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Hierarchical optimal control and stability of large-scale systems*

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Abstract

There are three parts in this paper: (1) a state-regulating problem is solved for a very-large-scale system (VLSS). The optimal-control laws are formulated with multiechelon-dynamical-hierarchical structure. A set of matrix equations which arises in formulating optimal-control laws is shown to be solvable in an alternative way; (2) the stability of such formulated multiechelon-hierarchical structure is analysed; (3) the number of levels of hierarchy needed in a multiechelon structure is calculated as also the order of dynamic-state regulator required.

Key words: Large-scale system, decentralised control, multiechelon hierarchy, coordinators, dynamic-state feedback, controllability and observability indices, Pontryagin's maximum principle, Second method of Liapunov.

1. Introduction and problem statement

One of the earliest formal quantitative treatments of hierarchical (multilevel) has been presented by Mesarovic *et al*[1]. Since then a great deal of work has been done in the field[2–14]. Two schemes, goal-coordination[1] and interaction-prediction[15], describe a 'coordination' process in hierarchical systems. In these schemes, only two-level controllers and their coordinations are proposed. The goal-coordination principle is concerned with open-loop control of hierarchical systems, whereas interaction-prediction has both open- and closed-loop forms of optimal control. There is another method[7–9,16] of closed-loop control of two-level hierarchical system which has linear state-feedback-control structure. In this method, a structural perturbation is employed through which the interactions among the subsystems are set to zero. This makes the system completely decomposed to subsystems so that local linear state-feedback-control laws can be generated. Because of structural perturbation, the performance criteria calculated are not the same as that for the overall system. Sundareshan[16] has shown that a class of interactions is said to be 'beneficial' if the performance criteria in decomposed case is greater than that in centralised case and the class of interactions is said to be 'neutral' if the calculated performance criteria for both decomposed and centralised cases are the same.

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In this paper, we consider a problem of designing optimal-control laws for a very-large-scale system (VLSS), using dynamical controllers and forming a multiechelon hierarchical structure. The performance criteria are taken as an integral of the square of the error in physical variables, so that the optimal-control laws designed shall regulate the states while minimising the cost function. Such multiechelon hierarchical structure with dynamical optimal-control laws is shown to be stable. The dynamical optimal-control laws are formulated with minimum information exchange amongst the levels of hierarchy as also the number of levels of hierarchy.

Consider a very-large-scale time-invariant system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^r$ the output and $u \in \mathbb{R}^m$ the input vector. The matrices, A, B, and C have appropriate dimensions. The mathematical model of a VLSS given by (1) can be written in another form as,

$$\begin{aligned} \dot{x}_l &= A_l x_l + \sum_{\substack{k=1 \\ k \neq l}}^{\gamma} A_{lk} x_{lk} + B_l u_l \\ y_l &= C_l x_l, \quad l = 1, \ldots, \gamma \end{aligned} \tag{1a}$$

where γ is the total number of areas, $x_l \in \mathbb{R}^{n_l}$ the state, $u_l \in \mathbb{R}^{m_l}$ the input to the lth-area large-scale system (LSS) and the matrices A_l, A_{lk} and B_l are of dimensions $(n_l \times n_l)$, $(n_l \times n_k)$ and $(n_l \times m_l)$, respectively, such that,

$$A = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1\gamma} \\ A_{21} & A_2 & \cdots & A_{2\gamma} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{\gamma 1} & A_{\gamma 2} & \cdots & A_{\gamma} \end{bmatrix}$$

$$\mathrm{B} = \text{block diag}[B_1, \ldots, B_\gamma]$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Thus a VLSS is decomposed into γ areas, with the interactions amongst the area LSSs being indicated by the matrix A_{lk}, $k = 1, \ldots, \gamma$, $l = 1, \ldots, \gamma$ and $k \neq l$. Such interaction matrix in any VLSS model is very sparse in nature, and therefore the interactions amongst the areas may be neglected. This enables us to have a simpler model of each area in the compact form as given below:

$$\begin{aligned} \dot{x}_l &= A_l x_l + B_l u_l \\ y_l &= C_l x_l \end{aligned} \tag{2}$$

where $x_l \in \mathbb{R}^{n_l}$ is the state, $u_l \in \mathbb{R}^{m_l}$ the input and $y_l \in \mathbb{R}^{r_l}$ the output vector of lth-area large-scale system. The model (2) can be expressed in detailed form as[17]

