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A new multidimensional rational approximant*

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Abstract

A new Padé-type rational approximant for a power series in two variables matched at the nodal points of a raster structure different from both Chisholm and Bose & Basu is developed. It becomes easier to define n-D rational approximants of a power series in n-variables for our choice of nodal points of n-D raster structure, which can be reduced to Padé approximants in one variable and Padé-type rational approximants in $(n-k)$ variables. Unlike Chisholm's approximants no special constraints are imposed to obtain these unique rational approximant for 'normal case'. All the properties satisfied by Chisholm's approximants are also satisfied by the new approximants.

Key words: Padé approximation, digital signal processing, rational approximant, power series, IIR digital filters.

1. Introduction

The role of one-dimensional Padé approximation theory in the areas of theoretical physics[1], numerical analysis[2], and electrical engineering[3] is well established. In multidimensional digital signal processing applications, the extended Padé approximation theory has been widely used[4].

A multidimensional rational approximant is the ratio of two n-variable polynomials constructed from the coefficients of power series in n-variables. Unlike Padé approximant the definition of rational approximant for more than one variable is not unique. It depends upon the choice of nodal points of n-D raster structure at which the coefficients of Taylor series expansion of the approximants are matched. Chisholm and McEwan[5] extended the definition of rational approximant of a power series in more than two variables corresponding to Chisholm structure[5]. Bose and Basu[6] also defined Padé-type rational approximants of a power series in two variables for a raster structure different from Chisholm. They employ block Hankel matrix structure for characterisation of the rational approximants and have shown that unlike 1-D scalar case, the solution to 2-D Padé approximation problem may not be unique when a solution is guaranteed to exist. In a recent communication[7], we have developed an explicit determinant algorithm for Chisholm approximants and used it for the design of IIR digital filters from a knowledge of given impulse response.

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In the following sections we develop new n-D rational approximants and discuss their properties.

2. Main results

For a power series in n-variables given by

$$f(z) = \sum_{k=0}^{\infty} c_k \prod_{p=1}^{n} z_p^{k_p} \tag{1}$$

the $[l/m]$ rational approximant is defined as

$$f_{l,m}(z) = \frac{A_l(z)}{B_m(z)} = \frac{\sum_{i=0}^{l} a_i \prod_{p=1}^{n} z_p^{i_p}}{\sum_{j=0}^{m} b_j \prod_{p=1}^{n} z_p^{j_p}} \tag{2}$$

where

$$[l/m]' = [l_1, l_2, \ldots, l_n/m_1, m_2, \ldots, m_n]$$

$$z = (z_1, z_2, \ldots, z_n);\ c_k = c_{k_1, k_2, \ldots, k_n}$$

$$a_i = a_{i_1, i_2, \ldots, i_n};\ b_j = b_{j_1, j_2, \ldots, j_n}$$

$$\sum_{k=0}^{\infty} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty};\ \sum_{i=0}^{l} = \sum_{i_1=0}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_n=0}^{l_n}$$

$$\sum_{j=0}^{m} = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_n=0}^{m_n}$$

$a_i = 0(i_p > l_p)$; $b_j = 0(j_p > m_p)$; $c_k = 0(k_p < 0)$ (for any given $p = 1, 2, 3, \ldots, n$).

Since multiplying the numerator and the denominator by any constant leaves $f_{l,m}(z)$ unchanged, we impose the normalization condition

$$B_m(0) = b_{0,0,\ldots,0} = 1. \tag{3}$$

In general, for normal case we can write

$$\left(\sum_{i=0}^{\infty} c_i \prod_{p=1}^{n} z_p^{i_p}\right)\left(\sum_{j=0}^{m} b_j \prod_{p=1}^{n} z_p^{j_p}\right) - \left(\sum_{i=0}^{l} a_i \prod_{p=1}^{n} z_p^{i_p}\right) = \left(\sum_{i=0}^{\infty} r_i \prod_{p=1}^{n} z_p^{i_p}\right) \tag{4}$$

where $r_k = 0$ at $k = (k_1, k_2, \ldots, k_n)$ nodal points of the raster structure at which $f(z)$ and $f_{l,m}(z)$ are exactly matched.

