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Short Communication

Explicit stiffness matrices of the serendipity rectangular elements for linear plane elastic continuum

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Abstract

The analytical integration formulae for the products of shape function local derivatives of parabolic and cubic elements over the square region are first obtained. By means of a transformation from global to local coordinates, the analytical integration formulae for the products of shape function global derivatives over the rectangular region are then established These results are then illustrated with reference to the evaluation of stiffness matrices in terms of explicit expressions for the parabolic and cubic elements of the serendipity family in rectangular shapes under the assumptions of linear plane elasticity.

Key words: Isoparametric quadrilateral element, explicit stiffness matrix, serendipity, finite element analysis.

1. Introduction

Recently, Babu and Pinder[1] and Mizukami[2] presented some integration formulae for the four-node linear isoparametric quadrilateral element. Using these formulae, one can easily obtain explicit stiffness matrix for the linear four-node rectangular element under the assumption of linear plane elasticity as illustrated by Robinson[3]. However, the linear four-node rectangular element is not very efficient and a large number of these elements are necessary to achieve sufficient accuracy. For this reason, in recent years there has been increasing usage of higher order elements for obtaining accurate solutions to continuum mechanics problems. The isoparametric elements of the serendipity family are among the most popular higher-order elements. Although the numerical integration for the evaluation of the element stiffness matrix is both simple and adaptable to computer programming, it increases the computations to be performed significantly. Thus, the standard numerical integration procedure is quite unattractive, especially for higher-order elements. Therefore, if the explicit (and simple) expressions of the element stiffness matrices are obtained, a big reduction (up to 99%) in computation time can be expected. Further, this will improve the efficiency of the standard finite element approach and make it more competitive with other numerical methods like finite difference and boundary integral equation methods. It is easily recognized that in this approach the computational convenience, speed and accuracy are enhanced many fold.

2. Element shape functions

Consider the rectangular elements of fig. 1 whose sides are parallel to the global coordinate axes (x_1, x_2). If we label each nodal point (x^e_{1i}, x^e_{2i}) by i and define two local normalized coordinates s and t by the equations:

$$s = (x_1 - x^e_{1c})/a, \quad \mathrm{d}s = \mathrm{d}x_1/a$$
$$t = (x_2 - x^e_{2c})/b, \quad \mathrm{d}t = \mathrm{d}x_2/b \tag{1}$$

where (x^e_{1c}, x^e_{2c}) denotes the position of the element centre in the (x_1, x_2) coordinate system and a, b denote half-side lengths of the rectangular element. In the normalized local coordinate system (s, t), the rectangular element is then defined by the square:

Table I
Numerical values of $I_s^{i,j}(i,j = 1,2,3,\ldots,8)$ (Parabolic element)

i/j	1	2	3	4	5	6	7	8
1	$\frac{26}{45}$							
2	$\frac{-8}{9}$	$\frac{16}{9}$						
3	$\frac{14}{45}$	$\frac{-8}{9}$	$\frac{26}{45}$					
4	$\frac{-1}{15}$	0	$\frac{1}{15}$	$\frac{8}{15}$				
5	$\frac{23}{90}$	$\frac{-4}{9}$	$\frac{17}{90}$	$\frac{1}{15}$	$\frac{26}{45}$			
6	$\frac{-4}{9}$	$\frac{8}{9}$	$\frac{-4}{9}$	0	$\frac{-8}{9}$	$\frac{16}{9}$		
7	$\frac{17}{90}$	$\frac{-4}{9}$	$\frac{23}{90}$	$\frac{-1}{15}$	$\frac{14}{45}$	$\frac{-8}{9}$	$\frac{26}{45}$	
8	$\frac{1}{15}$	0	$\frac{-1}{15}$	$\frac{-8}{15}$	$\frac{-1}{15}$	0	$\frac{1}{15}$	$\frac{8}{15}$

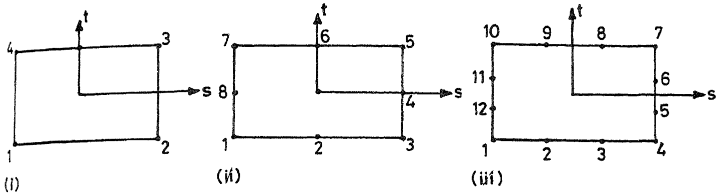

FIG 1. Rectangular elements of serendipity family (i) linear, (ii) parabolic, and (iii) cubic.