$$\dot{x}_l = A_l x_l + \sum_{i=1}^{\nu} B_{il} u_{il}$$

$$y_{il} = C_{il} x_l, \quad i = 1, \ldots, \nu \tag{2a}$$

where $u_{il} \in \mathbb{R}^{m_{il}}$ is the input and $y_{il} \in \mathbb{R}^{r_{il}}$ the output of the ith-control station in the lth area, such that

$$B_l = [B_{1l}, \ldots, B_{\nu l}], \; C_l^T = [C_{1l}^T, \ldots, C_{\nu l}^T] \tag{2b}$$

$$\text{with } m_l = \sum_{i=1}^{\nu} m_{il} \text{ and } r_l = \sum_{i=1}^{\nu} r_{il}. \tag{2c}$$

The mathematical model for the dynamic coordinator for VLSS is,

$$\dot{x}_c = A_c x_c + B_c u_c \tag{3}$$

and that for lth-area LSS is,

$$\dot{x}_{cl} = A_{cl} x_{cl} + B_{cl} u_{cl} \tag{3a}$$

such that,

$$A_c = \text{block diag}(A_{c1}, \ldots, A_{c\gamma}) \text{ and}$$

$$B_c = \text{block diag}(B_{c1}, \ldots, B_{c\gamma}) \tag{3b}$$

where $l = 1, \ldots, \gamma$ areas, $x_c \in \mathbb{R}^{n_c}$ the state, $u_c \in \mathbb{R}^{m_c}$ the input to the coordinator. Similarly, $x_{cl} \in \mathbb{R}^{n_{cl}}$ and $u_{cl} \in \mathbb{R}^{m_{cl}}$ denotes the state and input to the lth-area coordinator respectively. The matrices A_c, B_c, A_{cl} and B_{cl} are of dimensions $(n_c \times n_c)$, $(n_c \times m_c)$, $(n_{cl} \times n_{cl})$ and $(n_{cl} \times m_{cl})$, respectively.

The mathematical model of the supremal coordinator is also dynamic in nature and is given by

$$\dot{x}_{sc} = A_{sc} x_{sc} + B_{sc} u_{sc} \tag{4}$$

where $x_{sc} \in \mathbb{R}^{n_{sc}}$ is the state and $u_{sc} \in \mathbb{R}^{m_{sc}}$ the input to the supremal coordinator with A_{sc} and B_{sc} of appropriate dimensions.

The problem is to minimise the following cost function,

$$J = \int_{t_0}^{t_f} \frac{1}{2} (y^T Q_y y + u^T R u + x_c^T Q_c x_c + u_c^T R_c u_c + x_{sc}^T Q_{sc} x_{sc} + u_{sc}^T R_{sc} u_{sc})\, \mathrm{d}t \quad (5)$$

with $t_f \to \infty$, subject to constraints (1), (3) and (4).

2. Design of optimal-decentralised and hierarchical-control laws

The Hamiltonian for the above optimisation problem is expressed as a function of states, control inputs and costates as given in the following,

$$H(\mathrm{x}(t),\ x_c(t),\ x_{sc}(t),\ u(t),\ u_c(t),\ u_{sc}(t),\ p(t),\ p_c(t),\ p_{sc}(t))$$

$$= \frac{1}{2} [x^T C^T Q_y C x + u^T R u + x_c^T Q_c x_c + u_c^T R_c u_c + x_{sc}^T Q_{sc} x_{sc} + u_{sc}^T R_{sc} u_{sc}]$$

$$+ p^T(t) [Ax + Bu] + p_c^T(t) [A_c x_c + B_c u_c] + p_{sc}^T(t) [A_{sc} x_{sc} + B_{sc} u_{sc}].$$

Applying the set of necessary conditions for the Hamiltonian H to be minimum, which gives a set of costate equations and the following optimal-control laws:

$$u(t) = -R^{-1} B^T p(t); \quad (6a)$$

$$u_c(t) = -R_c^{-1} B_c^T p_c(t); \quad (6b)$$

$$u_{sc}(t) = -R_{sc}^{-1} B_{sc}^T p_{sc}(t). \quad (6c)$$

We now relate the costates $p(t)$, $p_c(t)$, $p_{sc}(t)$ as linear maps of system states $x(t)$, coordinator states $x_c(t)$ and supremal-coordinator states $x_{sc}(t)$, such that the hierarchical-feedback system forms a multiechelon (pyramid) structure. That is, we let

$$Q = C^T Q_y C; \quad (7)$$

$$p(t) = Kx(t) + K_c x_c(t); \quad (8a)$$

$$p_c(t) = Hx(t) + H_c x_c(t) + H_{sc} x_{sc}(t); \quad (8b)$$

$$p_{sc}(t) = F_c x_c(t) + F_{sc} x_{sc}(t). \quad (8c)$$

Differentiating (8a–c) and the algebraic manipulation of the necessary conditions give the following set of matrix equations.

