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Physiological-type flow in a circular pipe in the presence of a transverse magnetic field

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Abstract

A physiological-type flow in a straight circular tube in the presence of a uniform transverse magnetic field is investigated. The pressure gradient is mathematically modelled by a Fourier series of which the second partial sum resembles very closely with the observed form of pressure gradient. The governing equations are solved exactly by using the method of Laplace transforms. It is found that the effect of a magnetic field is to reduce the magnitude of velocity near the central region of the pipe. The coefficients of amplitude and phase lag of the mean velocity are also reduced in the presence of a magnetic field.

Key words: Physiological-type flow, transverse magnetic field.

1. Introduction

The problems of pulsatile flow through pipes have received much attention in literature due to their importance in blood-flow research. Womersley[1] has obtained an exact solution for a purely oscillatory flow in a rigid tube and brought out the importance of such flows in physiology. An exact solution of pulsating laminar flow superposed on the steady motion in a rigid circular pipe has been presented by Uchida[2]. Chandran *et al*[3], Chandran and Yearwood[4], and Yearwood and Chandran[5] have studied experimentally the physiological-type flow in a curved tube. The effect of unsteadiness on the flow has been investigated experimentally by Siouffi *et al*[6] using simple pulsatile- and physiological-type flows in tubes with stenosis and bifurcations.

The application of magnetohydrodynamic principles in medicine and engineering is of growing interest. In the investigations reported by Barnothy[7], it was observed that the biological systems, in general, are greatly affected by the application of an external magnetic field. Korchevskii and Marochnik[8] suggested the possibility of regulating movement of blood by the application of an external magnetic field. By modelling the blood vessel by a cylindrical tube of uniform cross-section with non-conducting walls, Vardanyan[9] showed that the application of magnetic field reduces the speed of blood flow.

An exact solution for the steady one-dimensional flow of an electrically conducting fluid through a circular pipe with non-conducting walls in the presence of a uniform transverse magnetic field has been obtained by Gold[10]. The corresponding unsteady problem with an exponentially decaying pressure gradient has been studied by Gupta and Bani Singh[11] for the special case of the Reynolds Number equal to magnetic Reynolds Number. Deshikachar and Ramachandra Rao[12] have studied the effect of a transverse magnetic field on the steady flow of blood in a channel of variable cross-section and the corresponding oscillatory flow has been investigated by Ramachandra Rao and Deshikachar[13]. They have highlighted the importance of magnetic field on the separation in the flow and the steady-streaming phenomena. It is well known that the flows in tubes of variable cross-section in the presence of an applied magnetic field are important in some biological flows. McMichael and Deutsch[14] have considered the steady flow in a circular tube of variable cross-section in the presence of a magnetic field in the axial direction of the tube and the corresponding oscillatory flow has been investigated by Deshikachar and Ramachandra Rao[15]. It would be interesting to study the effects of transverse magnetic field for the flows in tubes of uniform or non-uniform cross-section.

In the present paper, as a first attempt we have considered the effect of transverse magnetic field on a physiological-type flow in a uniform circular pipe. The form of the pressure gradient or the volume flux for the physiological-type flow presented by Siouffi *et al*[6] is mathematically modelled by a Fourier series and it is observed that the graph of the second partial sum of this series resembles very closely with the form of the volume flux used by them[6]. The governing equations are solved exactly by using the method of Laplace transforms. It is found that the effect of magnetic field is to reduce the velocity significantly near the centre of the pipe. The phase lag and amplitude of mean velocity are also reduced in the presence of a magnetic field.

2. Formulation of the problem

Consider a physiological-type of flow of an incompressible electrically conducting viscous fluid in a circular pipe in cylindrical polar coordinates (r', θ', z') with the axis of the pipe coinciding with the z'-axis. A uniform transverse magnetic field H_0 is applied perpendicular to z'-axis. The equations governing the unsteady flow under the usual magnetohydrodynamic approximations are[16]:

$$\rho'\left[\frac{\partial \bar{v}'}{\partial t'} + (\bar{v}'\cdot\nabla')\bar{v}'\right] = -\nabla' p' + \eta\nabla'^2 \bar{v}' + \mu(\bar{J}'\times\bar{H}'), \tag{1}$$

