J. Indian Inst. Sci., May–June 1988, 68, 167–184

On the asymptotic formulae for the spectral matrix of a differential operator

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Received on April 21, 1986; Revised on November 20, 1987.

Abstract

In the present paper we obtain certain asymptotic formula for the spectral matrix of a self-adjoint second-order differential system. We base our derivation of the formula on a Tauberian theorem due to N. Wiener.

Key words: Spectral matrix, isometric mapping, ϕ- and C-Fourier transforms, Cauchy-type equations, Riemann matrix function, majorize, convolution, Wiener's Tauberian theorem.

1. Introduction

Consider the differential system

$$MU = \lambda U \tag{1.1}$$

where

(i) $M = \begin{pmatrix} -D^2+p(x) & r(x) \\ r(x) & -D^2+q(x) \end{pmatrix}, D = \mathrm{d}/\mathrm{d}x$ and $U \equiv U(x,\lambda) = (u(x,\lambda), v(x,\lambda))^{\mathrm{T}}$

(ii) $Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$

is a real $C_{1-k}(0,b)$, $(k = 0,1)$, class matrix summable on $[0,b)$, where b is finite or infinite; by $C_k(\alpha,\beta)$-class matrices, we mean matrices which are k times differentiable with respect to the variable x over (α,β), the kth derivative being continuous in the interval.

(iii) λ is a complex parameter.

(iv) The boundary conditions at $x = 0$ and $x = b$ are respectively

$$a_{j1}u(0,\lambda)+a_{j2}u'(0,\lambda)+a_{j3}v(0,\lambda)+a_{j4}v'(0,\lambda) = 0;$$
$$b_{j1}u(b,\lambda)+b_{j2}u'(b,\lambda)+b_{j3}v(b,\lambda)+b_{j4}v'(b,\lambda) = 0; \tag{1.2}$$

$j = 1, 2$. a_{ij}, b_{ij} are real-valued constants independent of λ, satisfying

(i) rank(a_{ij}), rank$(b_{ij}) = 2$, $i = 1, 2$, $j = 1, 2, 3, 4$;

(ii) $a_{j1}\, a_{k2} + a_{j3}\, a_{k4} = 0$, $j, k = 1, 2$; $b_{11} b_{22} - b_{12} b_{21} + b_{13} b_{24} - b_{14} b_{23} = 0$;

(iii) for vectors $a_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4})\,(a_j, a_k) = \delta_{jk}, \delta_{jk}$, the kronecker delta.

By making b tend to infinity we obtain[1] a self-adjoint eigenvalue problem associated with the system (1.1) over the interval $[0, \infty)$.

The eigenvalue problem associated with the Fourier system corresponding to the general system (1.1) is obtained by considering the system (1.1) with $p(x) = q(x) = r(x) = 0$ whose solutions satisfy the same boundary conditions at $x = 0$ and $x = b$. For the treatment of the Fourier system over $[0, b]$ we impose the additional conditions

$$b_{j1} a_{k2} + b_{j3} a_{k4} = 0, \quad b_{j2} a_{k1} + b_{j4} a_{k3} = 0, \quad j, k = 1, 2 \tag{1.2a}$$

involving the constants a_{ij}, b_{ij} in the boundary conditions at $x = 0$, $x = b$.

Let $\phi_j(0|x, \lambda) = (u_j(0|x, \lambda), v_j(0|x, \lambda))^T$, $j = 1, 2$, be two linearly independent boundary-condition vectors at $x = 0$ *i.e.* $\phi_j(0|x, \lambda)$ are the solutions of (1.1) and $\phi_j(0|0, \lambda) = (a_{j2}, a_{j4})^T$, $\phi_j'(0|0, \lambda) = -(a_{j1}, a_{j3})^T$, $j = 1, 2$, a_{ij} are those which occur in (1.2).

Consider two other vectors

$\theta_k(0|x, \lambda) = (x_k(0|x, \lambda), y_k(0|x, \lambda))^T$ solutions of (1.1) connected with $\phi_j(0|x, \lambda)$ by means of the relations $[\phi_j, \theta_k] = \delta_{jk}$, the kronecker delta, where [..] is the bilinear concomitant defined for vectors $\alpha_j = (\alpha_{1j}, \alpha_{2j})^T$, $j = 1, 2$ by

$$[\alpha_1, \alpha_2] = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{11}' & \alpha_{12}' \end{vmatrix} + \begin{vmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{21}' & \alpha_{22}' \end{vmatrix}$$

For the problem of the interval $[0, b]$, $b > 0$ arbitrary, there occurs in the explicit expression for the Green's matrix $G(b, x, y, \lambda)$ a symmetrix matrix $(l_{rs}(\lambda))$, depending only on λ, b and the coefficients in the boundary conditions at $x = b$, which tends to $(m_{rs}(\lambda))$ as b tends to infinity. The matrix $(m_{rs}(\lambda))$ plays a vital role in the problem of the interval $[0, \infty)$ (*see* Chakravarty[1] for details; a regular eigenvalue problem is considered here for the interval $[0, b]$ and the problem of the singular interval $[0, \infty)$ is solved by making b tend to infinity through a suitable sequence.)

If λ_{nb} is a simple pole of $l_{rs}(\lambda)$, with residue R_{rs}, the normalized eigenvector is given by

$$\psi_n(b, x) = \sum_{r=1}^{2} R_{rr}^{\frac{1}{2}} \phi_r(0|x, \lambda_{nb}) \tag{1.3}$$

where $R_{12}^2 = R_{21}^2 = R_{11} R_{22}$.

