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Generalized Laplace Stieltjes integral

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Abstract

The concept of functions of bounded kth variation has been extended to an infinite range of intervals leading to the definition of the RS_k^* integral on an infinite segment that introduces a Laplace-type integral. A convergence formula along with some other properties has been presented.

Key words: Functions, bounded kth variation, RS_k^* integral, Laplace integral, Stieltjes integral.

1. Introduction

Russell[1] defines an integral, the RS_k integral, which is an extension of the Riemann-Stieltjes integral. Further properties of this integral including the convergence formulae are obtained in Das and Lahiri[2] and Das and Das[3]. Bhattacharyya and Das[4] extend this notion so as to define a Lebesgue-type integral, the LS_k integral. A Perron-type generalization of such integrals and its approximate and proximal extensions are obtained in Das and Das[5]. The purpose of the present paper is to obtain a Laplace-Stieltjes-type integral, the LapS_k integral, induced by the RS_k^* integral. To this end, it is desirable to set up the notions and results of BV_k functions[6], k-convex functions[7] and of the RS_k^* integrals[1,2] on an infinite segment. The concepts should also be extended so as to accommodate complex-valued functions. In the next section we obtain certain properties of the RS_k^* integral which are useful in sequel. Finally, in the last section we obtain the definition of our proposed integral, the LapS_k integral, which is an extension of the Laplace-Stieltjes integral[8]. A convergence formula for such an integral is an immediate consequence of its reduction to the Laplace-Stieltjes integral and in turn to the Laplace integral.

Let a and b be two real numbers such that $a < b$ and let k be a positive integer greater than 1. For notations and definitions we refer to Russell[1,6], Natanson[9] and Das and Lahiri[10]. However, we quote two basic definitions needed in this context.

Definition 1.1[6]. The total kth variation of g on $[a, b]$ is the number

$$V_k[g; a, b] = \sup_{\pi} \sum_{i=0}^{n-k} |Q_{k-1}(g; x_{i+1}, \ldots, x_{i+k}) - Q_{k-1}(g; x_i, \ldots, x_{i+k-1})|,$$

where the supremum is taken over all π: $a \le x_0 < x_1 < \ldots < x_n \le b$ sub-division of $[a,b]$. If $V_k[g;a,b] < +\infty$, then g is said to be of bounded kth variation, BV_k, on $[a,b]$ and we write $g \varepsilon BV_k[a,b]$. The symbol

$$Q_{k-1}(g;\alpha_0,\alpha_1,\ldots,\alpha_{k-1}) = \sum_{i=0}^{k-1} \frac{g(\alpha_i)}{\prod_{j=0,j\neq i}^{k} (\alpha_i-\alpha_j)}$$

stands for the $(k-1)$th divided difference of g.

Definition 1.2.[1] The RS_k^* integral of f with respect to g on $[a,b]$,

$$*\int_a^b f(x)\,\frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}}$$

is the real number I, if it exists uniquely, and if for each $\varepsilon > 0$ there is a real number $\delta(\varepsilon)$ such that when $x_i \le \xi_i \le x_{i+1}, i = 0,\ldots, n-k$

$$\left|I - \sum_{i=0}^{n-k} f(\xi_i)\,[Q_{k-1}(g;x_{i+1},\ldots,x_{i+k}) - Q_{k-1}(g;x_i,\ldots,x_{i+k-1})]\right| < \varepsilon$$

whenever $\|\pi\| < \delta(\varepsilon)$. If the integral exists we write $(f,g)\ \varepsilon\ RS_k^*[a,b]$.

It is observed[1] that if f is continuous and g is BV_k on $[a,b]$, then $(f,g)\ \varepsilon\ RS_k^*[a,b]$.

2. BV_k functions and RS_k^* integrals on infinite segments

Definition 2.1. Let g be a function defined for all x, $-\infty < x < \infty$. If $V_k[g;a,b]$ is finite for all $a < b$, and if $\sup_{a<b} V_k[g;a,b]$ is finite, then g is said to be of bounded kth variation, BV_k, on $(-\infty,\infty)$, and the number

$$V_k[g;-\infty,\infty] = \sup_{a<b} V_k[g;a,b]$$

is called the total kth variation of g on $(-\infty,\infty)$.

