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Multiple solutions of nonlinear probabilistic load flow equations*

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Abstract

In this paper the probabilistic load flow equations have been proved to belong to quartic equations. The given variables are the expected values and the moments of the second order of bus powers. The searched variables are the expected values and the moments of the second order of the rectangular components of bus voltages. The probabilistic solutions have many values and according to starting point the iterative process converges to different solutions. Because of the dimension of the problem the analysis of multiple solutions has been carried for 2-bus power system.

Key words: Power system, probability, load flow.

1. Introduction

The calculation of probabilistic load flows is of a great interest since it enables to avoid expensive simulation computations necessary while designing the power system work under the condition of random variations of bus powers[1–8].

While the deterministic load flow problem gives information on a single operating point, probabilistic load flow depicts the range of variations of the output quantities due to the random variations of the input quantities.

Probabilistic load flow equations are obtained from the nonlinear load flow equations with bus powers characterized by probability distributions or the expected values and the moments of the second order. In the earlier works concerning the calculation of probabilistic load flows, the probabilistic load flow equations were based on the direct current model of the electric power system[1], on the linearized load flow equations[2–4], and on the nonlinear load flow equations[5–8].

With linear models the shape of the output probability distributions can be computed by convolution techniques[3,4]. With nonlinear models, the evaluation of the output probability distributions is so difficult that the technique of statistic moments is used to determine the moments of the output quantities regardless of the shape of the input probability distributions[5–8].

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The nonlinear probabilistic equations are divided into equations for expected values and matrices of covariances; this, however, makes it impossible to fully maintain the nonlinearity of probabilistic load flow equations[7].

The solution of linear or nonlinear probabilistic load flow equations cannot avoid the possibility that load and generation random changes could drive the system into an unfeasible operating region. Hence the functional constraints *i.e.*, limits on generation, voltages, currents, etc., should be included in the analysis[8].

Literature survey does not show any work on the solvability of probabilistic load flow equations and their connection with deterministic load flow equations. It is only the use of the moments of the second order, instead of the covariances, which enables to maintain the full nonlinearity of probabilistic load flow equations and to obtain multiple solutions. The set of probabilistic load flow equations has many solutions and according to the starting point the iterative process converges to different solutions. The nature of these solutions could be predicted by an analysis of the probabilistic load flow equations of the particular electric power systems.

In this paper the analysis of PQ-and PU-bus power system has been carried out.

2. Formulation of the problem

The deterministic quadratic form of load flow equations for PQ- and PU-buses is well-known. The given variables are bus powers and bus voltage magnitude, the searched variables – rectangular components of bus voltages. As the electric power system works under random conditions, instead of precise bus power values, only the following quantities could be given: a) expected values of bus powers, b) covariances or the moments of the second order of bus powers. The following are to be searched: a) expected values of the rectangular components of bus voltages, b) the moments of the second order and the covariances of rectangular components of bus voltages.

In order to solve such a problem one should find the functional relation between the expected values and the second order moments of bus powers and rectangular components of bus voltages.

3. Deterministic load flow equations

The equations are of the following form

$$P_k = \sum_{l=1}^{w} (G_{kl}K_{kl} + B_{kl}L_{kl})$$

$$Q_k = \sum_{l=1}^{w} (-B_{kl}K_{kl} + G_{kl}L_{kl})$$

$$U_k^2 = K_{kk} \tag{1}$$

where

G_{kl}, B_{kl}: bus admitances,

$K_{kl} = U_{k1}U_{l1} + U_{k2}U_{l2}$, $L_{kl} = -U_{k1}U_{l2} + U_{k2}U_{l1}$

$k1 = k+k-1$, $k2 = k+k$, $l1 = l+l-1$, $l2 = l+l$

U_{k1}, U_{l1}: real components of bus voltages,

U_{k2}, U_{l2}: imaginary components of bus voltages,

U_k: voltage magnitude at bus k,

w: the number of buses.