Further, when $R_{11} R_{22} - R_{12}^2 > 0$, there are two normalized linearly independent eigenvectors $\psi_n^{(j)}(x), j = 1, 2$, corresponding to an eigenvalue λ_{nb}, such that

$$\psi_n^{(1)}(x) = R_{11}^{-\frac{1}{2}} (R_{11} \phi_1(0|x, \lambda_{nb}) + R_{12} \phi_2(0|x, \lambda_{nb}))$$

$$\psi_n^{(2)}(x) = -R_{11}^{-\frac{1}{2}} (R_{11} R_{22} - R_{12}^2)^{\frac{1}{2}} \phi_2(0|x, \lambda_{nb}) \tag{1.4}$$

(compare Tiwari[2,3]).

Let $\rho(b, t) = (\rho_{rs}(b, t))$ be a matrix such that $d\rho_{rs}(b, u) = R_{rs}$. Then $(d\rho_{rs}(b, u))$ is either positive definite or positive semi-definite.

Denote by $L_\rho^2(-\infty, \infty)$ the Hilbert space of matrices $h(x)$, square integrable over $-\infty < x < \infty$ with weight $d\rho(u)$ (*i.e.* $\int_{-\infty}^{\infty} h^T(x)\, d\rho\, h(x) < \infty$). Then following Titchmarsh[4] closely, by making b tend to infinity through a suitable sequence, Tiwari[2,3] shows that

(i) $\rho(b, u) = (\rho_{rs}(b, u))$ tends uniformly to $\rho(u) \equiv (\rho_{rs}(u))$, u real, where $d\rho \equiv (d\rho_{rs}(u))$ is either positive definite in the sense that the corresponding quadratic form is positive definite or $d\rho$ is positive semi-definite in the sense that the matrix $d\rho$ is singular and all its principal minors are non-negative.

(ii) $\rho(u)$ is given by

$$\rho(u) = (\rho_{rs}(u)) = \frac{1}{\pi} \lim_{\nu \to 0} \int_0^u (-\operatorname{im} m_{rs}(\mu + i\nu))\, d\mu, \quad \lambda = \mu + i\nu,$$

$\rho(u)$ is normalized in the sense that $\rho(0) = 0$.

(iii) if $f \in L_2[0, b]$, $\rho(u)$ generates an isometric mapping of $L_2[0, b]$ on to $L_\rho^2(-\infty, \infty)$ by means of the formulae

$$E(u, f) \equiv E(u) = \int_0^{\infty} f^T(x)\, \phi(0|x, u)\, dx$$

$$f(x) = \int_{-\infty}^{\infty} E^T(u)\, d\rho(u)\, \phi(0|x, u) \tag{1.5}$$

where $\phi(0|x, u) = \phi(x, u) = \phi = \begin{pmatrix} u_1(x, u) & u_2(x, u) \\ v_1(x, u) & v_2(x, u) \end{pmatrix}$.

a matrix whose jth column is the vector $\phi_j(0|x,u)$, u real. The integrals are convergent in the metrics $L_2[0,b]$ and $L^2_\rho(-\infty,\infty)$ respectively.

The Parseval relation is

$$\|f\|^2 = \int_0^b |f|^2 \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} E^T(u,f) \, d\rho(u) \, E(u,f) \tag{1.6}$$

and for two vectors f, g,

$$\int_0^b (f,g) \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} E^T(u,f) \, d\rho(u) \, E(u,g) \tag{1.7}$$

$E(\lambda,f) = \displaystyle\int_0^\infty f^T(x) \, \phi(0|x,\lambda) \, dx$, ($\lambda$-complex), may be called the ϕ-Fourier transform of f.

The matrix $\rho(u) \equiv (\rho_{rs}(u))$, u real, is the spectral matrix with usual properties, associated with the system (1.1) with boundary condition (1.2) at $x = 0$.

We now actually construct the spectral matrix for the Fourier system by first constructing the same for the regular eigenvalue problem for the interval $[0,b]$ and then for the singular interval $[0,\infty)$ by making b tend to infinity through a suitable sequence.

By using condition (1.2a) it can be easily verified that for the Fourier system over the interval $[0,b]$, associated with an eigenvalue $\lambda^F_{nb} = n^2\pi^2/b^2$, there are two linearly independent normalized eigenvectors $(2/\pi)^{\frac{1}{2}} \psi^F_{nj}(x), j = 1,2$. The explicit representation of $\psi^F_{nj}(x)$ is

$$\psi^F_{nj}(x) = \begin{pmatrix} a_{j2} \cos nx\,\pi/b \oplus a_{j1} \sin nx\,\pi/b \\ a_{j4} \cos nx\,\pi/b \oplus a_{j3} \sin nx\,\pi/b \end{pmatrix} \tag{1.8}$$

$(2/\pi)^{\frac{1}{2}} \psi^F_{nj}(x)$, $j = 1,2$, being two linearly independent sequences of normalized eigenvectors, form a basis of the space of eigenvectors. An element

$$\pi^{-\frac{1}{2}} (\psi^F_{n2}(x) - \psi^F_{n1}(x)) \tag{1.9}$$

may be chosen as a normalized eigenvector corresponding to the eigenvalue $\lambda^F_{nb} = n^2\pi^2/b^2$ for the Fourier system under consideration (compare Acharyya[5]).

The two linearly independent boundary condition vectors $\phi^F_j(0|x,\lambda)$ at $x = 0$ for the Fourier system are given by

$$\phi^F_j(0|x,\lambda) = \begin{pmatrix} a_{j2} \cos \lambda^{\frac{1}{2}}x - a_{j1}/\lambda^{\frac{1}{2}} \sin \lambda^{\frac{1}{2}}x \\ a_{j4} \cos \lambda^{\frac{1}{2}}x - a_{j3}/\lambda^{\frac{1}{2}} \sin \lambda^{\frac{1}{2}}x \end{pmatrix}, \; j = 1,2. \tag{1.10}$$

Substituting from (1.4) by (1.8) and (1.10) with $\lambda^F_{nb} = n^2\pi^2/b^2$ and observing that the resulting equation holds identically in x, we obtain, on slight simplification,

$R^F_{11} = 2/\pi$, $R^F_{12} = 0$ and $R^F_{22} = 2/\pi$, where R^F_{ij} are the residues of $l^F_{ij}(\lambda)$ (the equivalent of $l_{ij}(\lambda)$ of the general system) at a simple pole λ^F_{nb}.