Several properties of the function g of BV_k on $(-\infty,\infty)$ can be obtained analogous to those obtained by Russell[6] and Natanson[9].

Definition 2.2. Let f be a bounded continuous function and let g be BV_k on $(-\infty,\infty)$. We define

$$*\int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}} = \lim_{\substack{a\to-\infty\\ b\to+\infty}} *\int_a^b f(x)\frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}},$$

and if it is finite, then we say $(f,g) \varepsilon RS_k^*(-\infty,\infty)$.

Similarly, we define

$$*\int_a^{\infty} f(x)\frac{d^k g(x)}{dx^{k-1}} \quad \text{and} \quad *\int_{-\infty}^{b} f(x)\frac{d^k g(x)}{dx^{k-1}}$$

Results of Sections 2, 3, 6 of Russell[6], Lemma 2 and Theorem 6 of Das and Lahiri[2] and Lemma 3.1 of Das and Das[3] have obvious extensions for improper integrals. We note below the following observation.

Observation 2.1. Das and Das[3] (Theorem 3.4) obtain an analogue of Helly's second theorem:

Let f be continuous on $[a,b]$ and let $\{g_p\}$ be a sequence of functions which converges uniformly to a finite function g on $[a,b]$. If K is a fixed positive number and $V_k[g_p;a,b] \le K$ for all p, then

$$\lim_{p\to\infty} *\int_a^b f(x)\frac{d^k g_p(x)}{dx^{k-1}} = *\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$$

We show below that this theorem does not hold for all continuous integrands in $(-\infty,\infty)$. For example, let

$$\begin{aligned} G(x) &= 0 \text{ if } x \le 0 \\ &= x^k \text{ if } 0 < x \le 1 \\ &= \sum_{r=1}^{k} (-1)^{r-1}\binom{k}{r} x^{k-r} \quad \text{if } x > 1, \end{aligned}$$

and let $g_p(x) = G(x-p), p = 1,2,\ldots$. Clearly $\{g_p\}$ converges uniformly to $g \equiv 0$ in $(-\infty,\infty)$. Obviously for each $p, p = 1,2,\ldots, g_p^{(k-1)}(p) = 0$ and $g_p^{(k-1)}(p+1) = k!$. Then in view of Lemma 1 of Das and Lahiri[2], $V_k[g_p;p,p+1] = k$. Further in view of Lemma 1 of Russell[6] and Definition 2.1,

$$V_k[g_p;\ -\infty,p] = V_k[g_p;p+1,\infty] = 0.$$

So by an analogue of Theorem 7 of Russell[6], it follows that g_p is BV_k on $(-\infty,\infty)$ for each $p, p = 1,2,\ldots,$ and

$$V_k[g_p;\ -\infty,\infty] = V_k[g_p;\ p,p+1] = k.$$

Applying an analogue of the Corollary to Theorem 3 of Russell[1] and Lemma 2 of Das and Lahiri[2], we obtain

$$*\int_{-\infty}^{\infty} 1 \ \frac{\mathrm{d}^k g_p(x)}{\mathrm{d}x^{k-1}} = *\int_{p}^{p+1} 1 \ \frac{\mathrm{d}^k g_p(x)}{\mathrm{d}x^{k-1}} = k$$

for each $p = 1, 2, \ldots$. Consequently,

$$\lim_{p\to\infty} \ *\int_{-\infty}^{\infty} 1 \ \frac{\mathrm{d}^k g_p(x)}{\mathrm{d}x^{k-1}} = k \neq 0 = *\int_{-\infty}^{\infty} 1 \ \frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}}$$

This trouble may be overcome, as in Natanson[9], by considering those continuous functions f on $(-\infty, \infty)$ such that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. According to Natanson[9] (p. 240), we denote this class by C_∞.