Equations (1) can be rewritten in their general form as one equation:

$$s_k = \sum_{l=1}^{w} (b_{kl}K_{kl} + c_{kl}L_{kl}) \tag{2}$$

where

if $s_k = P_k$ then $b_{kl} = G_{kl}$, $c_{kl} = B_{kl}$

if $s_k = Q_k$ then $b_{kl} = -B_{kl}$, $c_{kl} = G_{kl}$

if $s_k = U_k^2$ then $b_{kk} = 1$, $c_{kk} = 0$.

Form (2) is very convenient while deriving probabilistic load flow equations since it enables to shorten the equations without losing any detail.

4. Probabilistic load flow equations

First the equations are to be derived for expected values and then for the moments of the second order.

4.1 *Equations for expected values*

Performing the operation of the expected value on the function (2) one obtains

$$\bar{s}_k = \sum_{l=1}^{w} (b_{kl}\bar{K}_{kl} + c_{kl}\bar{L}_{kl}) \tag{3}$$

where

$\bar{K}_{kl} = m_{k1l1} + m_{k2l2}$, $\bar{L}_{kl} = -m_{k1l2} + m_{k2l1}$

$m_{ij} = Ex_ix_j$, the moment of the second order.

Hence the equations for expected values are linear.

4.2 *Equations for the moments of the second order*

The equation is obtained by means of the operation of the expected value on the product s_ks_m, where k and m mean the bus numbers

$$\overline{s_k s_m} = \sum_{l=1}^{w} \sum_{n=1}^{w} \Big(b_{kl} b_{mn} \overline{K_{kl} K_{mn}} + b_{kl} c_{mn} \overline{K_{kl} L_{mn}} + c_{kl} b_{mn} \overline{L_{kl} K_{mn}} + c_{kl} c_{mn} \overline{L_{kl} L_{mn}} \Big)$$

where

$$\overline{K_{kl} K_{mn}} = m_{k1l1k2l2} + m_{k1l1m2n2} + m_{k2l2m1n1} + m_{k2l2m2n2}$$
$$\overline{K_{kl} L_{mn}} = -m_{k1l1m1n2} + m_{k1l1m2n1} - m_{k2l2m1n2} + m_{k2l2m2n1}$$
$$\overline{L_{kl} K_{mn}} = -m_{k1l2m1n1} - m_{k1l2m2n1} + m_{k2l1m1n1} + m_{k2l1m2n1}$$
$$\overline{L_{kl} L_{mn}} = m_{k1l2m1n2} - m_{k1l2m2n1} - m_{k2l1m1n2} + m_{k2l1m2n1}$$

$m_{ijkl} = E x_i x_j x_k x_l$, the moment of the fourth order.

The moments of the fourth order subject depend on expected values, moments of second and third order which means that the number of the unknown exceeds the number of equations. In order to equate the number of the unknown and the equations one must express the moments of third and fourth order by means of expected values and the moments of second order. This is possible when the probability distribution of the rectangular components of the bus voltages is a multidimensional normal distribution. Then

$$m_{ijk} = m_{ij} x_k + m_{ik} x_j + m_{jk} x_i - 2\bar{x}_i \bar{x}_j \bar{x}_k \tag{5}$$

$$m_{ijkl} = m_{ij} m_{kl} + m_{ik} m_{jl} + m_{il} m_{jk} - 2\bar{x}_i \bar{x}_j \bar{x}_k \bar{x}_l . \tag{6}$$

From (5) and (6) it follows that the assumption of normality of the probability distribution of bus voltages makes the numbers of the unknown and equations equal.

5. Analysis of probabilistic load flow equations describing 2-bus power system

Figure 1 presents the scheme of the system for which the analysis has been carried out. Complex voltage at PQ- and PU-bus equals $U_1 + jU_2$ and the slack bus voltage equals one.