$-1 \le s, t \le 1$. Following Zienkiewicz[4], the element shape functions N_i for the parabolic and the cubic elements of the serendipity family are given by:

(1) Parabolic element

Corner nodes: $N_i = \frac{1}{4}(1+s_0)(1+t_0)(s_0+t_0-1),\ i = 1,3,5,7$

Table II
Numerical values of $I_t^{i,j}$, $(i,j = 1,2,3,\ldots,8)$
(Parabolic element)

i/j	1	2	3	4	5	6	7	8
1	$\frac{26}{45}$							
2	$\frac{1}{15}$	$\frac{8}{15}$						
3	$\frac{17}{90}$	$\frac{1}{15}$	$\frac{26}{45}$					
4	$\frac{-4}{9}$	0	$\frac{-8}{9}$	$\frac{16}{9}$				
5	$\frac{23}{90}$	$\frac{-1}{15}$	$\frac{14}{45}$	$\frac{-8}{9}$	$\frac{26}{45}$			
6	$\frac{-1}{15}$	$\frac{-8}{15}$	$\frac{-1}{15}$	0	$\frac{1}{15}$	$\frac{8}{15}$		
7	$\frac{14}{45}$	$\frac{-1}{15}$	$\frac{23}{90}$	$\frac{-4}{9}$	$\frac{17}{90}$	$\frac{1}{15}$	$\frac{26}{45}$	
8	$\frac{-8}{9}$	0	$\frac{-4}{9}$	$\frac{8}{9}$	$\frac{-4}{9}$	0	$\frac{-8}{9}$	$\frac{16}{9}$

Midside nodes: $N_i = \frac{1}{2}\,(1-s^2)\,(1+t_0),\ i = 2,6$

$$N_i = \frac{1}{2}\,(1+s_0)\,(1-t^2),\ i = 4,8 \qquad (2)$$

(2) Cubic element

Corner nodes: $N_i = \frac{1}{32}\,(1+t_0)\,(1+s_0)\,[-10+9(s^2+t^2)],\ i = 1,4,7,10$

Midside nodes: $N_i = \frac{9}{32}\,(1+t_0)\,(1-s^2)\,(1+9s_0),\ i = 2,3,8,9$

$$N_i = \frac{9}{32}\,(1+s_0)\,(1-t^2)\,(1+9t_0),\ i = 5,6,11,12 \qquad (3)$$

where $t_0 = tt_i, s_0 = ss_i$ and s_i, t_i are the values of s and t respectively at the ith node.

Table III
Numerical values of $I_{st}^{i,j}(i,j = 1,2,3,\ldots,8)$
(Parabolic element)

i/j	1	2	3	4	5	6	7	8
1	$\frac{17}{36}$	$\frac{1}{9}$	$\frac{-1}{12}$	$\frac{-1}{9}$	$\frac{7}{36}$	$\frac{-1}{9}$	$\frac{1}{12}$	$\frac{-5}{9}$
2	$\frac{-5}{9}$	0	$\frac{5}{9}$	$\frac{-4}{9}$	$\frac{-1}{9}$	0	$\frac{1}{9}$	$\frac{4}{9}$
3	$\frac{1}{12}$	$\frac{-1}{9}$	$\frac{-17}{36}$	$\frac{5}{9}$	$\frac{-1}{12}$	$\frac{1}{9}$	$\frac{-7}{36}$	$\frac{1}{9}$
4	$\frac{-1}{9}$	$\frac{-4}{9}$	$\frac{-1}{9}$	0	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0
5	$\frac{7}{36}$	$\frac{-1}{9}$	$\frac{1}{12}$	$\frac{-5}{9}$	$\frac{17}{36}$	$\frac{1}{9}$	$\frac{-1}{12}$	$\frac{-1}{9}$
6	$\frac{-1}{9}$	0	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{-5}{9}$	0	$\frac{5}{9}$	$\frac{-4}{9}$
7	$\frac{-1}{12}$	$\frac{1}{9}$	$\frac{-7}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{-1}{9}$	$\frac{-17}{36}$	$\frac{5}{9}$
8	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	0	$\frac{-1}{9}$	$\frac{-4}{9}$	$\frac{-1}{9}$	0