To define $f_{l,m}(z)$ for 2, 3 and n dimensions we need to obtain a_is and b_js for these dimensions. Further, this necessitates description of various regions of an n-dimensional raster structure. To define regions $R_{\{S_d(n)\}} = R_{\{\ \}}$, let $S_n(n) = \{1, 2, \ldots, n\}$ be set of integers, and $S_d(n)$ be each and every subset of d ($d = 1, 2, \ldots, n$) number of elements contained in the set $S_n(n)$.

In the following subsections we obtain approximants for $n = 2$, 3 and n.

2.1. *New 2-D rational approximants*

For a formal power series in two variables given by (1) for $n = 2$, we define $[l_1, l_2/m_1, m_2]$ rational approximant in (2) for $n = 2$. With $l_1 = m_1 = l$, $l_2 = m_2 = m$ and constraints $|l_1 - l_2| = |m_1 - m_2| \leq 2$ for obtaining a simplified recursive relation. $A_l(z)$ and $B_m(z)$ can be written as

$$A_l(z) = \sum_{i_1=0}^{l} \sum_{i_2=0}^{m} a_{i_1, i_2} z_1^{i_1} z_2^{i_2}, \tag{5}$$

$$B_m(z) = \sum_{j_1=0}^{l} \sum_{j_2=0}^{m} b_{j_1, j_2} z_1^{j_1} z_2^{j_2}. \tag{6}$$

Since $b_{0,0} = 1$ from (3), we need to define $\{(l+1)\ (m+1)\}$ a_i coefficients and $\{(l+1)\ (m+1) - 1\}$ b_j coefficients. To obtain $\{2(l+1)\ (m+1) - 1\}$ a_i and b_j coefficients we need $\{2(l+1)\ (m+1) - 1\}$ independent equations. These are obtained from (4), modified corresponding to 2-D case, *i.e.*, for $n = 2$, by matching the coefficients of $\left(\prod_{p=1}^{2} z_p^{k_p}\right)$ at nodal points (k_1, k_2) of the 2-D raster where $r_k = 0$.

The $(l+1)\ (m+1)$ a_i coefficients are determined in terms of b_j coefficients satisfying (4), modified for $n = 2$, at nodal points corresponding to 2-tuple $k = (k_1, k_2)$, for $0 \leq k_1 \leq l$; $0 \leq k_2 \leq m$ and $r_k = 0$. The set of these equations is given as

$$\sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} c_{k-i} b_i = a_k;\ k = \{(k_1, k_2) : 0 \leq k_1 \leq l; 0 \leq k_2 \leq m\} \tag{7}$$

where $c_{k-i} = c_{(k_1-i_1),\ (k_2-i_2)}$.

The range of parameter k is shown in fig. 1 as shaded rectangle marked as R. The remaining $\{(l+1)\ (m+1) - 1\}$ equations are obtained in the regions $R_{\{S_d(n)\}}$. The nodal points in two-dimensional regions *i.e.* the region $R_{\{S_d(n)\}}$ for $d = 1$, are defined by

$$0 \leq k_i \leq (l_i + m_i)\, i = 1,\ 2$$

if $p \in S_1(2)$ then $(l_p + 1) \leq k_p \leq (l_p + m_p)$ but if $p \in S_2(2)$ and $\underset{q \neq p\epsilon}{q}$ $(S_2 \backslash S_1)(2)$;

$$k_p + k_q \leq (l_q + m_q) \tag{8}$$

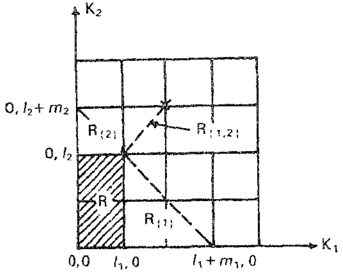

FIG. 1.

