariant theory, (Chapter 18), which was a major area of MacMahon's researches, comprising 22 papers. This topic has later been extended by others, and again, after many years, its usefulness has been felt. The 'Aitken letter', included as introduction to this chapter, gives a view of invariant theory in a broader perspective (as falling in the domain of group theory). For a more recent account about this topic, we refer to Dieudonne and Carrell[7]. As a sample of MacMahon's results here, keeping in mind the need to avoid more elaborate definitions, we mention, from the enumerative aspect, the following one: 'The generating function for the number of perpetuants of degree d is

$$\frac{x^{2^{d-1}-1}}{(1-x^2)(1-x^3)\ \ldots\ (1-x^d)}\ ;$$

i.e. the coefficient of x^n in the series expansion of this function is the number of perpetuants of degree d and weight n. This statement was conjectured (for general d) by MacMahon, who gave a conditional proof, and proved by Stroh. Later, MacMahon, too, gave an (algebraical) proof and further extended it into a more general context. (Here, we note a notational slip: on p. 479, p and j are to be same.)

Next we come to MacMahon's work on symmetric functions (Chapter 13) and determinants (Chapter 14). The papers in both these chapters are theoretical. The memoir on symmetric functions here corresponds to Section XI of *Combinatory analysis*[1]. This paper gives a theory of symmetric functions of several sytems of quantities, generalising many earlier results pertaining to a single system (for simplicity of exposition the theory is confined to two systems.) Chapter 14 contains five papers of MacMahon, on determinants, written after publication of ref. 1. In this connection the paper of Littlewood and Richardson[8], quoted from in the introductory section (but not reprinted, as mentioned in ref. 6, is the (natural) next paper to read. Here it may be mentioned that we learn (from editor's comments) that the formula for $N_{\sigma, n}$ – the number of symmetric functions appearing in the expansion of the σ-dimensional permanent on n letters – conjectured by MacMahon, had been proved by J. H. Redfield (but unpublished).

In Chapter 15, there are four papers of MacMahon dealing with repeating patterns. This topic, having a recreational aspect, would be of interest to a general reader, too; particularly the third paper (no. 99). However, MacMahon appears to have had a serious goal related to Hilbert's 18th problem: 'Whether polyhedra also exist that do not appear as fundamental regions by groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible?' But his publications in this subject abruptly ended in 1922–23, probably because he came to know in 1922 that methods of crystallographic group theory, in this context, had already been employed with some success. Reference 9 is its own complete reference, given as 'to appear' on p. 212. Incidentally, accounts of progress on Hilbert's celebrated problems can be found in ref. 10.

MacMahon's isolated papers are collected under Chapter 17. These include his initial papers on classical geometry, the sole paper in astronomy (mentioned in §2) and a couple concerning physics. The remaining ones are in topics from classical analysis. While each paper is interesting by itself, we select the generalisation of nine-point circle property of a triangle, as an example, since it seems to have a 'wider appeal' : For any triangle, the six points, where the lines from the circumcentre and the orthocentre making angles α and $\pi - \alpha$ $(0<\alpha<\pi)$ respectively with the sides meet them, lie on a circle, $C(\alpha)$ say. Further, MacMahon obtains six more points (corresponding to the three midpoints of segments joining the orthocentre with the vertices) on $C(\alpha)$, and shows that as $\alpha \to \frac{1}{2}\pi$ the above 12 points on $C(\alpha)$ coalesce into the 9 points of the nine-point circle. (In an appended note, M. Jenkins determines six more points on $C(\alpha)$.)

§4. Now we come to Chapter 16 having four papers in multiplicative number theory. The first paper investigates the consequences of the double identity:

$$\sum_{m=1}^{\infty} \frac{aq^m}{1-aq^m} = \sum_{m=1}^{\infty} \frac{a^m q^m}{1-q^m} = \sum_{n=1}^{\infty} (\sum_{d} a^d) q^n$$

where d runs through divisors of n, valid for arbitrary a, leading to several interesting identities. The second paper considers certain infinite series (in powers of q) such as

$$A_k = \sum_{n=0}^{\infty} a_{nk} q^n,$$

where $a_{nk} = \sum s_1 \ldots s_k$ with summation extended over positive integral k-tuples $(s_1, \ldots, s_k)$ satisfying $\sum s_j m_j = n$; here m_j's are arbitrary positive integers. It is shown, in particular,