3. Integration formulae for shape function global derivatives

Consider the integrals:

$$W^{i,j}_{mn} = \iint_R \frac{\partial N_i}{\partial x_m} \frac{\partial N_j}{\partial x_n} \, dR, i,j = 1,2,3,\ldots \gamma;\ m,n = 1,2 \tag{4}$$

where R = rectangular region, dR = rectangular area differential, and γ = number of nodes per element.
In terms of the local coordinates s, t defined in equation (1), the above integral of equation (4) can be rewritten as:

Table IV
Numerical values of $I_s^{i,j}(i,j = 1,2,3,\ldots,12)$ (Cubic element)

i/j	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{41}{42}$											
2	$\frac{-63}{40}$	$\frac{18}{5}$										
3	$\frac{9}{20}$	$\frac{-99}{40}$	$\frac{18}{5}$									
4	$\frac{25}{168}$	$\frac{9}{20}$	$\frac{-63}{40}$	$\frac{41}{42}$								
5	$\frac{-33}{560}$	0	0	$\frac{33}{560}$	$\frac{27}{70}$							
6	$\frac{3}{140}$	0	0	$\frac{-3}{140}$	$\frac{-27}{560}$	$\frac{27}{70}$						
7	$\frac{17}{168}$	$\frac{9}{40}$	$\frac{-63}{80}$	$\frac{155}{336}$	$\frac{-3}{140}$	$\frac{33}{560}$	$\frac{41}{42}$					
8	$\frac{9}{40}$	$\frac{-99}{80}$	$\frac{9}{5}$	$\frac{-63}{80}$	0	0	$\frac{-63}{40}$	$\frac{18}{5}$				
9	$\frac{-63}{80}$	$\frac{9}{5}$	$\frac{-99}{80}$	$\frac{9}{40}$	0	0	$\frac{9}{20}$	$\frac{-99}{40}$	$\frac{18}{5}$			
10	$\frac{155}{336}$	$\frac{-63}{80}$	$\frac{9}{40}$	$\frac{17}{168}$	$\frac{3}{140}$	$\frac{-33}{560}$	$\frac{25}{168}$	$\frac{9}{20}$	$\frac{-63}{40}$	$\frac{41}{42}$		
11	$\frac{-3}{140}$	0	0	$\frac{3}{140}$	$\frac{27}{560}$	$\frac{-27}{70}$	$\frac{-33}{560}$	0	0	$\frac{33}{560}$	$\frac{27}{70}$	
12	$\frac{33}{560}$	0	0	$\frac{-33}{560}$	$\frac{-27}{70}$	$\frac{27}{560}$	$\frac{3}{140}$	0	0	$\frac{-3}{140}$	$\frac{-27}{560}$	$\frac{27}{70}$

$$W_{mn}^{i,j} = ab \int_{-1}^{1} \int_{-1}^{1} \frac{\partial N_i}{\partial x_m} \frac{\partial N_j}{\partial x_n} \, ds \, dt \tag{5}$$

and

$$\frac{\partial N_k}{\partial x_1} = \frac{1}{a} \frac{\partial N_k}{\partial s}, \frac{\partial N_k}{\partial x_2} = \frac{1}{b} \frac{\partial N_k}{\partial t} \tag{6}$$

where $k = m, n = 1, 2$.