A similar consideration with (1.3), (1.9) and (1.10) leads to

$R^F_{11} = 1/\pi$, $R^F_{22} = 1/\pi$ and $R^F_{12} = 1/\pi$.

Put $d\sigma_{rs}(b,u) = R^F_{rs}$. Then the matrix $\sigma(b,u) = (\sigma_{rs}(b,u))$ has the explicit representation $2/\pi \cdot uI$, I, 2×2 unit matrix or $u/\pi \cdot I_1$, $I_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ according as the matrix (R^F_{rs}) is positive definite or positive semi-definite.

As before, $\sigma(b,u)$ can be extended to $\sigma(u)$ as b tends to infinity through a suitable sequence. $\sigma(u)$ is the spectral matrix for the Fourier system and

$$\sigma(u) = 2u/\pi \cdot I, \ (d\sigma(u) \text{ positive definite}), \tag{1.11}$$

and $$\sigma(u) = u/\pi \cdot I_1, \ (d\sigma(u) \text{ positive semi-definite}). \tag{1.12}$$

In what follows we assume $d\rho(u)$, $d\sigma(u)$ are both positive definite; the case when they are positive semi-definite follows similarly. It may be noted that $\sigma(u)$ for the Fourier system has properties similar to those of $\rho(u)$ for the general system.

When $u = \mu^2$, put $\rho(u) = \rho_1(\mu)$ and assume that $\rho_1(\mu)$ is extended to the negative half-line as an odd function. Similarly for $\sigma_1(\mu)$, where $\sigma_1(\mu) = \sigma(u)$.

Spectral theory of second- and higher-order differential equations and of Dirac equations is a subject which is being intensively investigated by mathematicians of many countries. But the spectral problems for the system

$$LY = \lambda MY \tag{A}$$

a system consisting of m differential equations each of order n have not received as much attention. Again, if the tensor interaction forces are taken into account, the Schrödinger equation for a deuteron (in the ground state) leads to the system

$$Y'' + \lambda^2 Y = (V(x) + 6x^{-2}P)Y, \ 0 < x < \infty, \ P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{B}$$

where $V(x)$ is a hermitian matrix satisfying certain conditions.

Our system (1.1) is a special case of (A) (with $m = n = 2$) but a generalized form of (B). We are therefore led to investigate the spectral problems associated with (1.1). In the present paper we investigate the asymptotic formulae for the spectral matrix $\rho_1(\mu)$ for large μ. We follow mostly the methods of Levitan[6,7] and Marchenko[8] in the investigation of the problem.

2. Some preliminary investigations

$$\text{Let } C_j(x,\lambda^{\frac{1}{2}}) = \begin{pmatrix} a_{j1} \sin \lambda^{\frac{1}{2}}x \oplus a_{j2} \cos \lambda^{\frac{1}{2}}x \\ a_{j3} \sin \lambda^{\frac{1}{2}}x \oplus a_{j4} \cos \lambda^{\frac{1}{2}}x \end{pmatrix}, \ j = 1,2. \tag{2.1}$$

$C(x, \lambda^{\frac{1}{2}})$ being the matrix whose jth column is the vector $C_j(x, \lambda^{\frac{1}{2}}), j = 1, 2$. Evidently, $C_j(x, \lambda^{\frac{1}{2}})$ are the solutions of the Fourier system.

$$\text{Let } S_j(x, \lambda^{\frac{1}{2}}) = \lambda^{\frac{1}{2}} \int_0^x C_j(x, \lambda^{\frac{1}{2}})\, dx \tag{2.2}$$

$S(x, \lambda^{\frac{1}{2}})$ being the matrix whose jth column is the vector $S_j(x, \lambda^{\frac{1}{2}})$.
Let λ be real, say $\lambda = u$.

When $u < 0$, let $e_j(x, |u|^{\frac{1}{2}}) = C_j(x, iu^{\frac{1}{2}})$ and $Z_j(x, |u|^{\frac{1}{2}}) = S_j(x, iu^{\frac{1}{2}})$, the corresponding matrices whose jth column vectors are $e_j(x, |u|^{\frac{1}{2}})$ and $Z_j(x, |u|^{\frac{1}{2}})$ being represented respectively by $e(x, |u|^{\frac{1}{2}})$ and $Z(x, |u|^{\frac{1}{2}})$.

We prove the following lemma.

Lemma 2.1. For $-\infty < u \leq 0$, and arbitrary but fixed x, say $x = x_0$,

$$\int_{-\infty}^{0} e^T(x_0, |u|^{\frac{1}{2}})\, d\rho(u)\, e(x_0, |u|^{\frac{1}{2}}) = O(1).$$

It has been established by Ray Paladhi[9] (p. 176) that there exists a matrix $M(x, t)$ such that $C(x, \lambda^{\frac{1}{2}})$ and $\phi(x, \lambda)$ are connected to each other by

$$C(x, \lambda^{\frac{1}{2}}) = \phi(x, \lambda) - \int_0^x M(x, t)\, \phi(t, \lambda)\, dt \tag{2.3}$$

where $M(x, t)$ satisfies a set of conditions elaborated on p. 177 of the paper under reference. $M(x, t)$ is finite in the sense that $M(x, t) = 0$ for $t > x$ and $M(x, 0) = 0$, x lying in an arbitrary but fixed interval.

