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Two-step design of proportional integral-feedback controllers for singularly perturbed systems*

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Abstract
The paper presents a new two-step design of proportional integral feedback controllers for singularly perturbed systems with constant disturbances. The control law is derived using a slow subsystem obtained through the iterative approach. It is shown that the integral gain computation can be separated from proportional control design. The proposed design procedure is demonstrated in controller design for a synchronous machine connected to an infinite-bus.

Key words: Proportional integral feedback, singularly perturbed systems, parameter perturbation.

1. Introduction

The design of control system is often confronted with high dimensionality of the dynamical system. The problem is further aggravated by the presence of 'parasitic' parameters resulting in stiff numerical problems[1]. To alleviate these difficulties, model-order reduction and time-scale decomposition of such systems are generally carried out using aggregation methods, singular perturbation theory and feedback controllers are designed using the reduced-order models[2,3]. Besides the above difficulties, system disturbances are quite common and the proportional control alone can not take care of any external disturbances and/or parameter uncertainties. Thus, proportional integral-feedback controllers are designed to deal with such situations[4]. However, the task of determining optimal-feedback gains for proportional integral control becomes complicated for high-order systems. The difficulties are further increased due to numerical ill-conditioning associated with the singularly perturbed systems.

This paper presents a new two-step design of proportional-integral-feedback controllers for singularly perturbed systems. The control law is designed using the slow subsystem obtained *via* the iterative approach[5]. It is shown that the proportional and integral-feedback gains can be derived separately using parameter perturbation technique[6]. The theory is demonstrated in controller design for a single machine infinite-bus power system. The results of two-step design are compared with those obtained *via* the single-step design procedure.

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2. Problem formulation

Consider the following singularly perturbed linear time-invariant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u + E_1 \mathrm{d} \tag{1a}$$

$$\varepsilon \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u + E_2 \mathrm{d} \tag{1b}$$

$$y = C_1 x_1 + C_2 x_2 \tag{1c}$$

where $\varepsilon > 0$ is a small singular perturbation parameter, $x_1 \varepsilon R^{n1}$, $x_2 \varepsilon R^{n2}$, $u \varepsilon R^r$, $y \varepsilon R^p$, and d represents a constant unmeasurable disturbance.

It is required to design a control of the form

$$u = K_1 x - K_2 \int y \,\mathrm{d}t \tag{2}$$

where $x^T = [x_1^T \; x_2^T]$ for the system (1) such that the effect of constant disturbances is nullified and $y \to y_d$ as $t \to \infty$.

The effect of disturbances on output is eliminated by introducing integral state of the controlled output variable

$$\dot{w} = y = C_1 x_1 + C_2 x_2 . \tag{3}$$

Thus the problem here is to design u to minimize the performance index

$$J = \int_0^\infty (x^T Qx + \rho^2 w^T Fw + u^T Ru) \,\mathrm{d}t \tag{4}$$

subject to the dynamical constraints (1) and (3), where $\rho > 0$ is a small parameter.

3. Iterative separation of time scales

Equation (1b) can be written as

$$\dot{x}_2 = \hat{A}_{21} x_1 + \hat{A}_{22} x_2 + \hat{B}_1 u + \hat{E}_2 \mathrm{d} \tag{1d}$$

where

$$\hat{A}_{21} = A_{21}/\varepsilon, \hat{A}_{22} = A_{22}/\varepsilon, \hat{B}_2 = B_2/\varepsilon \text{ and } \hat{E}_2 = E_2/\varepsilon . \tag{5}$$

Using the following transformation[5,7]

$$\xi_j = x_1 - H_{kj}\, \eta_k , \eta_k = x_2 - L_k x_1 \tag{6}$$

(1a), (1d) and (1c) reduce to

$$\dot{\xi}_j = A_{kj}\, \xi_j + A_{ofj}\, \eta_k + B_{1j} u + E_1 \mathrm{d} \tag{7a}$$

$$\dot{\eta}_k = A_{fok}\, \xi_j + A_{fkj}\, \eta_k + B_{2k} u + \hat{E}_2 \mathrm{d} \tag{7b}$$

$$y = C_{1k}\, \xi_j + C_{2j}\, \eta_k \tag{7c}$$

where

$$L_i = \hat{A}_{22}^{-1}\hat{A}_{21} + \hat{A}_{22}^{-1} L_{i-1}(A_{11} - A_{12}L_{i-1})$$