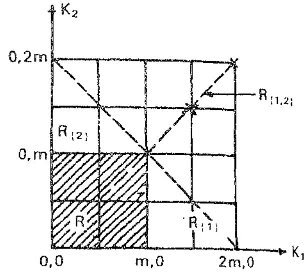

FIG. 2.

where for $n = 2$,

$$S_2(2) = \{1,2\} \text{ and } S_1(2) = \{1\}, \{2\}$$

if $S_1(2) = \{1\}$, then $(S_2 \backslash S_1)(2) = \{2\}$.

With parameter k defined by set of equation (8), $\frac{1}{2}[m(m+1)+l(l+1)]$ the number of equations containing b_l only can be obtained from (4) modified for $n = 2$ as

$$\sum_{i_1=0}^{l} \sum_{i_2=0}^{m} c_{(k_1-i_1),(k_2-i_2)} b_{i_1, i_2} = 0. \tag{9}$$

The range of parameter k in 2-D regions is shown as $R_{\{1\}}$ and $R_{\{2\}}$ in fig. 1.

The remaining $\frac{1}{2}[(l+m)-(l-m)^2]$ equations are obtained in 1-D region $R_{\{S_d(2)\}} = R_{\{1,2\}}$ for $d = 2$. The range of parameter k in 1-D region is defined as

$$\text{if } p, q \in S_2(2) \text{ then } k_p - k_q = l_p - l_q$$
$$\text{for } (l_p+1) \leqslant k_p \leqslant (m_p + l_p)$$
$$\text{and } (l_q+1) \leqslant k_q \leqslant (m_q + l_q). \tag{10}$$

The range of parameter k defined by (10) lying in 1-D region $R_{\{1,2\}}$ is marked 'x' in fig. 1. For this range of k with $r_k = 0$, we obtain from (4), modified for $n = 2$, the following set of equations for determining b_js

$$\sum_{i_1=0}^{l} \sum_{i_2=0}^{m} c_{(k_1-i_1),(k_2-i_2)} b_{i_1,i_2} = 0. \tag{11}$$

When any one of the z_i vanishes, (7) and (9) together with (11) reduce to equations corresponding to Padé approximants for a power series in 1-variable.

For $n = 2$, if $l_1 = l_2 = m_1 = m_2 = m$, (2) defines 2-D diagonal rational approximants. In such a case (7), (9) and (11), after proper modification, can be expressed in terms of double-complex contour integral of the function

$$R(z_1, z_2) = \left(\sum_{i=0}^{2m} c_i z_1^{i_1} z_2^{i_2}\right)\left(\sum_{j=0}^{m} b_j z_1^{j_1} z_2^{j_2}\right) - \left(\sum_{i=0}^{m} a_i z_1^{i_1} z_2^{i_2}\right). \tag{12}$$

Equations (7) and (9) can therefore be written as

$$\oint_{\Gamma_1}\oint_{\Gamma_2} R(z_1, z_2) z_1^{-i_1-1} z_2^{-i_2-1} dz_1 dz_2 = 0$$
$$(i_1 \geqslant 0, i_2 \geqslant 0; i_1 + i_2 \leqslant 2m) \tag{13}$$

where Γ_1 and Γ_2 are simple closed contours encircling the origin in the complex z_1- and z_2-planes, respectively. Equations defined by (11) can be written as

$$\oint_{\Gamma_1}\oint_{\Gamma_2} R(z_1, z_2) z_1^{-i_1-1} z_2^{-i_2-1} dz_1 dz_2 = 0$$
$$(m+1 \leqslant i_1 = i_2 \leqslant 2\, m). \tag{14}$$

The set of nodal points defined in (13) and (14) is shown in fig. 2. We observe that each c_k occurs in only one of the set of equations (7), (9) and (11) as coefficients of $b_{0,0}$.