Along with (2.3) we have also the integral equation

$$\phi(x, \lambda) = C(x, \lambda^{\frac{1}{2}}) + \int_0^x K(x, t)\, C(t, \lambda^{\frac{1}{2}})\, dt \tag{2.4}$$

where $K(x, t)$ is finite and is such that $U \equiv (X_j(x, t), Y_j(x, t))^T = \bar{A}_j K(x, t)$ where $\bar{A}_j = (a_{j2}, a_{j4})$ and $K(x, t) = \begin{pmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{pmatrix}$ satisfies *inter alia*, the Cauchy-type equation

$$\partial^2 U / \partial x^2 = \partial^2 U / \partial t^2 + Q(x) U \tag{2.4a}$$

with initial conditions

$$U(x,t)|_{t=0} = f(x) \neq 0 \text{ a twice differentiable function and}$$
$$\partial U(x,t)/\partial t|_{t=0} = 0 \tag{2.5}$$

for each $j = 1,2$. (*see* Ray Paladhi[9], pp. 174–175).

The solution of (2.4a)–(2.5) by the Riemann method yields

$$U(x,t) = \frac{1}{2}((f(x+t)+f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} W(x,t,s)\, f(s)\, ds \tag{2.6}$$

the Riemann matrix function $W(x,t,s)$ satisfying the inequality

$$|W(x,t,s)| \leq \frac{1}{2}\int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma \exp\left(\frac{1}{2}t\int_{x-t}^{x+t} |Q(\sigma)|\, d\sigma\right) \tag{2.7}$$

where by $|S|$ we mean the sum of the moduli of all the elements of the matrix S (*see* Chakravarty and Ray Paladhi[10]).

Since $f(x)$ is continuous in x lying in an arbitrary but finite interval, we can assume $|f(x \pm t)| \leq \alpha(t_0)$, where $t_0 \geq t$ is fixed but arbitrary. Hence from (2.7) and (2.6)

$$K(x,t) = O(|\alpha|) + O\left(\int_0^t |Q(\sigma)|\, d\sigma\right) \tag{2.8}$$

where $O(.)$ for a matrix means that each element of the matrix is $O(.)$. Since $M(x,t)$ also satisfies a Cauchy-type equation with initial conditions (like (2.4a) and (2.5)) (*see* Ray Paladhi[9], p. 177), we have also the relation

$$M(t,s) = O(|\alpha|) + O\left(\int_0^t |Q(\sigma)|\, d\sigma\right). \tag{2.9}$$

Integrating both the sides of (2.3) with respect to x over the interval $(0,x_0)$ and changing the order of integration on the right hand side, we obtain

$$\lambda^{-\frac{1}{2}} S(x_0,\lambda^{\frac{1}{2}}) = \int_0^{x_0} \left(I - \int_t^{x_0} M(t,s)\, ds\right) \phi(t,\lambda)\, dt$$

$$= \int_0^{x_0} H_0(x_0,t)\, \phi(t,\lambda)\, dt \tag{2.10}$$

where $H_0(x_0,t) = I - \int_t^{x_0} M(t,s)\, ds$, I, 2×2 unit matrix.

Now each column vector of $\lambda^{-\frac{1}{2}}S(x_0,\lambda^{\frac{1}{2}})$ is the ϕ-Fourier transform of the corresponding column vector of $H(x_0,t)$ for $t < x_0$, and equal to zero for $t \geq x_0$. Therefore by the Parseval relation (1.6)

$$1/\pi \int_{-\infty}^{\infty} u^{-1}\, S_j^T(x_0,u^{\frac{1}{2}})\, d\rho(u)\, S_k(x_0,u^{\frac{1}{2}})$$

$$= \int_0^{x_0} (H_{0j}, H_{0k})\, dt,\ j,k = 1,2, \tag{2.11}$$

where $H_{0j}(.)$ and $H_{0k}(.)$ are the jth and the kth column vectors of $H_0(.)$. Since the right hand side of (2.11) is obviously finite, it follows that

$$\int_{-\infty}^{\infty} u^{-1}\, S^T(x_0,u^{\frac{1}{2}})\, d\rho(u)\, S(x_0,u^{\frac{1}{2}}) = O(1) \tag{2.12}$$

leading to

$$\int_{-\infty}^{0} u^{-1}\, S^T(x_0,u^{\frac{1}{2}})\, d\rho(u)\, S(x_0,u^{\frac{1}{2}})$$

$$\ll \int_{-\infty}^{\infty} u^{-1} S^T(x_0,u^{\frac{1}{2}})\, d\rho(u)\, S(x_0,u^{\frac{1}{2}}) = O(1) \tag{2.13}$$

where the symbol $\ll$ means that the right hand side matrix majorizes that on the left. When $u < 0$, we obtain from (2.2) by the first mean value theorem of the integral calculus, $Z(x,|u|^{\frac{1}{2}}) = ix|u|^{1/2} e\,(\theta x,|u|^{\frac{1}{2}}),\ 0 < \theta < 1.$ (2.14)

The lemma then follows from (2.13) and (2.14).

For an $n \times n$ matrix $A = (a_{rs})$, let $\|A\| = \max_{1 \leq r,s \leq n} |a_{rs}|$.

Then the following proposition holds.

Proposition A. A necessary and sufficient condition that $\int A\, dx$ is absolutely convergent (*i.e.* each element of the matrix integral is absolutely convergent) is that $\int \|A\|\, dx$ is convergent.

The proof follows with little modification in the proof for the corresponding result for the series as given in Mirsky[11] (p. 331).

Theorem 2.1. Under the conditions of lemma 2.1

$$\int_{-\infty}^{0} \|e(x_0,|u|^{\frac{1}{2}})\|^2\, \|d\rho(u)\| < \infty, \text{ holds.} \tag{2.15}$$

The theorem follows from lemma 2.1 by using proposition A and noting that $d\rho(u)$ is positive.

The theorem generalizes Marchenko's lemma 2.2.1 (*see* Marchenko[8], p. 42). We now establish the following lemma which plays a basic role in our further investigations.