Theorem 2.1. Let $f \in C_\infty$ and let $\{g_p\}$ be a sequence of functions on $(-\infty, \infty)$ which converges uniformly to a finite function g on $(-\infty, \infty)$. If K is a fixed positive number and $V_k[g_p; -\infty, \infty] \leq K$ for all p, then

$$\lim_{p\to\infty} \ *\int_{-\infty}^{\infty} f(x) \ \frac{\mathrm{d}^k g_p(x)}{\mathrm{d}x^{k-1}} = *\int_{-\infty}^{\infty} f(x) \ \frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}}$$

Proof. We omit the proof. The proof can be carried out from that of Theorem 6 (p. 240) of Natanson[9], applying analogues of the Corollary to Theorem 3 of Russell[1] and Lemma 3.1 of Das and Das[3], and Theorem 3.4 of Das and Das[3] (in its original form) in appropriate steps.

We now present the definitions of functions of bounded kth variation and RS_k^* integrals for complex-valued functions.

Definition 2.3. (a) If $g = g_1 + ig_2$, where g_1 and g_2 are real-valued functions on $[a, b]$, then $g \in BV_k[a, b]$ if and only if $g_i \in BV_k[a, b]$, $i = 1, 2$, and

$$V_k[g; a, b] \leq V_k[g_1; a, b] + V_k[g_2; a, b].$$

(b) If $f = f_1 + if_2$ and $g = g_1 + ig_2$, where f_i and g_i are real-valued functions on $[a, b]$, then we define

$$*\int_a^b f(x) \ \frac{\mathrm{d}^k g(x)}{\mathrm{d}x^{k-1}} = *\int_a^b f_1(x) \ \frac{\mathrm{d}^k g_1(x)}{\mathrm{d}x^{k-1}} - *\int_a^b f_2(x) \ \frac{\mathrm{d}^k g_2(x)}{\mathrm{d}x^{k-1}}$$

$$+ *i \int_a^b f_2(x) \ \frac{\mathrm{d}^k g_1(x)}{\mathrm{d}x^{k-1}} + *i \int_a^b f_1(x) \ \frac{\mathrm{d}^k g_2(x)}{\mathrm{d}x^{k-1}},$$

provided all the integrals on the right exist.

These definitions can be extended on infinite segments.

3. Some results on RS_k^* integral

Russell[1] presents a reduction formula for the RS_k^* integral. We shall utilise the result to reduce an RS_k^* integral to an RS integral introducing a related normalized function of bounded variation. For the development of the context we require the definition of AC_k function of Das and Lahiri[10] and some of its standard properties which we refer to Das and Lahiri[10] and Das and Das[11]. We simply make a remark in view of Theorem 9 of Russell[6] and Definition 1.4 of De Sarkar and Das[12].

Remark 3.1. If $g \varepsilon BV_k[a,b]$, then $g^{(k-1)}$ exists and is BV on E such that $[a,b]-E$ is countable.

Definition 3.1. If $g \varepsilon BV_k[a,b]$, then define α on $[a,b]$ by

$$\begin{aligned} \alpha(x) &= 0 && \text{if } x = a \\ &= \frac{g_+^{(k-1)}(x)+g_-^{(k-1)}(x)}{2} - g_+^{(k-1)}(a) && \text{if } a < x < b \\ &= g_-^{(k-1)}(b) - g_+^{(k-1)}(a) && \text{if } x = b. \end{aligned} \tag{1}$$

Clearly, α is BV on $[a,b]$ and also

$$\alpha(a) = 0,$$

$$\alpha(x) = \frac{\alpha(x+)+\alpha(x-)}{2} \quad \text{if } a < x < b.$$

Hence, α is a normalized function[8] of bounded variation on $[a,b]$. Further, by Remark 3.1

$$\alpha(x) = g^{(k-1)}(x) - g_+^{(k-1)}(a) \tag{2}$$

on E where $[a,b]-E$ is countable. It readily follows that for $a \le x \le b$, $\alpha(x)+g_+^{(k-1)}(a)$ lies between the infimum and the supremum of $\{g_+^{(k-1)}(x), g_-^{(k-1)}(x)\}$.