2.2 *New 3-D diagonal rational approximants*

For a formal power series in three variables given by (1) for $n = 3$, we define $[l_1, l_2, l_3/m_1, m_2, m_3]$ rational approximant by (2) for $n = 3$. With $l_1 = l_2 = l_3 = m_1 = m_2 = m_3 = m$, a diagonal rational approximant is obtained corresponding to which $A_l(z)$ and $B_m(z)$ can be written as

$$A_l(z) = \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{m} a_{i_1, i_2, i_3} z_1^{i_1} z_2^{i_2} z_3^{i_3} \tag{15}$$

and

$$B_m(z) = \sum_{j_1=0}^{m} \sum_{j_2=0}^{m} \sum_{j_3=0}^{m} b_{j_1, j_2, j_3} z_1^{j_1} z_2^{j_2} z_3^{j_3}. \tag{16}$$

Since $b_{0,0,0} = 1$ from (3), for $n = 3$, we need to define $(m+1)^3$ a_i coefficients and $\{(m+1)^3 - 1\}$ b_j coefficients. To obtain $\{2(m+1)^3 - 1\}$ a_i and b_j coefficients we need $\{2(m+1)^3 - 1\}$ independent equations. These are obtained from (4), modified corresponding to 3-D diagonal case, by matching the coefficients of $\left(\prod_{p=1}^{3} z_p^{i_p}\right)$ at nodal points (k_1, k_2, k_3) of the raster structure where $r_k = 0$.

The $(m+1)^3$ a_i coefficients are determined in terms of b_j coefficients satisfying (4), modified corresponding to 3-D diagonal case, at nodal points corresponding to 3-tuple

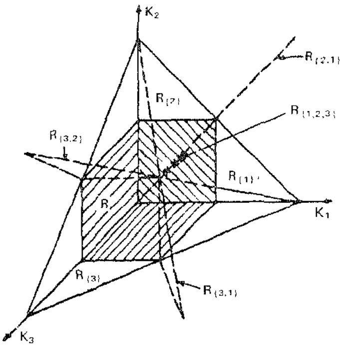

FIG. 3.

$k = (k_1, k_2, k_3)$, for $0 \leqslant k_p \leqslant m$, $p = 1, 2, 3$, and $r_k = 0$. The set of these equations is given as

$$\sum_{i=0}^{k} c_{(k-i)}\, b_i = a_k$$

for $k = \{(k_1, k_2, k_3) : 0 \leqslant k_p \leqslant m; p = 1, 2, 3\}$. (17)

The range of parameter k is shown in fig. 3 as shaded cube marked as R.

When any one of z_i vanishes, (17) reduces to equations corresponding to (7) for $l = m$ for 2-D diagonal rational approximant and when any two of z_i vanish then (17) reduces to corresponding equation for diagonal Padé approximants.

The remaining $\{(m+1)^3 - 1\}$ number of equations involving b_j only are obtained in the regions $R_{\{s_d(3)\}}$. The nodal points k of the raster structure corresponding to regions $R_{\{s_d(3)\}}$ are defined by the following expressions

$$0 \leqslant k_i \leqslant 2m \qquad i = 1, 2, 3. \tag{18}$$

If $p, q \,\varepsilon\, S_d(3)$; $d = 1, 2, 3$; then $(m+1) \leqslant k_p = k_q \leqslant 2m$ (19)

but if $p \,\varepsilon\, S_3(3)$; $\underset{q \neq p}{q} \;\varepsilon\, (S_3 \backslash S_d)\,(3)$; then $k_p + k_q \leqslant 2\,m$. (20)

The total number of nodal points, k, obtained from the last three expressions is given by

$$\sum_{d=1}^{3} \binom{3}{d} \sum_{k'=1}^{m} (k')^{3-d} \tag{21}$$

which is the exact number required to determine $\{(m+1)^3 - 1\}$ number of b_j coefficients.

The set of equations obtained from (4), modified corresponding to 3-D diagonal case by coefficient matching at nodal points described in (18–20) where $r_k = 0$ is given by

$$\sum_{i=0}^{m} c_{(k-i)} b_i = 0. \tag{22}$$

The different regions are shown in fig. 3 as $R_{\{1\}}$, $R_{\{2\}}$, $R_{\{3\}}$, $R_{\{1,2\}}$, $R_{\{2,3\}}$, $R_{\{3,1\}}$ and $R_{\{1,2,3\}}$. The number of regions $R_{\{S_d(3)\}}$ for a particular d is given by $\binom{3}{d}$.