$$2^{2k}(2k+1)!\, A_k = (-1)^k \frac{1}{J_1} J \prod_{j=1}^{k} (J^2-(2j-1)^2),$$

it being understood that after expansion J^r is to be replaced by

$$J_r = \sum_{n=0}^{\infty} (-1)^n (2n+1)^r \, q^{n(n+1)/2}.$$

The third paper primarily gives the table of primitive roots of the congruence $x^8 \equiv 1(\text{mod } M)$ for (second and third categories of) $M < 120$. (M is said to be in sth category if the congruence $x^2 \equiv 1(\text{mod } M)$ has exactly 2^s solutions.)

The last paper is particularly interesting, although an unsuccessful attempt on Goldbach problem (that every even integer (> 4) is a sum of two odd primes). It represents an algebraic approach to the problem. In fact, MacMahon gives several expressions for the number of representations, $N_{2n,2}$, of $2n$ as sum of two primes.

§5. In conclusion, it can certainly be said that the efforts of the editor, Professor George E. Andrews, have made for a pleasant reading of the volume. The investigative style of MacMahon's papers is quite captivating. We would like to think that these volumes in (at least) universities may prove an inspiration for inquisitive students.

References

1. MacMahon, P.A. *Combinatory analysis*, Cambridge Univ. Press, London, 1915, 1916; reprint, two vols. in one, Chelsea, New York, 1960.
2. MacMahon, P.A. *An introduction to combinatory analysis*, Cambridge Univ. Press, London, 1920; reprinted in *Famous problems and other monographs*, 2nd ed., Chelsea, New York, 1962.
3. MacMahon, P.A. *New mathematical pastimes*, Cambridge Univ. Press, Cambridge, 1921.
4. Baker, H.F. *J. Lond. Math. Soc.*, 1930, **5**, 307–318.
5. Turner, H.H. *Mon. Not. R. Astr. Soc.*, 1930, **90**, 373–378.
6. Stein, P.R. *Adv. Math.*, 1979, **31**, 350–355.
7. Dieudonné, P.R. and Carrell, J.B. *Adv. Math.*, 1970, **4**, 1–80.
8. Littlewood, D.E. and Richardson, A.R. *Philosophical Trans. R. Soc.*, 1934, **A-233**, 99–141.
9. Schattschneider, D. *Math. Mag.*, 1980, **53**, 224–233.
10. Browder, F.E., (ed.) Mathematical developments arising from Hilbert problems, *Proc. Symp. Pure Math.*, 1976, **28**.

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Seminar on geometric measure theory by R. Hardt and L. Simon. Birkhauser Verlag, CH-4010, Basel, Switzerland, 1986, pp. 117, S. Fr. 26. Indian orders to Springer Books (India) Pvt. Ltd., New Delhi 110 017.

The euclidean space $\mathbb{R}^n$ (where $n \geq 1$) admits, apart from the familiar Lebesgue measure, a large class of regular Borel measures which are invariant under all isometries. The Lebesgue measure and its multiples of course have the distinction of assigning finite measure to the unit cube and more generally any compact subset. The other measures are 'lower dimensional'; for them the subsets of a suitable dimension $m<n$ could have finite nonzero measure while any nonempty open subset would have infinite measure. The Hausdorff measures and the integral geometric measures are some of the examples. Though such a measure may seem rather strange to one steeped in classical geometry and calculus, it turns out that these measures play an important role in various geometrical problems, especially in singularity theory, calculus of variations, etc.

The theory of these measures, which evolved out of the pioneering work of Besicovitch about fifty years ago, has come a long way both in terms of the body of its own and also applications to various geometric problems, thanks to the work of several authors including Federer, Fleming, De Georgi, Almgren, Allard, etc.

The role played by differentiable functions in the classical set up is now taken over by Lipshitz functions and the theory often deals with images of bounded subsets of IR^m under Lipshitz maps into IR^n, $n \geq m$, called rectifiable sets, and countable unions of such sets. Various notions in calculus such as tangent space, gradient, currents have been generalised to the new set up (approximate tangent space, rectifiable current, etc.) and modern versions of various classical theorems as well as new structure theorems have been proved forming into a coherent theory, often with strikingly new consequences.