Lemma 2.2. If $Q(x)$ is summable in every finite interval, then there exist continuous matrices C, such that

(i) $\sup_{-\infty<\mu<\infty} \overset{\mu+1}{\underset{\mu}{V}} \rho_1(\mu) \ll C$, uniformly in μ; and equivalently

(ii) $\rho_1(b+\mu)-\rho_1(b-\mu) \ll C$, uniformly for b, $\mu \geq 0$, holds.
Similar results also hold for the spectral matrix $\sigma_1(\mu)$.

As in Levitan[6] (p. 212), consider

$$g_\varepsilon(t,a) = \varepsilon^{-2}(2\varepsilon-t)\cos at,\quad 0 \leq t \leq 2\varepsilon;$$
$$= 0 \qquad\qquad , t > 2\varepsilon.$$

Then $\psi_\varepsilon(\lambda,a) = \int_0^{2\varepsilon} g_\varepsilon(t,a)\, C(t,\lambda^{\frac{1}{2}})\, dt$

$$= \varepsilon^{-2}\int_0^{2\varepsilon} (2\varepsilon-t)\, C(t,\lambda^{\frac{1}{2}}) \cos at\, dt \tag{2.16}$$

When λ is real and equal to u, we have for $u < 0$,

$$\|\psi_\varepsilon(u,a)\| \leq \varepsilon^{-2} \int_0^{2\varepsilon} |2\varepsilon-t|\, |\cos at|\, \|C(t,i|u|^{\frac{1}{2}})\|\, dt \text{ (Mirsky}^{11}\text{, p. 343)}$$

$$\leq 4\|e(\xi,|u|^{\frac{1}{2}})\|,\quad 0<\xi<2\varepsilon.$$

Therefore, from (2.15) and proposition A, it follows that

$$\int_{-\infty}^0 \psi_\varepsilon^T(u,a)\, d\rho(u)\, \psi_\varepsilon(u,a) = O(1) \tag{2.17}$$

uniformly for a and ε, which may be small enough. Substituting for $C(t,\lambda^{1/2})$ in (2.16) by (2.3), we obtain on changing the order of integration

$$\psi_\varepsilon(\lambda,a) = \int_0^{2\varepsilon} (g_\varepsilon(s,a)I \ominus \int_s^{2\varepsilon} M(t,s)\, g_\varepsilon(t,a)\, dt)\, \phi(s,\lambda)\, ds$$

so that each column vector of $\psi_\varepsilon(\lambda,a)$ is the ϕ-Fourier transform of the corresponding column vector of

$H(s,a,\varepsilon) \equiv g_\varepsilon(s,a)\, I - \int_s^{2\varepsilon} M(t,s)\, g_\varepsilon(t,a)\, dt$ for $s \le 2\varepsilon$ and equal to zero for $s \ge 2\varepsilon$.

Hence by the Parseval theorem and relation (2.17)

$$\pi^{-1} \int_0^\infty \psi_\varepsilon^T(u,a)\, d\rho(u)\, \psi_\varepsilon(u,a)$$

$$= \int_0^{2\varepsilon} H^T(s,a,\varepsilon)\, H(s,a,\varepsilon)\, ds + O(1) \tag{2.18}$$

uniformly for a, ε, small enough.

Now, $\int_0^{2\varepsilon} H^T(s,a,\varepsilon)\, H(s,a,\varepsilon)\, ds$

$$= I \int_0^{2\varepsilon} g_\varepsilon^2(s,a)\, ds - 2\int_0^{2\varepsilon} g_\varepsilon(s,a) \left(\int_s^{2\varepsilon} M(t,s)\, g_\varepsilon(t,a)\, dt \right) ds$$

$$+ \int_0^{2\varepsilon} \left(\int_s^{2\varepsilon} M(t,s)\, g_\varepsilon(t,a)\, dt \right)^2 ds$$

$$= J_1 - 2J_2 + J_3, \text{ say.}$$

From definition $g_\varepsilon(t,a) = O(1/\varepsilon)$ uniformly for a and from (2.9)

$$\int_0^t M(t,s)\, ds = O(|\alpha|\, t) + O\left(t \int_0^t |Q(\sigma)|\, d\sigma \right)$$

On changing the order of integration

$$J_2 = O\left(\varepsilon^{-2} \int_0^{2\varepsilon} dt \int_0^t M(t,s)\, ds \right) = O(|\alpha|) + O\left(\int_0^{2\varepsilon} |Q(\sigma)|\, d\sigma \right) \tag{2.19}$$

Since $\int_s^{2t} M(t,s)\, g_\varepsilon(t,a)\, dt = O(|\alpha|) + O\left(\int_0^{2\varepsilon} |Q(\sigma)|\, d\sigma \right) = O(1)$, it follows that

$$J_3 = O\left(\int_0^{2\varepsilon} ds \left| \int_s^{2\varepsilon} M(t,s)\, g_\varepsilon(t,a)\, dt \right| \right) = O(|\alpha|\, \varepsilon) + O\left(\varepsilon \int_0^{2\varepsilon} |Q(\sigma)|\, d\sigma \right). \tag{2.20}$$

Hence from (2.18), (2.19) and (2.20)

$$1/2\pi \int_{-\infty}^{\infty} \psi_\varepsilon^T(\mu^2, a)\, d\rho_1(\mu)\, \psi_\varepsilon(\mu^2, a) = I \int_0^{2\varepsilon} g_\varepsilon(s, a)\, ds + O(|\alpha|)$$

$$+ O\left(\int_0^{2\varepsilon} |Q(\sigma)|\, d\sigma\right) + O(1). \qquad (2.21)$$

Since $C(t, \lambda^{\frac{1}{2}}) \cos at = \frac{1}{2}(C(t, \lambda^{\frac{1}{2}} + a) + C(t, \lambda^{\frac{1}{2}} - a))$ and