$$A^TK + KA - KBR^{-1}B^TK + Q - K_cB_cR_c^{-1}B_c^TH = 0; \tag{9a}$$

$$A^TK_c + K_cA_c - KBR^{-1}B^TK_c - K_cB_cR_c^{-1}B_c^TH_c = 0; \tag{9b}$$

$$-K_cB_cR_c^{-1}B_c^TH_{sc} = 0; \tag{9c}$$

$$A_c^TH + HA - HBR^{-1}B^TK - H_cB_cR_c^{-1}B_c^TH = 0; \tag{10a}$$

$$\begin{aligned} A_c^TH_c + H_cA_c - H_cB_cR_c^{-1}B_c^TH_c + Q_c - HBR^{-1}B^TK_c& \\ - H_{sc}B_{sc}R_{sc}^{-1}B_{sc}^TF_c = 0;& \end{aligned} \tag{10b}$$

$$A_c^TH_{sc} + H_{sc}A_{sc} - H_cB_cR_c^{-1}B_c^TH_{sc} - H_{sc}B_{sc}R_{sc}^{-1}B_{sc}^TF_{sc} = 0; \tag{10c}$$

$$-F_cB_cR_c^{-1}B_c^TH = 0; \tag{11a}$$

$$A_{sc}^TF_c + F_cA_c - F_cB_cR_c^{-1}B_c^TH_c - F_{sc}B_{sc}R_{sc}^{-1}B_{sc}^TF_c = 0; \tag{11b}$$

$$A_{sc}^TF_{sc} + F_{sc}A_{sc} - F_{sc}B_{sc}R_{sc}^{-1}B_{sc}^TF_{sc} + Q_{sc} - F_cB_cR_c^{-1}B_c^TH_{sc} = 0. \tag{11c}$$

The solution of these nine matrix equations, (9a–c), (10a–c), and (11a–c), gives the matrices, K, K_c, H, H_c, H_{sc}, F_c and F_{sc}.

Structures of gain matrices:

$$\begin{aligned} K &= \text{block diag}[K_1,\ldots,K_\gamma]_{n\times n}\\ H_c &= \text{block diag}[H_{c1},\ldots,H_{c\gamma}]_{n_c\times n_c}\\ F_{sc} &= \text{diagonal matrix}(n_{sc}\times n_{sc})\\ K_c &= \text{block diag}[K_{c1},\ldots,K_{c\gamma}]_{n\times n_c}\\ H &= \text{block diag}[H_1,\ldots,H_\gamma]_{n_c\times n}\\ H_{sc}^T &= [H_{sc1}^T,\ldots,H_{sc\gamma}^T]_{n_{sc}\times n_c}\\ F_c &= [F_{c1},\ldots,F_{c\gamma}]_{n_{sc}\times n_c}. \end{aligned} \tag{12}$$

Matrices K, H_c and F_{sc} are positive-definite and symmetric in nature. The other matrices, K_c, H, H_{sc} and F_c, act as observability matrices for the information (states) to flow from one level to the other. In this multiechelon-hierarchical structure, any two consecutive levels only exchange information and the number of controllers reduces as the levels go up.

3. Minimisation of information exchange amongst the levels of hierarchy and the order of the dynamic coordinator

Consider the description of lth-area LSS given by (2), (2a–c). Let $\bar{p}_i = \{p_1, \ldots, p_{m_i}\}$ be the set of controllability indices with respect to $u_{il} = [u_{1l}, \ldots, u_{m_{il}}]^T$ of the ith-control station in the lth area. Let $\hat{\bar{p}}_i = \{\hat{p}_1, \ldots, \hat{p}_{k_{ci}}\}$ be the set of largest indices with $\hat{\bar{p}}_i \subset \bar{p}_i$, such that

$$n_{il} = \sum_{j=1}^{k_{ci}} \hat{p}_j \text{ and}$$

$$n_l = \sum_{i=1}^{\nu} n_{il} = \sum_{i=1}^{\nu} \sum_{j=1}^{k_{ci}} \hat{p}_j.$$