When any one of z_i vanishes, (22) reduces to (9) and (11) for ($l = m$) corresponding to 2-D rational approximant, and when any two of z_i vanish, (22) reduces to corresponding equation for diagonal Padé approximants.

2.3 *New n-D diagonal rational approximants*

For an n-variable power series given by (1) the new n-D diagonal rational approximant is defined by (2) with the constraints

$$l_1 = l_2 = \ldots = l_n = m_1 = m_2 = \ldots = m_n = m. \tag{23}$$

Since $b_{0,0,\ldots,0} = 1$ from (3), we need to define $(m+1)^n$ a_i and $\{(m+1)^n - 1\}$ b_j coefficients. To obtain $\{2(m+1)^n - 1\}$ a_i and b_j coefficients we need $\{2(m+1)^n - 1\}$ independent equations. These are obtained from (4), modified for n-D diagonal approximants, by matching the coefficients of $\left(\prod_{p=1}^{n} z_p^{i_p}\right)$ at nodal points (k_1, $k_2, \ldots, k_n$) of the raster structure where $r_k = 0$.

The $(m+1)^n$ a_i coefficients are determined in terms of b_j coefficients satisfying (4) modified for diagonal approximants, at nodal points corresponding to n-tuples $k = (k_1, k_2, \ldots, k_n)$, for $0 \leqslant k_p \leqslant m$; $p = 1, 2, \ldots, n$ and $r_k = 0$. The set of these equations is

$$\sum_{i=0}^{k} c_{(k-i)} b_i = a_k;\ k = \{(k_1, k_2, \ldots, k_n):$$
$$0 \leqslant k_p \leqslant m, p = 1, 2, 3, \ldots, n\}. \tag{24}$$

The remaining $\{(m+1)^n - 1)\}$ equations involving b_j only are obtained in the regions $R_{\{S_d(n)\}}$. The number of regions for a particular d is equal to $\binom{n}{d}$. The nodal points k of the raster structure corresponding to regions $R_{\{S_d(n)\}}$ are defined by the following expressions

$$0 \leqslant k_i \leqslant 2m \quad i = 1, 2, \ldots, n \tag{25}$$

if $p, q \in S_d(n)$; $d = 1, 2, \ldots, n$ then

$$(m+1) \leqslant k_p = k_q \leqslant 2m \tag{26}$$

but if $p \in S_n(n)$; $\underset{q \neq p}{q \in (S_n \backslash S_d)}$ (n), then

$$k_p + k_q \leqslant 2m. \tag{27}$$

The total number of nodal points, k, defined from the last three expressions is given by

$$\sum_{d=1}^{n} \binom{n}{d} \sum_{k'=1}^{m} (k')^{n-d} \qquad (2$$

which is the exact number required to determine $(m+1)^n - 1$ number of b_j coefficient

The set of equations obtained from (4), modified for n-D diagonal approximants, t coefficients matching at nodal points, described in (25–27) where $r_k = 0$ is given t

$$\sum_{i=0}^{m} c_{(k-i)}\, b_i = 0. \qquad (2$$

In the following section we discuss the properties of diagonal rational approximant

3. Properties of new n-D rational approximants

All the properties satisfied by Chisholm diagonal rational approximants are also satisfie by the new diagonal rational approximants developed in this paper. For brevity we stat these properties in the following paragraphs. The proofs of these properties are discusse by Karan[8].

(1) *Symmetry*: The definition of the approximant is symmetrical between the variable z_k $(k = 1, 2, \ldots, n)$.

(2) *Existence and uniqueness*: An n-variable power series always defines a sequence o approximants. These approximants are in general unique.

(3) *Projection*: If any k $(< n)$ of the variables are equated to zero, the approximan reduces to corresponding approximants in $(n-k)$ variables formed from the powe series with the same variables equated to zero.