$$\nabla'\times\bar{E}' = -\mu\frac{\partial \bar{H}'}{\partial t'}, \tag{2}$$

$$\nabla' \times \bar{H}' = \bar{J}', \tag{3}$$

$$\bar{J}' = \sigma [\bar{E}' + \mu (\bar{v}' \times \bar{H}')], \tag{4}$$

$$\nabla' \cdot \bar{v}' = 0 = \nabla' \cdot \bar{H}', \tag{5}$$

where $\bar{v}'$ is the velocity, p' the pressure, ρ' the density, η coefficent of viscosity, $\bar{H}'$ the magnetic field, $\bar{E}'$ the electric field, $\bar{J}'$ the current density, σ the electrical conductivity of the fluid, μ the magnetic permeability, ∇' the del operator and ∇'^2 the Laplacian operator in cylindrical polar coordinates. Using (3)–(5), equations (1) and (2) can be reduced to the following equations:

$$\rho' \left[\frac{\partial \bar{v}'}{\partial t} + (\bar{v}' \cdot \nabla') \bar{v}' \right] - \mu (\bar{H}' \cdot \nabla') \bar{H}' = -\nabla' P' + \eta \nabla'^2 \bar{v}', \tag{6}$$

$$\mu\sigma \nabla' \times (\bar{v}' \times \bar{H}') + \nabla'^2 \bar{H}' = \mu\sigma \frac{\partial \bar{H}'}{\partial t'}, \tag{7}$$

where $P' = p' + |\bar{H}'^2|/2$. Assuming the motion to be purely axial, the velocity and magnetic fields are taken as

$$\bar{v}' = [0, 0, v_z(r', \theta', t')], \tag{8}$$

$$\bar{H}' = [H_0 \cos\theta', \ -H_0 \sin\theta', \ H'_z(r', \theta', t^1)]. \tag{9}$$

Substitution of (8) and (9) in (6) and (7) gives

$$\frac{\partial v_z}{\partial t'} = -\frac{1}{\rho'} \frac{\partial P'}{\partial z'} + \nu \nabla'^2 v'_z + \frac{\mu H_0}{\rho'} \left(\cos\theta' \frac{\partial H'_z}{\partial r'} - \frac{\sin\theta'}{r'} \frac{\partial H'_z}{\partial \theta'} \right) \tag{10}$$

$$\frac{\partial P'}{\partial r'} = 0 = \frac{\partial P'}{\partial \theta'}. \tag{11}$$

$$\mu\sigma \frac{\partial H'_z}{\partial t'} = \nabla'^2 H'_z + \mu\sigma H_0 \left(\cos\theta' \frac{\partial v'_z}{\partial r'} - \frac{\sin\theta'}{r'} \frac{\partial v'_z}{\partial \theta'} \right), \tag{12}$$

where

$$\nabla'^2 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} + \frac{1}{r'^2} \frac{\partial^2}{\partial \theta'^2}. \tag{13}$$

Equations (10) and (11) imply that the pressure gradient is a function of time alone.

Introducing the non-dimensional variables, defined by

$$v = v'_z/V_0,\ h = H'_z/H_0,\ \rho = r'/a,\ t = \omega t',\ \theta' = \theta, \tag{14}$$

where a is the radius of the tube and V_0 the characteristic velocity in equations (10) and (12) with the value of pressure gradient given in Appendix I, we obtain

$$\nabla^2 v + (M^2/Rm)\left(\frac{\sin\theta}{\rho}\frac{\partial h}{\partial\theta} - \cos\theta\frac{\partial h}{\partial\rho}\right) = \sum_{n=0}^{\infty} k_n e^{int} + \alpha^2\frac{\partial v}{\partial t}, \tag{15}$$

$$\nabla^2 h - Rm\left(\frac{\sin\theta}{\rho}\frac{\partial v}{\partial\theta} - \cos\theta\frac{\partial v}{\partial\rho}\right) = \alpha^2\eta_1\frac{\partial h}{\partial t}, \tag{16}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}, \tag{17}$$

$M^2 = \mu^2 H_0^2 a^2(\sigma/\rho' v)$ (Hartmann Number),

$Rm = \mu\sigma a V_0$ (Magnetic Reynolds Number),

$\alpha^2 = \omega a^2/\nu$ (Womersley parameter),

$\eta_1 = \nu/\kappa$ is the ratio of viscous diffusivity to magnetic diffusivity, $\kappa = 1/\mu\sigma$, and

$k_n = a^2 P_n/\eta V_0$ is a non-dimensional number.