From equations (5) and (6), it can be shown that

Table V
Numerical values of $I_t^{i,j}(i,j = 1,2,3,\ldots,12)$ (Cubic element)

i/j	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{41}{42}$											
2	$\frac{33}{560}$	$\frac{27}{70}$										
3	$\frac{-3}{140}$	$\frac{-27}{560}$	$\frac{27}{70}$									
4	$\frac{155}{336}$	$\frac{-3}{140}$	$\frac{33}{560}$	$\frac{41}{42}$								
5	$\frac{-63}{80}$	0	0	$\frac{-63}{40}$	$\frac{18}{5}$							
6	$\frac{9}{40}$	0	0	$\frac{9}{20}$	$\frac{-99}{40}$	$\frac{18}{5}$						
7	$\frac{17}{168}$	$\frac{3}{140}$	$\frac{-33}{560}$	$\frac{25}{168}$	$\frac{9}{20}$	$\frac{-63}{40}$	$\frac{41}{42}$					
8	$\frac{3}{140}$	$\frac{27}{560}$	$\frac{-27}{70}$	$\frac{-33}{560}$	0	0	$\frac{33}{560}$	$\frac{27}{70}$				
9	$\frac{-33}{560}$	$\frac{-27}{70}$	$\frac{27}{560}$	$\frac{3}{140}$	0	0	$\frac{-3}{140}$	$\frac{-27}{560}$	$\frac{27}{70}$			
10	$\frac{25}{168}$	$\frac{-33}{560}$	$\frac{3}{140}$	$\frac{17}{168}$	$\frac{9}{40}$	$\frac{-63}{80}$	$\frac{155}{336}$	$\frac{-3}{140}$	$\frac{33}{560}$	$\frac{41}{42}$		
11	$\frac{9}{20}$	0	0	$\frac{9}{40}$	$\frac{-99}{80}$	$\frac{9}{5}$	$\frac{-63}{80}$	0	0	$\frac{-63}{40}$	$\frac{18}{5}$	
12	$\frac{-63}{40}$	0	0	$\frac{-63}{80}$	$\frac{9}{5}$	$\frac{-99}{80}$	$\frac{9}{40}$	0	0	$\frac{9}{20}$	$\frac{-99}{40}$	$\frac{18}{5}$

$$W_{11}^{i,j} = \beta I_s^{i,j} ,$$

$$W_{22}^{i,j} = \alpha I_t^{i,j} , \qquad (7)$$

$$W_{12}^{i,j} = I_{st}^{i,j} ,$$

where $\alpha = a/b$, $\beta = b/a$,

$$I_s^{i,j} = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial N_i}{\partial s} \frac{\partial N_j}{\partial s} \, ds \, dt,$$

$$I_t^{i,j} = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial N_i}{\partial t} \frac{\partial N_j}{\partial t} \, ds \, dt, \text{ and}$$

$$I_{st}^{i,j} = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial N_i}{\partial s} \frac{\partial N_j}{\partial t} \, ds \, dt. \qquad (8)$$

Using equations (2), (3) and (8), the closed form expressions in terms of s_i, t_i, s_j, t_j for integrals $I_s^{i,j}, I_t^{i,j}$ and $I_{st}^{i,j}$ can now be derived. The numerical constants of these integrals for the parabolic and cubic elements of the serendipity family can then be computed (Tables I–VI).

4. Application to linear plane elasticity

Following Taylor[5], the stiffness submatrix of a rectangular element relating nodal pairs i and j for the linear anisotropic elastic solid under plane stress-plane strain conditions can be written as:

$$[K_{i,j}]_e = \left[\begin{array}{c|c} h S_{2i-1,2j-1} & h S_{2i-1,2j} \\ \hline h S_{2i,2j-1} & h S_{2i,2j} \end{array} \right] \qquad (9)$$

where

$$\begin{aligned} S_{2i-1,2j-1} &= (D_{11}\beta I_s^{i,j} + D_{33}\alpha I_t^{i,j}) + D_{13}(I_{st}^{i,j} + I_{st}^{j,i}) \\ S_{2i-1,2j} &= (D_{13}\beta I_s^{i,j} + D_{23}\alpha I_t^{i,j}) + (D_{12} I_{st}^{i,j} + D_{33} I_{st}^{j,i}) \\ S_{2i,2j-1} &= (D_{13}\beta I_s^{i,j} + D_{23}\alpha I_t^{i,j}) + (D_{33} I_{st}^{i,j} + D_{12} I_{st}^{j,i}) \\ S_{2i,2j} &= (D_{33}\beta I_s^{i,j} + D_{22}\alpha I_t^{i,j}) + D_{23}(I_{st}^{i,j} + I_{st}^{j,i}) \end{aligned} \qquad (10)$$