Then the minimum number of inputs influencing all the states of the ith-control station is k_{ci}, and the minimum number of inputs required to influence all the states of the lth area will be,

$$k_{cl} = \sum_{i=1}^{\nu} k_{ci}. \tag{13a}$$

Let $\bar{q}_i = \{q_1, \ldots, q_{r_i}\}$ be the set of observability indices with respect to $y_{il} = [y_{1l}, \ldots, y_{r_{il}}]^T$ of the ith-control station in the lth area. By similar reasoning as in the case of controllability, the minimum number of outputs required to observe all the states of the lth area will be,

$$k_{ol} = \sum_{i=1}^{\nu} k_{oi}. \tag{13b}$$

Thus, in the sense of minimum information flow between the coordinator and the lth-area control stations, the minimum number of output states to be directed to the coordinator is k_{ol} and the minimum number of inputs to be driven by the coordinator is k_{cl}. Though the order of the dynamic coordinator can be chosen arbitrarily the justified order in view of (13a) and (13b) can be taken as,

$$n_{cl} = \frac{k_{cl} + k_{ol}}{2} + 1 \tag{14}$$

where n_{cl} is the order of dynamic coordinator for the lth area.

4. Solution of matrix equations for gain matrices in the costate expressions

It is well-known that any linear time-invariant multivariable controllable-observable system can be equivalently transformed into controllable-observable companion canonical (phase variable) triple $(\hat{C}, \hat{A}, \hat{B})$ form, from which it can be noted that the columns of $\hat{C}^T$ are orthogonal to the columns of $\hat{B}$, that is $\hat{C}\hat{B}=0$. Utilising this fact, the nine matrix equations, (9a–c), (10a–c) and (11a–c), can be solved in an alternative way. The coordinator and the supremal-coordinator dynamics are in choice. Therefore, the input matrices, B_c and B_{sc}, and output matrices, K_c, F_c and H_{sc}, of the coordinator and the supremal-coordinator can be so chosen that the following becomes true. That is,

$$K_c B_c = 0, \quad F_c B_c = 0, \quad H_{sc} B_{sc} = 0.$$

The output matrix, H, can also be chosen so that $HB=0$, but because B is restricted (being a plant-input matrix), the relationship $HB=0$ may not however come out to be true. If, $HB \neq 0$ then usually in the case of large-scale systems the result is extremely a sparse matrix and therefore may be neglected. Thus the nine matrix equations can be individually solved in the following manner:

(a) solve (9a) to give K
(b) solve (9b) to give K_c
(c) solve (10b) to give H_c
(d) solve (10a) to give H
(e) solve (10c) to give H_{sc}
(f) solve (11c) to give F_{sc}
(g) solve (11b) to give F_c.

If $HB \neq 0$, then the steps, (c) and (d), are combined to give both H_c and H as solutions. It is to note that steps (a) and (b) involve the dimension of VLSS, *i.e.* n. This may be further simplified by employing Siljak's approach of disconnecting the area LSS, since the interaction matrices amongst the area LSSs are sparse. Thus the steps, (a) and (b), may be carried out for each area separately. Since one coordinator is allotted to each area and there is no interaction between the coordinators, therefore the steps, (c) to (e), are carried out for each coordinator separately. And in the end, the steps, (f) to (g), give out only for one supremal unit.

The gain matrices obtained as solutions of (9a–c), (10a–c) and (11a–c) when substituted into (8a–c) give the optimum-costate trajectory, using which through (6a–c), leads to optimum-control laws as follows,

$$u(t) = -R^{-1}B^T K x(t) - R^{-1}B^T K_c x_c(t), \tag{15a}$$

$$u_c(t) = -R_c^{-1}B_c^T H x(t) - R_c^{-1}B_c^T H_c x_c(t) - R_c^{-1}B_c^T H_{sc} x_{sc}(t), \tag{15b}$$

$$u_{sc}(t) = -R_{sc}^{-1}B_{sc}^T F_c x_c(t) - R_{sc}^{-1}B_{sc}^T F_{sc} x_{sc}(t). \tag{15c}$$