$$i = 1,2,\ldots,k \text{ with } L_1 = \hat{A}_{22}^{-1}\hat{A}_{21} \tag{8}$$

$$H_{ki} = A_{12}A_{fk}^{-1} + (A_{k,i-1}H_{k,i-1} - H_{k,i-1}A_{fok})A_{fk}^{-1}$$

$$i = 1,2,\ldots,j \text{ with } H_{k1} = A_{12}A_{fk}^{-1} \tag{9}$$

$$\begin{aligned}
A_{kjk} &= A_k - H_{kj}A_{fok}, & A_k &= A_{11} - A_{12}L_k\\
A_{ojj} &= A_{12} - H_{kj}A_{fk} + A_{kj}H_{kj}, & A_{fk} &= \hat{A}_{22} + L_k A_{12},\\
A_{fkj} &= A_{fk} + A_{fok}H_{kj}, & A_{fok} &= \hat{A}_{21} - \hat{A}_{22}L_k + L_k A_k,\\
B_{1j} &= B_1 - H_{kj}B_{2k}, & B_{2k} &= \hat{B}_2 + L_k B_1\\
C_{1k} &= C_1 - C_2 L_k, & C_{2j} &= C_2 + C_{1k}H_{kj}
\end{aligned} \tag{10}$$

and j and k are the iteration counts on the slow and fast states, respectively.

Since A_{ojj} and $A_{fok} \to 0$ for large values of j and k, the slow and fast variables are decoupled and (7a), (7b) reduce to

$$\dot{\xi}_j = A_{kj}\xi_j + B_{1j}u + E_{1d} + 0(\varepsilon^{\min(j,k+1)}) \tag{11a}$$

$$\dot{\eta}_k = A_{fkj}\eta_k + B_{2k}u + \hat{E}_2 d + 0(\varepsilon^{\min(j,k)}). \tag{11b}$$

Neglecting the stable-fast modes and assuming A_{fkj} nonsingular, equation (11b) yields the slow part of η_k due to slow-input u_s as

$$\eta_{ks} = -A_{fkj}^{-1}B_{2k}u_s. \tag{12}$$

From (7c), (11a) and (12), the slow subsystem is obtained as

$$\dot{\xi}_j = A_{kj}\xi_j + B_{1j}u_s + E_{1d} \tag{13a}$$

$$y_s = C_{1k}\xi_j + D_o u_s \tag{13b}$$

where

$$D_o = -C_{2j}A_{fkj}^{-1}B_{2k}.$$

Since $y \approx y_s$, we can write

$$\dot{w} = C_{1k}\xi_j + D_o u_s. \tag{13c}$$

4. Two-step design of proportional integral controllers

The proportional-integral-feedback controller is designed using the slow subsystem (13) and considering the minimization of the following functional

$$J = \int_{t_0}^{\infty} (\xi_i^T Q_m \xi_i + \rho^2 w^T F w + u_s^T R u_s)\, dt. \tag{14}$$

The solution to the above problem is given in Kwakernaak and Sivan[8] as

$$u_s = - R^{-1} \hat{B}^T \hat{S} \begin{vmatrix} \xi_i \\ w \end{vmatrix} \tag{15}$$

where $\hat{S}$ is the symmetric positive definite solution of

$$\hat{S}\hat{A} + \hat{A}^T\hat{S} - \hat{S}\hat{B}R^{-1}\hat{B}^T\hat{S} + \hat{Q} = 0 \tag{16}$$

and

$$\hat{A} = \begin{bmatrix} A_{kj} & 0 \\ C_{1k} & 0 \end{bmatrix},\ \hat{B} = \begin{bmatrix} B_{1j} \\ D_o \end{bmatrix},\ \hat{Q} = \begin{bmatrix} Q_m & 0 \\ 0 & \rho^2 F \end{bmatrix}. \tag{17}$$

Partitioning $\hat{S}$ as

$$\hat{S} = \begin{bmatrix} \hat{S} & \hat{S} \\ \hat{S}_2^T & \hat{S}_3 \end{bmatrix} \tag{18}$$

and introducing (17), (18) into (16) we obtain the following set of equations.