Until otherwise stated by α we shall mean the normalized function of bounded variation on $[a,b]$ relative to $g \varepsilon BV_k[a,b]$.

Theorem 3.1. If f is continuous on $[a,b]$ and $g \varepsilon BV_k[a,b]$, then

$$(k-1)!\ {*}\!\int_a^b f(x)\ \frac{d^k g(x)}{dx^{k-1}} = (RS) \int_a^b f(x)\, d\alpha(x). \tag{3}$$

Proof. By Theorem 2.5 of De Sarkar and Das[12], $g^{(r)}$ is AC_{k-1-r} on $[a,b]$ and so, in view of Theorems 3.3 and 3.4 of Das and Das[11], $g^{(r)}$ is the $(k-1-r)$ fold Lebesgue integral of $g^{(k-1)}$. Utilising (2), we have for each $r = 0, 1, \ldots, k-2$

$$g^{(r)}(x) = \int_c^x \int_c^{x_{k-2-r}} \cdots \int_c^{x_1} \alpha(t)\, dt\, dx_1 \ldots dx_{k-2-r} + P_r(x-c),$$

where $a \le c \le b$ and $P_r(x-c)$ is a polynomial of degree $(k-2)$ at the most. By repeated applications of Theorem 18 of Russell[1] (modified for RS_k^* integral), we obtain (3) and thus the theorem is proved.

Corollary 3.1. If f is continuous and $g \varepsilon BV_k[a,b]$, then the function

$$F(x) = *\int_a^x f(t)\, (d^k g(t)/dt^{k-1})$$

is a normalized function of bounded variation on $[a,b]$.

Proof. By Theorem 3.1,

$$F(x) = \frac{1}{(k-1)!}\ (RS) \int_a^x f(t)\, d\alpha(t)$$

where α is the normalized function of bounded variation on $[a,b]$ relative to $g\,\varepsilon\, BV_k[a,b]$. The proof now follows from Theorem 8b (p. 14) of Widder[8].

Theorem 3.2. If f is continuous in $a \le x < \infty$ and if g is BV_k on $a \le x \le R$ for every $R > a$, then

$$(k-1)!\ *\int_a^\infty f(x)\, \frac{d^k g(x)}{dx^{k-1}} = \int_a^\infty f(x)\, d\alpha(x), \tag{4}$$

provided the first integral converges.

Proof. By Theorem 3.1

$$(k-1)!\ *\int_a^R f(x)\, \frac{d^k g(x)}{dx^{k-1}} = \int_a^R f(x)\, d\alpha(x). \tag{5}$$

Since each point of $(k-1)$th differentiability of g is a point of continuity of α, it follows that α has atmost a countable points of discontinuity in $a \le x < \infty$. Since the integral on the left of (4) converges so we may assume the integral as the limit of the integrals on the left of (5) as $R \to \infty$ over the set of points of continuity of α, E (say). We thus have

$$(k-1)! \quad {}^*\!\int_a^\infty f(x) \; \frac{d^k g(x)}{dx^{k-1}} = \lim_{\substack{R\to\infty \\ R\varepsilon E}} \int_a^R f(x)\, d\alpha(x).$$

Since the integral on the right of (5) considered as a function of R is normalized, we can apply Theorem 8c (p. 14) of Widder[8] and obtain for each $R > a$

$$(k-1)! \quad {}^*\!\int_a^\infty f(x) \; \frac{d^k g(x)}{dx^{k-1}} = \lim_{R\to\infty} \int_a^R f(x)\, d\alpha(x) = \int_a^\infty f(x)\, d\alpha(x).$$

This completes the proof.