With these optimal-control laws, the closed-loop decentralised and dynamical-hierarchical VLSS becomes,

$$\dot{x}(t) = (A - SK)x(t) - SK_c x_c(t), \tag{16a}$$

$$\dot{x}_c(t) = -S_c H x(t) + (A_c - S_c H_c) x_c(t) - S_c H_{sc} x_{sc}(t), \tag{16b}$$

$$\dot{x}_{sc}(t) = -S_{sc} F_c x_c(t) + (A_{sc} - S_{sc} F_{sc}) x_{sc}(t), \tag{16c}$$

where, $S = BR^{-1}B^T$

$$\begin{aligned} S_c &= B_c R_c^{-1} B_c^T \\ S_{sc} &= B_{sc} R_{sc}^{-1} B_{sc}^T. \end{aligned} \tag{16d}$$

In equation (16a) the term, $SKx(t)$, forms a decentralised closed-loop feedback, whereas the term, $SK_c x_c(t)$, indicates a hierarchical drive from one-level up. In equation (16b), the term, $S_c H_c x_c(t)$, forms a closed-loop feedback to the dynamic coordinators situated at one-level up above the VLSS. The terms, $S_c H x(t)$ and $S_c H_{sc} x_{sc}(t)$, indicate drive from the VLSS and that from the supremal-dynamical coordinator, respectively. The supremal-dynamical coordinator is situated at one-level up above the dynamic coordinators. Thus, in (16c), the term, $S_{sc} F_{sc} x_{sc}(t)$, forms a closed-loop feedback to the supremal unit with the term, $S_{sc} F_c x_c(t)$, acting as a drive from the dynamic coordinators.

It is now therefore quite evident that the closed-loop description given by (16a–c) shows a multiechelon-hierarchical structure of a VLSS.

5. Stability analysis

In this section we now study the asymptotic stability analysis of the hierarchical closed-loop system (16a–c). Let the Liapunov function be defined as,

$$V[\bar{x}(t)] = \int_t^{\infty} \left\{ \bar{x}^T \begin{bmatrix} Q & & 0 \\ & Q_c & \\ 0 & & Q_{sc} \end{bmatrix} \bar{x} + \bar{u}^T \begin{bmatrix} \cdot R & & 0 \\ & R_c & \\ 0 & & R_{sc} \end{bmatrix} \bar{u} \right\} \mathrm{d}t \tag{17}$$

where,

$$\begin{aligned} \bar{x}^T &= [x^T,\ x_c^T,\ x_{sc}^T] \\ \bar{u}^T &= [u^T,\ u_c^T,\ u_{sc}^T] \end{aligned} \tag{18}$$

and the optimal-control laws be given by,

$$\bar{u}(\bar{x}) = \bar{P}\bar{x} \tag{19}$$

where,

$$\bar{P} = \begin{bmatrix} \bar{K} & \bar{K}_c & 0 \\ \bar{H} & \bar{H}_c & \bar{H}_{sc} \\ 0 & \bar{F}_c & \bar{F}_{sc} \end{bmatrix}. \tag{20}$$

On the basis of this assumption the closed-loop multiechelon-hierarchical structure becomes,

$$\dot{x}(t) = (A - B\bar{K})x(t) - B\bar{K}_c x_c(t), \tag{21a}$$

$$\dot{x}_c(t) = -B_c\bar{H}x(t) + (A_c - B_c\bar{H}_c)x_c(t) - B_c\bar{H}_{sc}x_{xc}(t), \tag{21b}$$

$$\dot{x}_{sc}(t) = -B_{sc}\bar{F}_c x_c(t) + (A_{sc} - B_{sc}\bar{F}_{sc})x_{sc}(t). \tag{21c}$$

Thus, using (19), equation (17) may be written as,

$$V[\bar{x}(t)] = \int_t^{\infty} \bar{x}^T[\bar{\mathrm{Q}} + \bar{\mathrm{P}}^T\bar{\mathrm{R}}\bar{\mathrm{P}}]x \; \mathrm{d}t \tag{22}$$

where,

$$\bar{Q} = \begin{bmatrix} Q & & 0 \\ & Q_c & \\ 0 & & Q_{sc} \end{bmatrix}, \qquad \bar{R} = \begin{bmatrix} \mathrm{R} & & 0 \\ & R_c & \\ 0 & & R_{sc} \end{bmatrix}. \tag{23}$$