(4) *Homographic invariance*: The definition of the approximants is invariant under al transformations of the group

$$z_r = \frac{A w_r}{1 - B_r w_r} (A \neq 0;\ r = 1, 2, \ldots, n).$$

The homographic invariance group does not include relative scale transformation $z_r = A_r w_r$ with $A_r \neq 0$ $(r = 1, 2, \ldots, n)$ and $A_r \neq A_s$ for some r, s; no analogue o relative scale transformations exists for Padé approximants.

(5) *Reciprocal invariance*: An approximant formed from the reciprocal of a power series is the reciprocal of the corresponding power series approximant.

(6) *Factorization*: If the given series is the product of two power series, one in k variables and the other in the remaining $(n-k)$ variables, then an approximant is the product of the corresponding approximant to the two power series.

(7) *Additivity*: If the given series is the sum of two power series, one in k-variables and the other in the remaining $(n-k)$ variables, then an approximant is the sum of the corresponding approximants formed from the two power series.

Table I
Data for two-dimensional sample response

$h(k_1, k_2)$							
k_1k_2	0	1	2	3	4	5	6
0	1.000	0.500	0.250	0.125	0.063	0.031	0.016
1	0.700	0.400	0.225	0.125	0.069	0.038	0.020
2	0.490	0.315	0.195	0.117	0.069	0.040	0.023
3	0.343	0.245	0.165	0.106	0.066	0.041	0.024
4	0.240	0.189	0.136	0.093	0.061	0.039	0.024
5	0.168	0.144	0.111	0.080	0.055	0.036	0.024
6	0.117	0.109	0.090	0.067	0.047	0.330	0.022

3.1 *Example*

Consider the data given in Table I for two-dimensional sample response.

Based on the results developed by Karan[8] for obtaining determinant form of 2-D rational approximants on the basis of the approximants developed in this paper, we obtain $H(z_1, z_2)$ for the data in Table I, for $n = 2$ and $l_1 = l_2 = m_1 = m_2 = 1$ as

$$H(z_1, z_2) = \frac{\begin{bmatrix} 0 & 0 & h_{01} & h_{02} \\ 0 & h_{10} & 0 & h_{20} \\ h_{11} & h_{12} & h_{21} & h_{22} \\ h_{00}z_1z_2 & (h_{00}z_1 + h_{01}z_1z_2) & (h_{00}z_2 + h_{10}z_1z_2) & (h_{00} + h_{01}z_2 + h_{10}z_1 + h_{11}z_1z_2) \end{bmatrix}}{\begin{bmatrix} 0 & 0 & h_{01} & h_{02} \\ 0 & h_{10} & 0 & h_{20} \\ h_{11} & h_{12} & h_{21} & h_{22} \\ z_1z_2 & z_1 & z_2 & 1 \end{bmatrix}}. \tag{30}$$

Substitution of data in (30) and solving the determinants, we obtain

$$H(z_1, z_2) = \frac{1}{1 - 0.5z_2 - 0.7z_1 + 0.3z_1z_2}. \tag{31}$$

The system is found to be stable by applying the stability test reported by the authors[9].

4. Conclusion

In this paper, two- and multi-dimensional rational approximant are developed, which are different from the approximants of both Chisholm and Bose and Basu. The difference occurs mainly because of the choice of nodal points of a raster structure where the given power series matches with the rational approximants. The choice of raster point is natural and it correlates 1-D to n-D diagonal approximant without forcing symmetrization as has been done in Chisholm's case. The drawback of Bose and Basu's method is that the choice of raster points poses serious difficulty in generalization to n-D case.

To define off diagonal rational approximants for 2-D case we have restricted ourselves to such a case where the difference in degree of z_1 and z_2 in approximants is utmost 2. If this difference in degree is more than 2 then defining off diagonal rational approximants becomes difficult as the number of nodal points available in the region $R_{\{2\}}$ & $R_{\{1\}}$ is more than the required. With the restrictions imposed we could easily derive recursive relation for 2-D case as developed by Karan[8].

An explicit determinant form of representing rational approximants has been developed by Karan[8] and a 2-D IIR digital filter has been realised by using these algorithms. The stability of filters so realised can be conveniently checked by employing a new stability test for 2-D case reported recently by the authors[9].

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