PQ-bus input data

$\bar{P} = 0.5$, $\bar{Q} = 0.25$, $\sigma_P = 20\%$, $\sigma_Q = 20\%$, case 1 $\rho_{PQ} = -0.8$,
case 2 $\rho_{PQ} = 0.8$.

PU-bus input data

$\bar{P} = 1$, $\sigma_P = 50\%$, U-deterministic value, case 1 $U = 1$,
case 2 $U = 0.6$,

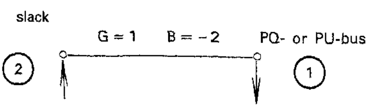

FIG. 1. Scheme of 2-bus power system.

where σ is the standard deviation, and ρ the correlation coefficient.

5.1 *Deterministic load flow equations*

Parameter values b_{kl} and c_{kl} are the following:

if $s_k = P$ then $b_{11} = 1$, $c_{11} = -2$, $b_{12} = -1$, $c_{12} = 2$;
if $s_k = Q$ then $b_{11} = 2$, $c_{11} = 1$, $b_{12} = -2$, $c_{12} = 1$;
if $s_k = U^2$ then $b_{11} = 1$, $c_{11} = 0$.

Hence the deterministic load flow equations are of the following form:

PQ-bus

$$P = U_1^2 + U_2^2 - U_1 + 2U_2$$
$$Q = 2U_1^2 + 2U_2^2 - 2U_1 - U_2 \tag{7}$$

PU-bus

$$P = U_1^2 + U_2^2 - U_1 + 2U_2$$
$$U^2 = U_1^2 + U_2^2. \tag{8}$$

Equations (7) and (8) are quadratic and have two solutions.

5.2 *Probabilistic load flow equations: PQ-bus*

Making use of the following substitutions

$$Y_1 = \bar{P},\ Y_2 = \bar{Q},\ Y_3 = m_{PP},\ Y_4 = m_{QQ},\ Y_5 = m_{PQ}$$
$$X_1 = \bar{U}_1,\ X_2 = \bar{U}_2,\ X_3 = m_{11},\ X_4 = m_{22},\ X_5 = m_{12}$$

where

$$m_{ij} = \bar{x}_i\bar{x}_j + \rho_{ij}\sigma_i\sigma_j$$

and of the formulae (3) and (4) one obtains five quartic equations

$$\begin{aligned}
Y_1 &= -X_1 + 2X_2 + X_3 + X_4;\\
Y_2 &= -2X_1 - X_2 + 2X_3 + 2X_4; \qquad (9)\\
Y_3 &= -2X_1^4 - 4X_1^2X_2^2 - 2X_2^4 + 4X_1^3 - 8X_1^2X_2 + 4X_1X_2^2 - 8X_2^3 - 6X_1X_3\\
&\quad - 2X_1X_4 + 8X_1X_5 + 8X_2X_3 + 12X_2X_4 - 4X_2X_5 + 3X_3^2 + 3X_4^2\\
&\quad + 2X_3X_4 + 4X_5^2 + X_3 + 4X_4 - 4X_5;\\
Y_4 &= -8X_1^4 - 16X_1^2X_2^2 - 8X_2^4 + 16X_1^3 + 8X_1^2X_2 + 16X_1X_2^2 + 8X_2^3 - 24X_1X_3\\
&\quad - 8X_1X_4 - 8X_1X_5 - 4X_2X_3 - 12X_2X_4 - 16X_2X_5 + 12X_3^2 + 12X_4^2\\
&\quad + 8X_3X_4 + 16X_5^2 + 4X_3 + X_4 + X_5;\\
Y_5 &= -4X_1^4 - 8X_1^2X_2^2 - 4X_2^4 + 8X_1^3 - 6X_1^2X_2 + 8X_1X_2^2 - 6X_2^3 - 12X_1X_3\\
&\quad - 4X_1X_4 + 6X_1X_5 + 3X_2X_3 + 9X_2X_4 - 8X_2X_5 + 6X_3^2 + 6X_4^2 + 4X_3X_4\\
&\quad + 8X_5^2 + 2X_3 - 2X_4 - 3X_5.
\end{aligned}$$