Plateau type problems, where an $(m-1)$-dimensional submanifold of IR^n, $m \leq n$, is sought to be realised as the boundary of an m-dimensional submanifold of minimum possible area (*viz.* m-dimensional measure), have been a major theme in applications. This is indeed a problem of minimising currents. The general theory yields minimising rectifiable currents under very general conditions and one then analyses their smoothness to find a smooth solution, or conclude nonexistence. A lot of the recent work notably by Allard, Almgren and the authors addresses the latter aspect of concluding regularity.

The book under review gives an overview of the subject. It is stated to be a faithful account of a seminar held at Schloss Mickeln, Dusseldorff, F.R.G., and serves 'as a base from which one may pursue further literature' as suggested by the authors. The suggested references for follow up include Federer's book (*Geometric measure theory*, Springer Verlag, 1969) which gives a detailed (rather encyclopaedic) account of the subject as it stood in the late 1960s, and a set of notes by one of the authors (L. Simon, *Lectures in geometric measure theory*, Proc. of the Center for Mathematical Analysis, Australian National University, Vol. 3, 1983).

The technical workout of the overview is meticulous. Proofs are included selectively, for well-chosen assertions. Several illustrative diagrams are included. The book however has a curious drawback. There is hardly any attempt to place the subject under discussion in a wider context or to enthuse an 'onlooker' into the subject. Hence without some initiation, which was perhaps automatic at the seminar, the reader is apt to feel it rather drab. However for one who has an overall interest in the subject (but is overawed by Federer's book!) this is a good place to start.

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A first look at perturbation theory by J. G. Simmonds and J. E. Mann, Jr. The MIT Press, 28, Carleton Street, Cambridge, Massachusetts, 02142, USA, 1986, pp. 134, $ 12.50. Indian orders to Affiliated East-West Press Pvt. Ltd., 6, Roselyn Gardens Apartments, 20/1A, Barnaby Road, Madras 600 010.

In many practical situations, one comes across problems of ordinary differential equations (ODE) or partial differential equations (PDE) which involve a large parameter N or a small parameter ϵ either in the equation or in the boundary condition or in the initial condition or in the domain. The task is to analyse the behaviour of the solution on $N \rightarrow \infty$ or $\epsilon \rightarrow 0$. Perturbation methods are useful in this context. The idea is to suggest an appropriate asymptotic expansion for the problem at hand and produce an approximate solution. One could think of using some numerical procedures to do the same but the stability of such schemes may be in danger because of the appearance of large and small parameters. Indeed, it will be wise to combine these two procedures (perturbation and numerical) cleverly to obtain the desired result.

In the present volume, the authors present the method in the context of some polynomial equations and ODEs. The difference between a regular perturbation and a singular perturbation is illustrated in a very simple situation. Due to the presence of infinite domains, the asymptotic expansion suggested may not be uniformly valid. To overcome this difficulty, the authors suggest a few methods, namely, (i) Two-scale method (Chapter V); (ii) Method of strained variable (Chapter IV); and (iii) Combination of both (Chapter V).

There may be singularity in the solution of boundary layer type which is due to the loss of highest order terms in the ODE. In Chapter VIII, the authors study the so-called inner, outer and matched asymptotic expansion techniques to tackle such problems.

Instead of singularities, there may be oscillations in the solution. (example: nth eigenvector of a Strum-Liouville problem). In such situations, one can use WKBJ approximation for high frequencies. This is illustrated in Chapter VI.

Apart from these, the reader could see some other methods and there is a chapter on application too. All these methods are classical by now and well understood in the literature. There are plenty of volumes treating various examples and with rigorous theory. The urgent need of the present book is thus not clear. The only plus point about the book is that all these methods are illustrated in the simplest of all situations and most of the time it is explained why a regular expansion will not work. For this reason, the book is recommended to the beginners who wish to learn some basics of perturbation methods.

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Game theory in the social sciences: Concepts and solutions by Martin Shubik. The MIT Press, 28, Carleton Street, Cambridge, Massachusetts, 02142, USA, 1982, pp. 442, $ 13.74. Indian orders to Affiliated East-West Press Pvt. Ltd., 6, Roselyn Gardens Apartments, Madras 600 010.

The book offers an excellent analysis of game theoretic concepts and methods as well as their application to the behavioural sciences – in particular to general economic theory. The coverage is extensive, the exposition meticulous and thorough, and the mathematical background required of the reader modest.