Then the value of the performance criteria for a trajectory starting at $\bar{x}(t_0)$ is given by $V[x(t_0), x_c(t_0), x_{sc}(t_0)]$. The total time derivative of $V(\bar{x})$ as given by (22) is,

$$\dot{V}(\bar{x}) = \mathrm{d}V(\bar{x})/\mathrm{d}t = \dot{V}(x, x_c, x_{sc}) = -\bar{x}^T(\bar{Q} + \bar{P}^T\bar{R}\bar{P})\bar{x}. \tag{24}$$

Since, $\dot{V}(\bar{x})$ is quadratic in $\bar{x}$, and because the dynamical-hierarchical-control system and the plant equations are linear, let $V(\bar{x})$ be also given by a quadratic form. Thus, instead of minimising $V(\bar{x})$, we pick a quadratic form of $V(\bar{x})$ and find the corresponding $\bar{u}(\bar{x})$. Therefore, we have,

$$V(\bar{x}) = \bar{x}^T P \bar{x} \tag{25}$$

where P is assumed to be any known positive-definite matrix. Matrix P consists of ordered-linear maps for the multiechelon-hierarchical structure as given below.

$$P = \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix} \tag{26}$$

where, K, H_c and F_{sc} are positive-definite and symmetric matrices and K_c, H, H_{sc} and F_c act as observability matrices for the information to flow from one to another level. The condition for the positive definiteness of matrix P is given in Appendix I. The time derivative of $V(\bar{x})$ (25) is given below.

$$\dot{V}(\bar{x}) = \dot{\bar{x}}^T P \bar{x} + \bar{x}^T P \dot{\bar{x}}.$$

Using (21a–c) and (26) we get after substitution the following matrix equation:

$$\dot{V}(\bar{x}) = \bar{x}^T \begin{bmatrix} (A-B\bar{K})^T & -\bar{H}^T B_c^T & 0 \\ -\bar{K}_c^T B^T & (A_c - B_c\bar{H}_c)^T & -\bar{F}_c^T B_{sc}^T \\ 0 & -\bar{H}_{sc}^T B_c^T & (A_{sc} - B_{sc}\bar{F}_{sc})^T \end{bmatrix} \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix} \bar{x}$$

$$+ \bar{x}^T \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix} \begin{bmatrix} (A-B\bar{K}) & -B\bar{K}_c & 0 \\ -B_c\bar{H} & (A_c - B_c\bar{H}_c) & -B_c\bar{H}_{sc} \\ 0 & -B_{sc}\bar{F}_c & (A_{sc} - B_{sc}\bar{F}_{sc}) \end{bmatrix} \bar{x}.$$

But $\dot{V}(\bar{x})$ is also equal to the negative integrand of the performance index (24); therefore, equating these two equations of $\dot{V}(\bar{x})$, for arbitrary $\bar{x}(t)$, we get the following matrix equations:

$$A^T K + KA - \bar{K}^T B^T K - KB\bar{K} + Q + \bar{K}^T R\bar{K} - \bar{H}^T B_c^T H + \bar{H}^T R_c \bar{H} - K_c B_c \bar{H} = 0; \tag{28a}$$

$$A^T K_c + K_c A_c - \bar{K}^T B^T K_c - KB\bar{K}_c + \bar{K}^T R\bar{K}_c - \bar{H}^T B_c^T H_c - K_c B_c \bar{H} + \bar{H}^T R_c \bar{H}_c = 0; \tag{28b}$$

$$-\bar{H}^T B_c^T H_{sc} + \bar{H}^T R_c \bar{H}_{sc} - K_c B_c \bar{H}_{sc} = 0; \tag{28c}$$

$$A_c^T H + HA - \bar{H}_c^T B_c^T H - H_c B_c \bar{H} + \bar{H}_c^T R_c \bar{H} - \bar{K}_c^T B^T K + \bar{K}_c^T R\bar{K} - HB\bar{K} = 0; \tag{29a}$$

$$A_c^T H_c + H_c A_c - \bar{H}_c^T B_c^T H_c - H_c B_c \bar{H}_c + \bar{H}_c^T R_c \bar{H}_c + Q_c - \bar{K}_c^T B^T K_c + \bar{K}_c^T R\bar{K}_c - \bar{F}_c^T B_{sc}^T F_c + \bar{F}_c^T R_{sc} \bar{F}_c - HB\bar{K}_c - H_{sc} B_{sc} \bar{F}_c = 0; \tag{29b}$$