$$\hat{S}_1 A_{kj} + \hat{S}_2 C_{1k} + A_{kj}^T \hat{S}_1 + C_{1k}^T \hat{S}_2^T - (\hat{S}_1 B_{1j} + \hat{S}_2 D_o) R^{-1}.$$

$$(B_{1j}^T \hat{S}_1 + D_o^T \hat{S}_2^T) + Q_m = 0 \tag{19a}$$

$$\hat{S}_2^T A_{kj} + \hat{S}_3 C_{1k} - (\hat{S}_2^T B_{1j} + \hat{S}_3 D_o) R^{-1}.\ (B_{1j}^T \hat{S}_1 + D_o^T \hat{S}_2^T) = 0 \tag{19b}$$

$$-(\hat{S}_2^T B_{1j} + \hat{S}_3 D_o) R^{-1} (B_{1j}^T \hat{S}_2 + D_o^T \hat{S}_3) + \rho^2 F = 0. \tag{19c}$$

Since ρ is small, the solution of (19) can be obtained through parameter perturbation technique[6] in the following form

$$\hat{S}_1 = S_1^o + \rho S_1^1 + \ldots \tag{20a}$$

$$\hat{S}_2 = \rho S_2^1 + \ldots \tag{20b}$$

$$\hat{S}_3 = \rho S_3^1 + \ldots. \tag{20c}$$

Introducing (20) into (19) and equating the coefficients of like powers of ρ, we obtain

$$S_1^o A_{kj} + A_{kj}^T S_1^o - S_1^o B_{1j} R^{-1} B_{1j}^T S_1^o + Q_m = 0 \tag{21}$$

$$(S_2^{1'} B_{1j} + S_3^1 D_o) R^{-1} (B_{1j}^T S_2^1 + D_o^T S_3^1) + F = 0 \tag{22}$$

$$S_2^{1'} A_{kj} + S_3^1 C_{1k} - (S_2^{1'} B_{1j} + S_3^1 D_o) R^{-1} B_{1j}^T S_1^o = 0 \tag{23}$$

$$S_1^1 A_{kj} + S_2^1 C_{1k} + A_{kj}^T S_1^1 + C_{1k}^T S_2^{1'} - (S_1^1 B_{1j} +$$

$$S_2^1 D_o) R^{-1} B_{1j}^T S_1^o - S_1^o B_{1j} R^{-1} (B_{1j}^T S_1^1 + D_o^T S_o^{1T}) = 0 \tag{24}$$

where equation (21) represents the reduced-order regulator problem with no integral terms. Equations (22) and (23) are set of coupled nonlinear equations and can be solved for S_2^1 and S_3^1 from the known value of S_1^o. We can express (24) as the following Liapunov equation for S_1^1

$$S_1^1 A_{CL} + A_{CL}^T S_1^1 + (S_2^1 H + H^T S_2^{1T}) = 0 \tag{25}$$

where

$$A_{CL} = (A_{kj} - B_{1j} R^{-1} B_{1j}^T S_1^o) \tag{26}$$

$$H = (C_{1k} - D_o R^{-1} B_{1j}^T S_1^o). \tag{27}$$

Once S_2^1 is known, equation (25) can be solved for S_1^1.

Using (15), (18) and (20) the closed-loop system can be represented as

$$\begin{bmatrix} \dot{\xi}_j \\ \dot{w} \end{bmatrix} = \hat{A}_{CL} \begin{bmatrix} \xi_j \\ w \end{bmatrix} \tag{28}$$

where

$$\hat{A}_{CL} = \hat{A}_{CL}^o + \hat{A}_{CL}^1 \tag{29}$$

and

$$\hat{A}_{CL}^o = \begin{bmatrix} A_{kj} - B_{1j} R^{-1} B_{1j}^T S_1^o & 0 \\ C_{1k} - D_o R^{-1} B_{1j}^T S_1^o & 0 \end{bmatrix} \tag{30}$$

$$\hat{A}_{CL}^1 = \begin{bmatrix} -B_{1j} R^{-1} (B_{1j}^T S_1^1 + D_o^T S_2^{1T}) & -B_{1j} R^{-1} (B_{1j}^T S_2^1 + D_o^T S_3^1) \\ -D_o R^{-1} (B_{1j}^T S_1^1 + D_o^T S_2^{1T}) & -D_o R^{-1} (B_{1j}^T S_2^1 + D_o^T S_3^1) \end{bmatrix} \tag{31}$$