The book can be divided into three parts. The first part is devoted to the art of modelling and associated issues. Chapters 1–3 discuss the decision problems of the players and the rules of the game. In particular are discussed the extensive and strategic game forms, the information, control and payoff aspects of decision makers and coalitions. Chapters 4 and 5 are devoted to individual and group preferences used in the study of rational behaviour. Relevant aspects of utility theory and optimality notions are stressed.

The second part presents the entire gamut of solution concepts for multiperson games. Chapters 6 and 7 deal with the cooperative game solutions such as the core, the (von Neumann–Morgenstern) stable sets, and the Shapley value. Chapters 8 and 9 are addressed to the noncooperative solutions, first in two-person zero-sum games where no cooperation is possible and then in nonantagonistic games where cooperation is possible, but not assumed. Chapters 10 and 11 present the remaining solution concepts to complete the spectrum. The relation between the different solution concepts is brought to light through well-compiled discussions and many effective examples. The author provides fresh insights into the conceptual richness of the subject.

The third part (Chapter 12) discusses the applications of game theory to a wide range of topics in social sciences.

It has a very wide reader appeal, and is an excellent book for the uninitiated. It is equally good for the game theorist as well as the practitioner. The notes at the end of each chapter and the bibliography make the subject matter discussed complete in all respects. It will have a satisfied clientele for a long time to come.

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Norbert Wiener: Collected works, Vol. 4, edited by P. Masani. The MIT Press, 28, Carleton Street, Cambridge, Massachusetts, 02142, 1985, pp. 1083, $ 70. Indian orders to Affiliated East-West Press, 6, Roselyn Gardens Apartments, 20/1A, Barnaby Road, Madras 600 010.

This is the final volume of the collected works of the late Prof. Norbert Wiener. Its bulk (1083 pages) and contents (cybernetics, science and society; ethics, aesthetics and literary criticism; book reviews and obituaries), following three previous volumes that contained Wiener's contributions to mathematical philosophy, the "Wiener" integrals, generalised harmonic analysis, turbulence, statistical mechanics, the "Wiener–Hopf" integral equation, prediction and filtering, quantum mechanics, relativity, etc., show the prodigious scope of Wiener's intellect.

About three quarters of the present volume (which should be of special interest to engineers) are devoted to reproductions of Wiener's writings on cybernetics, a subject which he for all practical purposes founded. All the classic papers are here, along with many not so well-known, often written for popular journals or delivered as lectures (some of them unfortunately repetitive); but Wiener's contributions to cybernetics have now become part of the received wisdom of the subject, so it is perhaps unnecessary to dwell on them at great length. Seeing all these contributions at one place, however, one agrees with the editor on the amazing cohesion in Wiener's thought over his whole life. Unlike Bertrand Russell (for example) and many other well-known intellects, Wiener hardly ever had to abandon his earlier positions; it is almost as if his scientific life was an unfolding and elaboration of ideas and concerns that he had right from his youth. At the age of 11, the young Norbert Wiener already wrote a paper on the theory of ignorance, at 16 another on the philogenetic development of the brain, and only a couple of years later one on the place of teleology in science. And he was writing on similar subjects, hardly ever having to correct himself, till almost the end of his life.

A volume like this can be particularly interesting when it reveals contributions that were either unknown or did not get the attention they deserved when first published. Given the remarkable breadth of Wiener's interests and the facility with which he wrote and published, it is no surprise to discover in this volume some papers which demand wider notice. I will pick only two examples, partly because they happen to be of personal interest.

The first is a reproduction (for the first time) of a memorandum on the mechanical solution of partial differential equations, submitted to Dr. Vannevar Bush in 1940. The memorandum is preceded by a letter to Bush, responding to his request (earlier that day!) for suggestions from MIT faculty on what they could contribute to the war effort. The memorandum is basically a proposal suggesting the construction of a digital computer to solve Laplace's equations, in particular for the flow past an aircraft wing. Dr. Bush did not pursue the project as he considered that it would drain manpower needed for more urgent war work. As we now know, the digital computer had to wait for the end of the war and the advocacy of von Neumann before it became a reality. It is interesting however that all the elements of what is now called a von Neumann machine were already there in Wiener's proposal, except for the concept of the stored program. If the Wiener memorandum had been widely circulated by Bush, it is quite possible that the von Neumann machine would today have been known after Wiener.