5.3 *Probabilistic load flow equations: PU-bus*

Making use of the previous substitution and $Y_2 = \bar{U}^2$, $Y_4 = m_{U^2U^2}$, $Y_5 = m_{PU^2}$ one obtains five quartic equations

Y_1 = the same as for PQ-bus;

$$Y_2 = X_3 + X_4;$$

Y_3 = the same as for PQ-bus;

$$Y_4 = -2X_1^4 - 4X_1^2X_2^2 - 2X_2^4 + 3X_3^2 + 2X_3X_4 + 4X_5^2 + 3X_4^2;$$

$$Y_5 = -2X_1^4 - 4X_1^2X_2^2 - 2X_2^4 + 2X_1^3 + 2X_1X_2^2 - 4X_1^2X_2 - 4X_2^3 + 3X_3^2 + 3X_4^2 + 4X_5^2 + 2X_3X_4 - 3X_1X_3 - 2X_2X_5 - X_1X_4 + 6X_3X_4 + 4X_1X_5 + 2X_2X_3.$$

Equations (9) and (10) can be written in their general form

$$Y = F(X) \tag{11}$$

where Y is the vector of given quantities, X the vector of searched quantities, and F the quartic function. Equation (11) can be solved iteratically using Newton method.

5.4 *Multiple solutions*

The results have been presented in Table I for PQ-bus and in Table II for PU-bus. From the results presented it follows that:

1. Probabilistic load flow equations have two multiple solutions similar to the deterministic load flow equations.

2. Some of the solutions are characterized by the correlation coefficient greater than one or imaginary value which means that probabilistic solutions may contain such area where there are no solutions of deterministic load flow equations.

Table I
PQ-bus load flow solutions

Variable	Case 1		Case 2	
	Sol. 1	Sol. 2	Sol. 1	Sol. 2
$\bar{U}_1 = X_1$	0.644	0.356	0.550	0.450
$\bar{U}_2 = X_2$	−0.15	−0.15	−0.15	−0.15
$\sigma_1 = \sqrt{X_3 - X_1^2}$	0.066	0.066	0.155	0.155
$\sigma_2 = \sqrt{X_4 - X_2^2}$	0.048	0.048	0.033	0.033
$\rho = (X_5 - X_1X_2)/\sigma_1\sigma_2$	0.9	−0.9	2.8	−2.8
			inadmissible	

Table II
PU-bus load flow solutions

Variable	Case 1		Case 2	
	Sol. 1	Sol. 2	Sol. 1	Sol. 2
$\bar{U}_1 = X_1$	0.872	−0.872	0.291	−0.547
$\bar{U}_2 = X_2$	0.436	−0.436	0.466	0.046
$\sigma_1 = \sqrt{X_3 - X_1^2}$	0.095	0.095	0.203	j 0.077
$\sigma_2 = \sqrt{X_4 - X_2^2}$	0.199	0.199	0.131	0.257
$\rho = (X_5 - X_1X_2)/\sigma_1\sigma_2$	−1.00	−1.00	−1.30	$-j$ 0.09
			inadmissible	

3. Inadmissable solution for PU-bus may mean that the power system is steady-state unstable because of too low value of voltage magnitude.

4. The value of standard deviations and correlation coefficients have an effect on the solution. In case 2 of PQ-bus the change of the sign of the correlation coefficient gives the inadmissable solution.

6. Conclusion

Probabilistic load flow equations are quartic and can be derived for expected values and the moments of the second order of rectangular components of bus voltages. In order to solve the probabilistic equations one should assume a normal multidimensional probability distribution of the rectangular components of bus voltages.

Probabilistic load flow equations have multiple solutions, some of which may be inadmissible. According to the chosen starting point one can obtain different solutions with the same input data.

Further investigation is required to extend the probabilistic analysis to larger power systems.

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