$$A_c^T H_{sc} + H_{sc} A_{sc} - \bar{H}_c^T B_c^T H_{sc} + \bar{H}_c^T R_c \bar{H}_{sc} - H_c B_c \bar{H}_{sc} - \bar{F}_c^T B_{sc}^T F_{sc} + \bar{F}_c^T R_{sc} \bar{F}_{sc} - H_{sc} B_{sc} \bar{F}_{sc} = 0; \tag{29c}$$

$$-\bar{H}_{sc}^T B_c^T H + \bar{H}_{sc}^T R_c \bar{H} - F_c B_c \bar{H} = 0; \tag{30a}$$

$$A_{sc}^T F_c + F_c A_c - \bar{F}_{sc}^T B_{sc}^T F_c + \bar{F}_{sc}^T R_{sc} \bar{F}_c - F_{sc} B_{sc} \bar{F}_c - \bar{H}_{sc}^T B_c^T H_c + \bar{H}_{sc}^T R_c \bar{H}_c - F_c B_c \bar{H}_c = 0; \tag{30b}$$

$$A_{sc}^T F_{sc} + F_{sc} A_{sc} - \bar{F}_{sc}^T B_{sc}^T F_{sc} - F_{sc} B_{sc} \bar{F}_{sc} + \bar{F}_{sc}^T R_{sc} \bar{F}_{sc} + Q_{sc} + \bar{H}_{sc}^T R_c \bar{H}_{sc} - \bar{H}_{sc}^T B_c^T H_{sc} - F_c B_c \bar{H}_{sc} = 0. \tag{30c}$$

At this stage, it is required to determine the unknown matrices $\bar{K}$, $\bar{K}_c$, $\bar{H}$, $\bar{H}_c$, $\bar{H}_{sc}$, $\dot{\bar{F}}_c$ and $\bar{F}_{sc}$, which are the elements of matrix $\bar{P}$ as given by (20). Because it is desired to find the control strategies as a function of state variables, then through Hamilton-Jacobi formulation it would not be impertinent to choose matrix $\bar{P}$ as given below:

$$\bar{P} = \begin{bmatrix} R^{-1}B^T & 0 & 0 \\ 0 & R_c^{-1}B_c^T & 0 \\ 0 & 0 & R_{sc}^{-1}B_{sc}^T \end{bmatrix} P.$$

When such a choice is being substituted in the set of equations (28a–c), (29a–c) and (30a–c), they reduce to the set of equations (9a–c), (10a–c) and (11a–c), respectively. This leads to the original problem of solving the set of nine equations, the solutions of which are given by (26). Thus, the decentralised and hierarchical-optimal-control laws given by (15a–c) do indeed stabilise the VLSS.

6. Number of levels of hierarchy

Depending upon the sparsity of interconnections in a VLSS, the number of area LSS can be decided. For each area, only one dynamic coordinator need to be designed. Thus the number of areas is equal to the number of coordinators. The VLSS and the coordinator dynamics are considered to be situated at levels 0 and 1 in the hierarchical systems. It is now quite evident that the total number of levels excluding level 0, will be equal to the total number of areas or coordinators. However, the number of levels in hierarchical systems can be reduced. In the earlier section, it has already been discussed about the order of the coordinator being chosen. The order of the supremal coordinator (at level 2) is also chosen in the same way by considering the controllability and observability indices at level 1. Thus no matter whatever be the number of coordinators at level 1, there can be only one supremal coordinator at level 2. If the order of the supremal-dynamic coordinator at level 2 becomes larger than the largest coordinator at level 1, then only one more level (*i.e.* level 3) is recommended.

7. Conclusion

The problem of designing dynamic controllers for hierarchical-feedback control of very-large-scale systems has been solved. The levels of hierarchy to exist is shown to be more than one. The multiechelon-hierarchical structure is formed through costate equations, which leads to solving the nine matrix equations (9a–c), (10a–c) and (11a–c), to give feedback-coefficient matrices as solutions. The lth-area LSS description (2) achieved by neglecting the interconnections among the area LSS is shown to serve two purposes: (a) to ease solving the first two, (9a), (9b), matrix equations, and (b) to give the number of area LSS as also the number of dynamic coordinators required. The well-known properties of controllable-observable companion forms have been utilised to simplify the solution algorithm for the nine nonlinear-matrix equations. Further, the order of the dynamic coordinators can be decided as shown in section 3, through controllability and observability indices. In the stability analysis, it has been shown that there exists a non-symmetric positive-definite matrix P through which hierarchical-dynamical-optimal control laws do stabilise the VLSS. The number of levels of hierarchy is being shown to depend upon the largest order at a certain level and the complexity in handling the states at that level.