h = element thickness,

Table VI
Numerical values of $I_{st}^{i,j}(i,j = 1,2,3,\ldots,12)$ (Cubic element)

i/j	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{17}{32}$	$\frac{69}{320}$	$\frac{-93}{320}$	$\frac{7}{160}$	$\frac{-69}{320}$	$\frac{93}{320}$	$\frac{-19}{160}$	$\frac{93}{320}$	$\frac{-69}{320}$	$\frac{-7}{160}$	$\frac{3}{320}$	$\frac{-159}{320}$
2	$\frac{-159}{320}$	0	$\frac{81}{160}$	$\frac{-3}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{93}{320}$	$\frac{-81}{160}$	0	$\frac{69}{320}$	$\frac{9}{64}$	$\frac{9}{64}$
3	$\frac{3}{320}$	$\frac{-81}{160}$	0	$\frac{159}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{-69}{320}$	0	$\frac{81}{160}$	$\frac{-93}{320}$	$\frac{9}{64}$	$\frac{9}{64}$
4	$\frac{-7}{160}$	$\frac{93}{320}$	$\frac{-69}{320}$	$\frac{-17}{32}$	$\frac{159}{320}$	$\frac{-3}{320}$	$\frac{7}{160}$	$\frac{69}{320}$	$\frac{-93}{320}$	$\frac{19}{160}$	$\frac{-93}{320}$	$\frac{69}{320}$
5	$\frac{-69}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{-69}{320}$	0	$\frac{81}{160}$	$\frac{-93}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{-93}{320}$	$\frac{81}{160}$	0
6	$\frac{93}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{93}{320}$	$\frac{-81}{160}$	0	$\frac{69}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{69}{320}$	0	$\frac{-81}{160}$
7	$\frac{-19}{160}$	$\frac{93}{320}$	$\frac{-69}{320}$	$\frac{-7}{160}$	$\frac{3}{320}$	$\frac{-159}{320}$	$\frac{17}{32}$	$\frac{69}{320}$	$\frac{-93}{320}$	$\frac{7}{160}$	$\frac{-69}{320}$	$\frac{93}{320}$
8	$\frac{93}{320}$	$\frac{-81}{160}$	0	$\frac{69}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{-159}{320}$	0	$\frac{81}{160}$	$\frac{-3}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$
9	$\frac{-69}{320}$	0	$\frac{81}{160}$	$\frac{-93}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{3}{320}$	$\frac{-81}{160}$	0	$\frac{159}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$
10	$\frac{7}{160}$	$\frac{69}{320}$	$\frac{-93}{320}$	$\frac{19}{160}$	$\frac{-93}{320}$	$\frac{69}{320}$	$\frac{-7}{160}$	$\frac{93}{320}$	$\frac{-69}{320}$	$\frac{-17}{32}$	$\frac{159}{320}$	$\frac{-3}{320}$
11	$\frac{-93}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{-93}{320}$	$\frac{81}{160}$	0	$\frac{-69}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{-69}{320}$	0	$\frac{81}{160}$
12	$\frac{69}{320}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{69}{320}$	0	$\frac{-81}{160}$	$\frac{93}{320}$	$\frac{-9}{64}$	$\frac{-9}{64}$	$\frac{93}{320}$	$\frac{-81}{160}$	0

and $D_{11}, D_{12}, D_{13}, D_{22}, D_{23}, D_{33}$ are the coefficients of the elasticity matrix $[D]$ given by:

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{Symmetric} & & D_{33} \end{bmatrix} \tag{11}$$

5. Conclusions

The closed form integration of integrals $W_{mn}^{i,j}$ could be performed easily in the present case as the determinant of the Jacobian matrix is constant ($= ab$). In this context, it should be noted that the determinant of the Jacobian matrix is a function of local coordinates for nonparallelogram-shaped elements and in that case the standard alternative is to resort to the approximate and costly numerical integration procedures. Even though the numerical values of integrals $I_s^{i,j}$, $I_t^{i,j}$ and $I_{st}^{i,j}$ described in Tables I–VI are applied in the present paper with reference to linear plane elasticity problems, it is easy to conceive their applicability in other areas of finite element analysis.

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