A second such paper, of particular interest to Indian readers, is about the "lonely nationalism" of Rudyard Kipling, written with K. Deutsch. This fascinating piece of literary criticism points out how Kipling celebrated the in-group, and how his writings reveal a horror of ambiguity and a frantic belief in the all-or-nothing character of group allegiance. Paradoxically, however, Kipling himself hardly ever belonged to such in-groups; from his school days, when he was scorned because of his poetic inclinations (presumably considered too 'soft' in his British boarding school), through the years in India where he could never join the Army because of his poor eyesight, to the days he spent in the U.S. where he wrote *Kim* and the *Jungle Book* among hostile or indifferent New Englanders, Kipling could never feel a sense of belonging; he was an alien wherever he was. The famous ballad about East and West ("Never the twain shall meet") was inspired by an actual incident where the twain had in fact met; the "western" hero of the ballad was in the historical encounter the grandson of an Afghan princess married to a British officer. Wiener points out how Kipling's passion for sharp distinctions goes with his worship of all determinate systems, whether it was the steam engine or the army.

The volume also contains a large number of miscellaneous snippets which add to its charm. Wiener writes about his visits to Europe (recalling a day when the U.S. had not yet become the power-house of science and technology that it now is); about the decline of cook-book engineering and the necessity for maintaining the "austere decencies" of the scientist; about the alphabet, aesthetics, atomic energy and anti-aircraft guns. His mind was encyclopaedic: I recall that when I was introduced to him at MIT in 1959, his first response was the enquiry "You must be Dravidian?". He cultivated friendship and collaboration not only with mathematicians but with engineers, biologists, physicists, doctors, artists and men of letters. This breadth of interest and scholarship is attractively demonstrated in this volume.

One unusual feature of the book is that the papers reproduced are generally followed by a commentary, reminding one of ancient Sanskrit texts. Written often by leading scientists in the area of the paper, the commentary is intended to provide a later-day perspective. Some of the commentaries do succeed in this aim: *e.g.* the one on the Wiener memorandum on computing machines. Others are not so successful, and some degenerate into short summaries of the paper, written with less force and clarity than the original–after all, Wiener's pen was as quick as his mind.

I strongly recommend the book as something that you might wish to look through on the variety of subjects that happened to interest a prodigious and lively mind.

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Women in science by Marilyn Bailey Ogilvie. The MIT Press, 28, Carleton Street, Cambridge, Massachusetts, 02142, USA, 1986, pp. 285, $ 25. Indian orders to Affiliated East-West Press Pvt. Ltd., 6, Roselyn Gardens Apartments, 20/1A, Barnaby Road, Madras 600 010.

The opening page of the book by Marilyn Ogilvie has the following quotation by Hertha Ayrton: "Personally I do not agree with sex being brought into science at all. The idea of

'Women and Science' is entirely irrelevant. Either a woman is a good scientist, or she is not; in any case she should be given opportunities, and her work should be studied from the scientific, not the sex, point of view". Being a woman scientist I am afraid I have myself very strong opinions against books dealing with women and science tending to agree with Hertha. But, on going through this particular book by Marilyn Ogilvie, I cannot help appreciating the tremendous effort put in by the author to collect the biographical notes about the European and American women of science from antiquity through the nineteenth century. It required an enormous reference work to bring an up-to-date resource book for all those concerned with history of science. It is an excellent achievement.

Unlike men, the women's activities in the fields related to science at different periods of history, have been closely interlinked with the status of women in the society during that period. Thus to understand the role of women in science the author has given a brief and interesting historical view reflecting the general status of women in the society at diffferent periods of time: Antiquity, The middle ages, The fifteenth, sixteenth and seventeenth centuries, The eighteenth century, The nineteenth and twentieth centuries.

The biographical notes are collected from various sources and are compiled in a lucid style. An effort is made to give a detailed bio-data depending on the availability. After each biographical account are listed the major or representative works by the subject.

An appendix to biographical accounts includes nineteenth-century women whose contributions appear to merit further study but for whom the author could collect only partial information. This offers as a starting place for further research.

The bibliography provides a very comprehensive list of biographical dictionaries, encyclopedias, catalogues, general history books with biographical information on women scientists and collective biographies. This makes this reference book very useful for research workers in the history of science.

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