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Appendix I

Matrix P consisting of ordered-linear maps for the multiechelon-hierarchical structure given in (26) is reproduced below:

$$P = \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix}.$$

where, the diagonal block elements, K, H_c and F_{sc}, are positive-definite symmetric matrices and the off-diagonal block elements, K_c, H, H_{sc} and F_c, are input-output maps between the levels of hierarchy. These mappings are indicated in the costate expressions (8a–c).

Theorem: Matrix P shown above is said to be positive-definite, if

$$\det\left\{\begin{bmatrix} K & K_c \\ H & H_c \end{bmatrix} - \begin{bmatrix} 0 \\ H_{sc} \end{bmatrix} F_{sc}^{-1}[0 \quad F_c]\right\} > 0 \qquad \text{(A)}$$

and $\det\,[K - K_c H_c^{-1} H] > 0$;

OR if

$$\det\left\{ K - [K_c \quad 0]\begin{bmatrix} H_c & H_{sc} \\ F_c & F_{sc} \end{bmatrix}^{-1} \begin{bmatrix} H \\ 0 \end{bmatrix} \right\} > 0$$

(B)

and $\det[H_c - H_{sc} F_{sc}^{-1} F_c] > 0$.

Proof: By partitioning matrix P we have,

$$P = \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix}.$$

The determinant of matrix P is,

$$\det P = \det\left\{ \begin{bmatrix} K & K_c \\ H & H_c \end{bmatrix} - \begin{bmatrix} 0 \\ H_{sc} \end{bmatrix} F_{sc}^{-1}[0 \;\; F_c] \right\} \det \begin{bmatrix} K & K_c \\ H & H_c \end{bmatrix}.$$

For matrix P to be positive-definite $\det P > 0$ which means,

$$\det\left\{ \begin{bmatrix} K & K_c \\ H & H_c \end{bmatrix} - \begin{bmatrix} 0 \\ H_{sc} \end{bmatrix} F_{sc}^{-1}[0 \;\; F_c] \right\} > 0; \tag{Aa}$$

$$\text{and } \det \begin{bmatrix} K & K_c \\ H & H_c \end{bmatrix} = \det[K - K_c H_c^{-1} \mathrm{H}] \det K > 0.$$

Because K is positive-definite and symmetric, it requires,

$$\det[K - K_c H_c^{-1} H] > 0. \tag{Ab}$$

Thus, for matrix P to be positive-definite, both the conditions, (Aa) and (Ab), must be satisfied.

Again, by partitioning matrix P in a different way, we have,

$$P = \begin{bmatrix} K & K_c & 0 \\ H & H_c & H_{sc} \\ 0 & F_c & F_{sc} \end{bmatrix}.$$

Then the determinant of matrix P is,

$$\det P = \det\left\{ K - [K_c \;\; 0] \begin{bmatrix} H_c & H_{sc} \\ F_c & F_{sc} \end{bmatrix}^{-1} \begin{bmatrix} H \\ 0 \end{bmatrix} \right\} \det K. \tag{Ba}$$

For matrix P to be positive-definite,

$\det P > 0$.

Because K is positive-definite and symmetric, it requires,

$$\det\left\{K-[K_c \quad 0]\begin{bmatrix}H_c & H_{sc}\\ F_c & F_{sc}\end{bmatrix}^{-1}\begin{bmatrix}H\\ 0\end{bmatrix}\right\}>0 \text{ and} \tag{Ba}$$

$$\det\begin{bmatrix}H_c & H_{sc}\\ F_c & F_{sc}\end{bmatrix}=\det[H_c-H_{sc}F_{sc}^{-1}F_c]\det H_c>0.$$

Because H_c is positive-definite and symmetric, therefore

$$\det[H_c-H_{sc}F_{sc}^{-1}F_c]>0. \tag{Bb}$$

Thus, for matrix P to be positive-definite, both the conditions, (Ba) and (Bb), must be satisfied.

Therefore, for matrix P to be positive-definite either one of the pairs of conditions, (A) or (B), must be satisfied.

Q.E.D.