$\lambda^{\frac{1}{2}} \int_0^x S(t, \lambda^{\frac{1}{2}})\, dt = -C(x, \lambda^{\frac{1}{2}}) + xA + B$, where

$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{13} & a_{23} \end{pmatrix}$ and $B = \begin{pmatrix} a_{12} & a_{22} \\ a_{14} & a_{24} \end{pmatrix}$, it follows from (2.16) by integration by parts

$$\psi_\varepsilon(\lambda, a) = \tfrac{1}{2}\varepsilon^{-2}(\lambda^{\frac{1}{2}} + a)^{-2}(2\varepsilon A + B - C(2\varepsilon, \lambda^{\frac{1}{2}} + a))$$
$$+ \tfrac{1}{2}\varepsilon^{-2}(\lambda^{\frac{1}{2}} - a)^{-2}(2\varepsilon A + B - C(2\varepsilon, \lambda^{\frac{1}{2}} - a)) = J_{11}(\lambda) + J_{12}(\lambda), \text{ say.}$$

When λ is real, say $\lambda = u$, we estimate separately $J_{11}(u)$ and $J_{12}(u)$ by using the explicit expressions for the elements of the matrix $C(2\varepsilon, u^{\frac{1}{2}} \pm a)$ as obtained from (2.1). Then it is easy to deduce that

$$\psi_\varepsilon(u, a) \gg K(\sin(u^{\frac{1}{2}} + b)\varepsilon/(u^{\frac{1}{2}} + b)\varepsilon)^2, K(\sin(u^{\frac{1}{2}} - b)\varepsilon/(u^{\frac{1}{2}} - b)\varepsilon)^2$$

where K are different constant matrices. K are non-singular, since the rank of the matrix (a_{ij}) of the coefficients in the boundary condition (1.2) is two. Also b have different constant values. (The symbol $\gg$ means that the matrix on the left majorizes that on the right.) Putting $u = \mu^2$ and for convenience $\varepsilon = 1$, it follows from (2.21) that

$$\int_{-\infty}^{\infty} (\sin \mu/\mu)^4\, d\rho_1(b + \mu), \int_{-\infty}^{\infty} (\sin \mu/\mu)^4\, d\rho_1(\mu - b) \ll C,$$

where C is a suitable constant matrix independent of b.

The lemma therefore follows, since $\sin \mu/\mu \geq 2/\pi$ for $0 \leq \mu \leq \frac{1}{2}\pi$.

3. Some auxiliary results

$$\text{Let } F(\lambda^{\frac{1}{2}}) \equiv F(\lambda^{\frac{1}{2}}, f) = \int_0^{\infty} f^T(x)\, C(x, \lambda^{\frac{1}{2}})\, dx \qquad (3.1)$$

the C-Fourier transform of the vector $f \varepsilon L_2[0, \infty)$.

When $u = \mu^2$, let $F(u^{1/2}) = F_1(\mu)$, if $\mu \geq 0$

$= -F_1(\mu)$, if $\mu < 0$.

Consider an arbitrary $f \varepsilon C^1(0, X)$ such that $f(x) = 0$ for all $x \geq X$. Then obviously $f(x) \varepsilon L_2[0, X]$ and hence $f(x) \varepsilon L_2[0, \infty)$.

Taking scalar product of (2.3) by $f^T(x)$, integrating over $[0, \infty)$ and then changing the order of integration, we have

$$\int_0^\infty f^T C(x, \lambda^{\frac{1}{2}})\, dx = \int_0^\infty (f(x) - h(x))^T \phi(x, \lambda)\, dx$$

where $h(x) = \int_x^\infty M^T(x, y) f(y)\, dy$.

Then $F(\lambda^{\frac{1}{2}}, f) = E(\lambda, f) - E(\lambda, h) = E(\lambda, g)$, where $g = f - h$.

Now $||g||^2 = \int_0^\infty |g|^2\, dx = 1/\pi \int_{-\infty}^\infty E^T(u, g)\, d\rho(u)\, E(u, g)$

$$= 1/\pi \int_{-\infty}^\infty F^T(u^{\frac{1}{2}}, f)\, d\rho(u)\, F(u^{\frac{1}{2}}, f) \tag{3.2}$$

by the Parseval relation.

Therefore from $|\, ||f||^2 - ||g||^2 \,| \leq ||h||\, (2||f|| + ||h||)$

$$\left|\, ||f||^2 - 1/2\pi \int_{-\infty}^\infty F_1^T(\mu, f)\, d\rho_1(\mu)\, F_1(\mu, f) \right|$$

$$\leq 1/\pi \int_{-\infty}^0 F_1^T(\mu, f)\, d\rho_1(\mu)\, F_1(\mu, f) + ||h||\, (2||f|| + ||h||) \tag{3.3}$$

which extends Marchenko's[8] result (p. 46) to the present system.
We establish the following lemmas.

Lemma 3.1. For an arbitrary $f \varepsilon C^1(0, X)$ which vanishes for all $x \geq X$,

$$\lim_{a \to \infty} 1/4\pi \int_{-\infty}^\infty (F_1(\mu + a) + F_1(\mu - a))^T\, d\rho_1(\mu)\, (F_1(\mu + a) + F_1(\mu - a)) = ||f||^2,$$

if $Q(x)$ satisfies the conditions of lemma 2.2.

The lemma holds, if we replace $\rho_1(\mu)$ by $\sigma_1(\mu)$, the spectral matrix for the Fourier system.

The lemma is proved by an adaptation of the analysis of Marchenko[8] (p. 46–48).