Theorem 3.3. If f and ϕ are continuous on $[a, b]$ and $g \varepsilon BV_k[a, b]$, and if

$$\beta(x) = {}^*\!\int_c^x \phi(t) \; \frac{d^k g(t)}{dt^{k-1}},$$

where $a \le x \le b$, $a \le c \le b$, then

$$\int_a^b f(x)\, d\beta(x) = {}^*\!\int_a^b f(x)\, \phi(x) \, \frac{d^k g(x)}{dx^{k-1}}. \tag{6}$$

Proof. Clearly $\beta \varepsilon BV[a, b]$ and so $(f, \beta) \varepsilon RS[a, b]$. That $(f\phi, g) \varepsilon RS_k^*[a, b]$ follows from Theorem 11 of Russell[1]. We may therefore consider $\pi(x_0, x_1, \dots, x_n)$ subdivision of $[a, b]$ with $x_i \varepsilon E = \{x : a \le x \le b$ and $g^{(k-1)}(x)$ exists$\}$. We write

$$\sigma_\pi = \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{x_{i+1}} \phi(t) \; \frac{d^k g(t)}{dt^{k-1}}.$$

Then in view of Theorem 1 and corollary to Theorem 3 of Russell[1], we have

$$\sigma_\pi - {}^*\!\int_a^b f(x)\, \phi(x) \; \frac{d^k g(x)}{dx^{k-1}} = \sum_{i=0}^{n-1} {}^*\!\int_{x_i}^{x_{i+1}} \{f(x_i) - f(x)\} \; \phi(x) \; \frac{d^k g(x)}{dx^{k-1}}.$$

Consequently, by Lemma 3.1 of Das and Das[3], we have

$$\left| \sigma_\pi - * \int_a^b f(x)\, \phi(x) \frac{d^k g(x)}{dx^{k-1}} \right|$$

$$\leq \sum_{i=0}^{n-1} \max_{x_i \leq x \leq x_{i+1}} |f(x_i) - f(x)| \; V_k(g; x_i, x_{i+1})$$

$$\leq M_\pi \; V_k(g; a, b),$$

where M_π is the largest of the numbers $\max_{x_i < x < x_{i+1}} |f(x_i) - f(x)|$, $i = 0, 1, \ldots, n-1$.

Since f is uniformly continuous on $[a, b]$ it follows that M_π tends to zero as the norm of π tends to zero. Since σ_π tends to the left integral of (6) as the norm of π-sub-division tends to zero, the theorem is proved.

4. The LapS$_k$ integral

Let $g(t)$ be a complex-valued function of the real variable t defined on the interval $0 \leq t < \infty$. Denote its real and imaginary parts by $g_1(t)$ and $g_2(t)$ respectively, $g(t) = g_1(t) + ig_2(t)$. Let $g \in BV_k[0, R]$ for every $R > 0$. Let s be a complex variable with real and imaginary parts σ and τ respectively, $s = \sigma + i\tau$. It follows from the existence theorem of the RS_k^* integral, Theorem 11 of Russell[1], that the integral

$$* \int_0^R e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

exists for each positive R and for every complex s.

Definition 4.1. Consider the improper integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}} = \lim_{R \to \infty} * \int_0^R e^{-st} \frac{d^k g(t)}{dt^{k-1}}. \tag{7}$$

If the limit exists for a given value of s, we say the integral on the left converges for that value of s. If the limit on the right does not exist, the integral on the left diverges. When the integral converges it defines a function of s which we denote by $f(s)$. This function $f(s)$ is called the k-generalized Laplace-Stieltjes transform of $g(t)$. The function, $f(s)$, will also be called the generating function and $g(t)$ will, sometimes, be called the determining function.

Theorem 4.1. If $s_0 = \sigma_0 + i\tau_0$ and if

$$\sup_{0 \le u < \infty} \left| * \int_0^u e^{-s_0 t} \frac{d^k g(t)}{dt^{k-1}} \right| = M < +\infty, \tag{8}$$

then the integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

converges for every s for which $\sigma > \sigma_0$, and

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}} = (s - s_0) \int_0^\infty e^{-(s-s_0)t} \beta(t)\, dt \tag{9}$$

where

$$\beta(u) = * \int_0^u e^{-s_0 t} \frac{d^k g(t)}{dt^{k-1}}$$

and the integral on the right of (9) converging absolutely.