Table I
Feedback gains and eigenvalues of the system

Feedback gain matrix $[K^T]$		Eigenvalues		
Single-step design	Two-step design	Open-loop	Closed-loop	
			Single-step	Two-step design
0·4348	0·8853	−0·2047	−1·5148	−1·5548
		−0·5556 + j9·958	−4·9308 + j14·508	−4·5383 + j12·573
−0·8985	−0·5225	−0·5556 − j9·958	−4·9308 − j14·508	−4·5383 − j12·573
		−31·0553	−2915·3278	−2801·5403
1·3802	1·3255	−38·1128	−38·1147	−38·1136
		−11·2898 + j313.564	−11·091 + j313·453	−11·1061 + j313·466
0·1011	0·1166	−11·2898 − j313·564	−11·091 − j313·453	−11·1061 − j313·466
		—	−0·0245	−0·0376

Thus the proportional integral control design can be carried out in two steps: (i) LQR design for the slow subsystem without integral control, (ii) computation of integral gains *via* perturbation methods.

5. Simulation results

We consider a synchronous machine connected to an infinite-bus[5,6] to illustrate the two-step design procedure. The system matrix, A, the input matrix, B, and the output matrix, C, for an operating point (P_o, O_o) are as follows:

$$A = \begin{bmatrix} 0{\cdot}0 & 1{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}0 & -23{\cdot}991 & -23{\cdot}991 & -52{\cdot}356 & -0{\cdot}928 & -94{\cdot}638 \\ -190{\cdot}139 & 0{\cdot}844 & -1{\cdot}468 & 24{\cdot}176 & 276{\cdot}396 & 3{\cdot}914 & 378{\cdot}422 \\ -347{\cdot}867 & 1{\cdot}544 & 1{\cdot}292 & -30{\cdot}200 & 505{\cdot}679 & 7{\cdot}161 & 692{\cdot}339 \\ -732{\cdot}006 & -1{\cdot}734 & -812{\cdot}175 & -812{\cdot}175 & -38{\cdot}741 & -1100{\cdot}370 & 11{\cdot}056 \\ 550{\cdot}415 & -2{\cdot}443 & 0{\cdot}130 & 4{\cdot}446 & -800{\cdot}115 & -11{\cdot}331 & -1095{\cdot}460 \\ 749{\cdot}755 & 1{\cdot}776 & 831{\cdot}868 & 831{\cdot}868 & 28{\cdot}295 & 1127{\cdot}050 & -11{\cdot}324 \end{bmatrix} \quad (32)$$

$$B = [0{\cdot}0 \quad 0{\cdot}0 \quad 2096{\cdot}86 \quad -1845{\cdot}52 \quad 0{\cdot}0 \quad -185{\cdot}5 \quad 0{\cdot}0]^T \quad (33)$$

$$C = [-0{\cdot}331 \quad 0{\cdot}0 \quad 0{\cdot}0 \quad 0{\cdot}0 \quad 0{\cdot}0 \quad -0{\cdot}197 \quad -0{\cdot}121]^T. \quad (34)$$

The slow subsystem model is obtained using the iterative approach with $j = k = 3$. The state and control weighting matrices are selected as $Q = \text{diag}\,[1, 1, 1, {\cdot}01]$ and $R = 1$ and the small perturbation parameter is chosen as $\rho = 0{\cdot}1$.

The proportional-integral-feedback controller is designed as outlined in the previous sections. The feedback gains and eigenvalues of the open- and closed-loop systems are compared with those obtained using single-step design procedure in Table I. It is observed that the proposed design approach yields quite satisfactory results. From simulation it is found that two-step design method reduces the computational burden by designing proportional and integral gains separately as compared to single-shot design.

The application of iterative time-scale separation to multimachine power systems for simulation in two time scale has been demonstrated in various research publications[10,11] The proposed two-step design technique can be applied to multimachine power systems and the proportional and integral control design can be carried out separately using the slow subsystem models as derived in Winkelman *et al*[10].

6. Conclusions

Two-step design of proportional-integral-feedback controllers for singularly perturbed systems is proposed to deal with unknown constant disturbances present in the system. It is shown that the integral control design can be separated from the design of proportional control law. The proposed method is illustrated in controller design for a synchronous machine connected to an infinite-bus.

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