Replace f by $f(a,x) = f\cos ax$, $-\infty < x < \infty$, for which $F_1(\mu,f)$ and $h(x)$ are replaced, respectively, by $F_1(a,\mu,f)$ and $h(a,x)$. Then from (3.3)

$$\lim_{a\to\infty} \left| \|f(a,x)\|^2 - 1/2\pi \int_{-\infty}^{\infty} F_1^T(a,\mu,f)\,d\rho_1(\mu)F_1(a,\mu,f) \right|$$

$$\leq \lim_{a\to\infty} 1/\pi \int_{-\infty}^{0} F_1^T(a,\mu,f)\,d\rho_1(\mu)F_1(a,\mu,f)$$

$$+ \lim_{a\to\infty} \|h(a,x)\| \,(2\|f(a,x)\| + \|h(a,x)\|). \tag{A}$$

By using the explicit form of $C(x,\mu)$ and the Riemann Lebesgue lemma in $F_1(a,\mu,f)$ it follows that

$$\lim_{a\to\infty} F_1(a,\mu,f) = 0.$$

When $\lambda \leq 0$,

$$|F(a,\lambda^{\frac{1}{2}},f)| \leq 4 \int_0^\infty |f|\,\|C(x,i|\lambda|^{\frac{1}{2}})\|\,dx = 4 \int_0^\infty |f|\,\|e(x,|\lambda|^{\frac{1}{2}})\|\,dx.$$

Now $f = 0$ outside $(0,X)$. Then it follows from above by using the Schwarz inequality that $F(a,\lambda^{\frac{1}{2}},f)|^2$ converges uniformly to zero in each sub-interval and is majorized by $16\|f\|^2\|e(X,|\lambda|^{\frac{1}{2}})\|$ which is integrable over $-\infty < \lambda \leq 0$ with weight $\|d\rho(\lambda)\|$, by theorem 2.1.

Hence $\lim_{a\to\infty} \int_{-\infty}^{0} |F^T(a,\lambda,f)|^2\,\|d\rho(\lambda)\|$

$$= \int_{-\infty}^{0} \lim_{a\to\infty} |F^T(a,\lambda,f)|^2\,\|d\rho(\lambda)\| = 0.$$

Now $h(a,x) = \int_x^X M^T(x,y)\,f(a,y)\,dy = \int_x^X M^T(x,y)\,f(y)\cos ay\,dy$, which tends to zero as a tends to infinity, by the Riemann Lebesgue lemma; $M^T(x,y)$ is bounded in $(0,X)$ when $Q(x)$ satisfies conditions stated in the lemma (*see* relation (2.9)).

Therefore $\lim_{a\to\infty} ||h(a,x)||^2 = \lim_{a\to\infty} \int_x^X |h(a,u)|^2 \, du = 0.$

Also $\lim_{a\to\infty} ||f(a,x)||^2 = \lim_{a\to\infty} \int_0^\infty |f|^2 \cos^2 ax \, dx = \frac{1}{2} ||f||^2$, by the Riemann Lebesgue lemma. Hence from (A)

$$\lim_{a\to\infty} 1/\pi \int_{-\infty}^{\infty} F_1^T(a,\mu,f) \, d\rho_1(\mu) F_1(a,\mu,f) = ||f||^2$$

Since $F_1(a,\mu,f) = \frac{1}{2}(F_1(\mu+a) + F_1(\mu-a))$, the lemma follows from above.

Lemma 3.2. If $Q(x)$ satisfies the condition of lemma 2.2 and $f(x)$ that of lemma 3.1, then

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} F_1^T(\mu+a) \, d\rho_1(\mu) \, F_1(\mu-a) = \lim_{a\to\infty} \int_{-\infty}^{\infty} F_1^T(\mu-a) \, d\rho_1(\mu) \, F_1(\mu+a)$$

$= 0$, uniformly in μ.
The lemma remains true when $\rho_1(\mu)$ is replaced by $\sigma_1(\mu)$.

$$\text{Let } G(\mu,a) = \int_{-\infty}^{\infty} F_1^T(\mu+a) \, d\rho_1(\mu) \, F_1(\mu-a)$$

$$= \left(\int_{-\infty}^{0} + \int_0^\infty\right) F_1^T(\mu+a) \, d\rho_1(\mu) \, F_1(\mu-a) = I_1 + I_2, \text{ say.}$$

Using the inequality

$(\Sigma a_{\mu\nu} x_\mu y_\nu)^2 \leq \Sigma a_{\mu\nu} x_\mu x_\nu \, \Sigma a_{\mu\nu} y_\mu y_\nu$, if $\Sigma a_{\mu\nu} x_\mu x_\nu$, $a_{\mu\nu} = a_{\nu\mu}$ is a positive quadratic form (with real but not necessarily positive coefficients), (Hardy, *et al*[12], ch. 29, p. 33) we obtain

$$|I_2|^2 \leq \int_0^\infty F_1^T(\mu+a) \, d\rho_1(\mu) \, F_1(\mu+a) \int_0^\infty F_1^T(\mu-a) \, d\rho_1(\mu) \, F_1(\mu-a).$$

By integration by parts

$$F_1(\mu,f) = \int_0^\infty f^T C(x,\mu) \, dx = 1/\mu \int_0^\infty f^T dS(x,\mu) = O(1/\mu).$$

Therefore

$$\int_0^\infty F_1^T(\mu+a)\,d\rho_1(\mu)\,F_1(\mu+a) = O\left(\int_0^\infty \|d\rho_1\|/(\mu+a)^2\right)$$

$$= O\left(\sum_{k=0}^{\infty} 1/(k+a)^2\right), \text{ by lemma 2.2, where}$$

$$\|d\rho_1\| = \max_{1\le r,s\le 2} |(\rho_1)_{rs}(\mu)|$$

Now $\sum_{k=0}^{\infty} 1/(k+a)^2 = \sum_{k=0}^{N} 1/(k+a)^2 + \sum_{k=N+1}^{\infty} 1/(k+a)^2$

$$\le \sum_{k=0}^{N} 1/(k+a)^2 + \sum_{k=N+1}^{\infty} 1/k^2.$$

The usual limit technique can now be applied so as to obtain

$$\lim_{a\to\infty} \int_0^\infty F_1^T(\mu+a)\,d\rho_1(\mu)\,F_1(\mu+a) = 0. \tag{3.4}$$

Similarly $\lim_{a\to\infty} \int_{-\infty}^0 F_1^T(\mu-a)\,d\rho_1(\mu)\,F_1(\mu-a) = 0.$ (3.5)

Again $\int_0^\infty F_1^T(\mu-a)\,d\rho_1(\mu)\,F_1(\mu-a) \le \int_{-\infty}^\infty F_1^T(\mu-a)\,d\rho_1\,F_1(\mu-a)$

$$\le K\left(1+\sum_{-\infty}^{-1} 1/k^2+\sum_{1}^{\infty} 1/k^2\right) = O(1) \tag{3.6}$$

where K is a constant (compare Levitan[7], p. 240).