Proof. By Theorem 3.3, we have

$$* \int_0^R e^{-st} \frac{d^k g(t)}{dt^{k-1}} = \int_0^R e^{-(s-s_0)t}\, d\beta(t)$$

$$= e^{-(s-s_0)R} \beta(R) + (s - s_0) \int_0^R e^{-(s-s_0)t} \beta(t)\, dt.$$

Utilising (8), we obtain, for $\sigma > \sigma_0$

$$\lim_{R \to \infty} e^{-(s-s_0)R} \beta(R) = 0.$$

Consequently,

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}} = (s-s_0) \int_0^\infty e^{-(s-s_0)t} \beta(t)\, dt$$

which proves (9).

Also,

$$\left| \int_0^\infty e^{-(s-s_0)t} \beta(t)\, dt \right| \le M \int_0^\infty e^{-(\sigma-\sigma_0)t}\, dt = \frac{M}{\sigma-\sigma_0}$$

for $\sigma > \sigma_0$, and so the integral on the right of (9) converges absolutely for $\sigma > \sigma_0$. This completes the proof.

Definition 4.2. The number σ_c, such that the integral (7) converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$ is called the abscissa of convergence. The line $\sigma = \sigma_c$ is called the axis of convergence.

If (7) converges for no point we have $\sigma_c = +\infty$ and if (7) converges for every point we have $\sigma_c = -\infty$.

Theorem 4.2. If $g \varepsilon BV_k[0,R]$ for every arbitrary $R > 0$, and if $g_+^{(k-1)}(t) = 0(e^{\gamma t})$ as $t \to \infty$ for some real number γ then the integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

converges for $\sigma > \gamma$.

Proof. Clearly $g_+^{(k-1)}(t)\ \varepsilon\ BV[0,R]$ for every $R > 0$. In fact, for every $R > 0$ there is $R_1, R_1 > R > 0$ and $g \varepsilon BV_k[0,R_1]$. Obviously then $g_+^{(k-1)}(R)$ exists and so $g_+^{(k-1)}(t)\ \varepsilon\ BV[0,R]$ for every arbitrary $R > 0$.

Following the proof of Theorem 3.1, it follows that

$$(k-1)!\ * \int_0^R e^{-st} \frac{d^k g(t)}{dt^{k-1}} = \int_0^R e^{-st}\, dg_+^{(k-1)}(t). \tag{10}$$

Since $g \varepsilon BV_k[0,R]$ for every arbitrary $R > 0$ and since $g_+^{(k-1)}(t) = 0(e^{\gamma t})$ as $t \to \infty$, there exists a constant M such that

$$|g_+^{(k-1)}(t)| \le M e^{\gamma t}, \quad 0 \le t < \infty.$$

Hence the integral

$$\int_0^\infty e^{-st} \; g_+^{(k-1)}(t)\, dt \tag{11}$$

is dominated, in absolute value, by

$$M \int_0^\infty e^{-(\sigma-\gamma)t}\, dt$$

which equals $\dfrac{M}{\sigma-\gamma}$ if $\sigma > \gamma$.

This shows that integral (11) converges absolutely for $\sigma > \gamma$. Now

$$\int_0^R e^{-st}\, dg_+^{(k-1)}(t) = e^{-sR} g_+^{(k-1)}(R) - g_+^{(k-1)}(O) + s * \int_0^R e^{-st} g_+^{(k-1)}(t)\, dt.$$

We observe that

$$e^{-sR} g_+^{(k-1)}(R) = 0(1) \text{ as } R \to \infty \text{ and } \sigma > \gamma.$$

Hence

$$*\int_0^\infty e^{-st}\, dg_+^{(k-1)}(t) = s \int_0^\infty e^{-st} g_+^{(k-1)}(t)\, dt - g_+^{(k-1)}(0). \tag{12}$$

The right-hand member of (12) is dominated, in absolute value, by

$$(\sigma^2+\tau^2)^{1/2} \; \frac{M}{\sigma-\gamma} + |g_+^{(k-1)}(0)|$$

and so the integral on the left of (12) is convergent. Hence, in view of equality (10), it follows that

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

converges, and thus the theorem is proved.

Note 4.1. Theorem 4.2 provides a consistency of Theorem 3.2.

That the exact converse of Theorem 4.2 is not true is shown by considering $g(t) = t^k/k!$. In fact, here $g_+^{(k-1)}(t) = g^{(k-1)}(t) = t$ for all t.

The integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}} = \frac{1}{(k-1)!} \int_0^\infty e^{-st} \, dt$$

converges for $\sigma > 0$, but $g_+^{(k-1)}(t) = t$ is not bounded.

As soon as relation (10) is obtained it is not difficult to prove the following results in view of the corresponding results in Widder[8], simply replacing $\alpha(t)$ there by $g_+^{(k-1)}(t)$.

Theorem 4.3. If $g \in BV_k[0, R]$ for every $R > 0$ and if the integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

converges for $s = s_0 = \gamma + i\delta$ with $\gamma > 0$, then

$$g_+^{(k-1)}(t) = o(e^{\gamma t}) \text{ as } t \to \infty.$$

Theorem 4.4. If $g \in BV_k[0, R]$ for every $R > 0$ and if the integral

$$* \int_0^\infty e^{-st} \frac{d^k g(t)}{dt^{k-1}}$$

converges for $s = s_0 = \gamma + i\delta$ with $\gamma < 0$, then $g_+^{(k-1)}(\infty)$ exists and $g_+^{(k-1)}(t) - g_+^{(k-1)}(\infty) = o(e^{\gamma t})$ as $t \to \infty$.

By the use of the above results we may frequently express LapS$_k$ integral in terms of an ordinary Laplace integral as in the following two theorems. We omit the easy proofs.

Theorem 4.5. If $g \in BV_k[0,R]$ for every $R>0$ and if the integral

$$f(s_0) = *\int_0^\infty e^{-s_0 t}\,\frac{d^k g(t)}{dt^{k-1}}$$

converges for $s_0=\gamma_0+i\delta_0$ with $\gamma_0>0$, then $(k-1)!\,f(s_0) = s_0\int_0^\infty e^{-s_0 t} g_+^{(k-1)}(t)\,dt$

$-g_+^{(k-1)}(0)$ and the integral $*\int_0^\infty e^{-s_0 t}\,\frac{d^k g(t)}{dt^{k-1}}$ converges absolutely if s_0 is replaced by any number with larger real part.

Theorem 4.6. If $g \in BV_k[0,R]$ for every $R>0$ and if the integral

$$f(s_0) = *\int_0^\infty e^{-s_0 t}\,\frac{d^k g(t)}{dt^{k-1}}$$

converges with $\gamma_0<0$ where $s_0=\gamma_0+i\delta_0$, then $g_+^{(k-1)}(\infty)$ exists and

$$(k-1)!\,f(s_0) = g_+^{(k-1)}(\infty)-g_+^{(k-1)}(0)+s_0\int_0^\infty e^{-s_0 t}[g_+^{(k-1)}(t)-g_+^{(k-1)}(\infty)]\,dt.$$

Also the integral $*\int_0^\infty e^{-s_0 t}\,\frac{d^k g(t)}{dt^{k-1}}$ converges absolutely if s_0 is replaced by any number with larger real part.

Following Widder[8] it is not difficult to establish the formula for abscissa of convergence, namely

Theorem 4.7. Let $1 = \overline{\lim_{t\to\infty}}\,\frac{\log|g_+^{(k-1)}(t)|}{t}$;

(a) if $1 \neq 0$ then $\sigma_c = 1$,

(b) if $1 = 0$ and $g_+^{(k-1)}(t)$ has no limit as $t\to\infty$ then $\sigma_c = 0$,

(c) if $\sigma_c \geq 0$ then $\sigma_c = 1$

(d) if $\sigma_c < 0$ then $\sigma_c = \overline{\lim_{t\to\infty}}\,\frac{\log|g_+^{(k-1)}(\infty)-g_+^{(k-1)}(t)|}{t}$.

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