Similarly $\int_{-\infty}^0 F_1^T(\mu+a)\,d\rho_1(\mu)\,F_1(\mu+a) = O(1).$ (3.7)

All the results (3.4)–(3.7) hold uniformly for μ.

Hence $\lim_{a\to\infty} I_1 = \lim_{a\to\infty} I_2 = 0.$

Similarly for $\int_{-\infty}^{\infty} F_1^T(\mu-a)\, d\rho_1(\mu)\, F_1(\mu+a)$ and for the case when ρ_1 is replaced by σ_1. The lemma therefore follows.

Put $\rho_1(\mu)-\sigma_1(\mu) = \Phi(\mu) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12} & \Phi_{22} \end{pmatrix}$, where Φ is symmetric, since ρ_1, σ_1 are so. Let $\Phi(\mu)$ be extended to negative μ as an odd function.

Lemma 3.3. *If* $Q(x)$ satisfies the condition of lemma 2.2, then for the C-Fourier transform $F_1(\mu)$ of an arbitrary vector f of lemma 3.1,

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} F_1^T(\mu-a)\, d\Phi(\mu)\, F_1(\mu-a) = 0, \text{ holds uniformly for } \mu \geq 0.$$

Since $F_1(\mu)$ is extended to negative μ as an odd function, the lemma follows from lemmas 3.1 and 3.2.

4. Derivation of the asymptotic formulae

In what follows we shall require the Wiener-Tauberian theorem[13] (pp. 73–74) as modified by Levitan[7] (pp. 241–242) *i.e.* the following theorem.

Theorem A. Let $h(\mu)$, $h_1(\mu)$ be two bounded measurable functions satisfying

(i) $h(\mu)$, $h_1(\mu)$ are each $O(1/\mu^2)$ for large values of μ;

(ii) the Fourier transform of $h(\mu)$ never vanishes.
Suppose further that $\theta(\mu)$ is a function satisfying the condition

$$\sup_{-\infty<\mu<\infty} \overset{\mu+1}{\underset{\mu}{V}} \theta(\mu) < \infty.$$

Then $\lim_{a\to\infty} \int_{-\infty}^{\infty} h(\mu-a)\, d\theta(\mu) = 0$ implies $\lim_{a\to\infty} \int_{-\infty}^{\infty} h_1(\mu-a)\, d\theta(\mu) = 0.$

(*see* also Titchmarsh[14], p. 371, where a different formulation is given.)

The following theorem is now established.

Theorem 4.2. If $Q(x)$ satisfies condition of lemma 2.2 and $\mu_0 > 0$, fixed but arbitrary, then

$$\lim_{a\to\infty} (\rho_1(\mu_0+a)-\rho_1(a)) = 2/\pi \,.\, \mu_0 I, \text{ where } I \text{ is the } 2\times 2 \text{ unit matrix.}$$

Since $f(x)$ is arbitrary and rank (a_{ij}), a_{ij} coefficients in the boundary condition (1.2), are two, it follows from the explicit form of $C(x,\mu)$ in the definition of $F_1(\mu)$ that the components of $F_1(\mu)$ are linearly independent. Hence if $F_1 = (F_{11}, F_{12})^T$, it follows from lemma 3.3 that

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} F_{1j}(\mu-a)\, F_{1k}(\mu-a)\, d\Phi_{jk}(\mu) = 0,\ j,k = 1,2. \tag{4.1}$$

Also $F_{1j}F_{1k} = 0(1/\mu^2)$, $\overset{\mu+1}{\underset{\mu}{V}}\ \Phi_{jk}(\mu) < \infty$, by lemma 2.2.

Again, the Fourier transform of convolution $F_{11} \star F_{12}$ is the product of the Fourier transforms of F_{11} and F_{12}. The theorem is obtained from Tauberian theorem A by closely following the analysis of Levitan[7] (pp. 241–243).
Finally we establish the following theorem.

Theorem 4.2. The spectral matrix $\rho_1(\mu)$ associated with the differential system (1.1) and appearing in the inversion formula (1.9) has the asymptotic representation

$\rho_1(\mu) = 2/\pi\,.\,\mu I + o(\mu)$, as μ tends to infinity.

Here I is the 2×2 unit matrix and $Q(x)$ satisfies the condition of lemma 2.2.

In theorem 4.1 put $\mu_0 = 2$ and $a = n+2k-1$; then there exists an integer $n > N$, such that

$$\rho_1(n+2k+1) - \rho_1(n+2k-1) = (4/\pi + \varepsilon_k)I.$$

where $|\varepsilon_k| < \varepsilon$, a pre-assigned positive number and $k \geq 0$ is arbitrary but fixed.

Putting $k = 0, 1, 2, \ldots, m$ (fixed) and summing, we obtain in the usual manner (compare Marchenko[8], p. 53)

$$\lim_{\mu\to\infty} 1/\mu\ \rho_1(\mu) = 2/\pi\,.\,I.$$

Hence the theorem.

Acknowledgements

The author is grateful to Dr S. K. Acharyya of the Department of Pure Mathematics, Calcutta University, and to the referees for valuable comments.

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