In general, it is difficult to obtain the exact solutions of equations (15) and (16). Approximate solution of this problem for small Hartmann Number has been presented by Gupta[17]. In what follows an exact solution for a special case in which $\eta_1 = 1$, which implies that the Reynolds Number ($R = aV_0/\nu$) and the magnetic Reynolds Number are the same, is obtained. The limitations inherent to this equality have been discussed by Gupta and Bani Singh[11].

The initial and boundary conditions on velocity and magnetic fields are

$$v(\rho, \theta, 0) = 0; \quad h(\rho, \theta, 0) = 0, \tag{18}$$

and

$$v(1, \theta, t) = 0; \quad h(1, \theta, t) = 0. \tag{19}$$

The first of the boundary conditions (19) represents the no-slip condition at the wall of the pipe $\rho = 1$ and the second implies that the walls are non-conducting.

3. Solution of the problem

Taking the Laplace transform of equations (15) and (16) with $\eta_1 = 1$ and using the initial conditions (18), we obtain

$$\nabla^2\bar{v} - \frac{M^2}{R}\left(\frac{\sin\theta}{\rho}\frac{\partial\bar{h}}{\partial\theta} - \cos\theta\frac{\partial\bar{h}}{\partial\rho}\right) = \sum_{n=0}^{\infty}\frac{k_n}{(s-in)} + \alpha^2 s\bar{v}, \tag{20}$$

$$\nabla^2 \bar{h} = R\left(\frac{\sin\theta}{\rho}\,\frac{\partial \bar{v}}{\partial \theta} - \cos\theta \frac{\partial v}{\partial \rho}\right) = \alpha^2 s \bar{h}, \tag{21}$$

where

$$\bar{v} = \int_0^\infty v \exp(-st)\,dt. \tag{22}$$

Using the transformations

$$\phi = \bar{v} + \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)} + \frac{M}{R}\bar{h}, \tag{23}$$

$$\psi = \bar{v} + \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)} - \frac{M}{R}\bar{h}, \tag{24}$$

in equations (20) and (21) and simplifying, we obtain

$$\nabla^2 \phi - M\left(\frac{\sin\theta}{\rho}\,\frac{\partial \phi}{\partial \theta} - \cos\theta \frac{\partial \phi}{\partial \rho}\right) = \alpha^2 s \phi, \tag{25}$$

$$\nabla^2 \psi - M\left(\frac{\sin\theta}{\rho}\,\frac{\partial \psi}{\partial \theta} - \cos\theta \frac{\partial \psi}{\partial \rho}\right) = \alpha^2 s \psi. \tag{26}$$

Again using the transformations

$$f(\rho, \theta) = e^{\beta\rho\cos\theta}\phi, \tag{27}$$

$$g(\rho, \theta) = e^{-\beta\rho\cos\theta}\psi, \tag{28}$$

equations (25) and (26) are reduced to

$$\nabla^2 f = \gamma^2 f, \tag{29}$$

$$\nabla^2 g = \gamma^2 g, \tag{30}$$

where

$$\gamma^2 = \beta^2 + \alpha^2 s, \quad M = 2\beta. \tag{31}$$

The boundary conditions for f and g are derived from (27) and (28) using (19), (23) and (24) and are given by

$$f(1, \theta) = e^{\beta\cos\theta} \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)},$$

$$g(1, \theta) = e^{-\beta\cos\theta} \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}. \tag{32}$$

The solutions of equations (29) and (30) satisfying the boundary conditions (32) are given by

$$f(\rho, \theta) = \sum_{m=0}^{\infty} A_m I_m(\rho\gamma)\cos m\theta, \tag{33}$$

and

$$g(\rho, \theta) = \sum_{m=0}^{\infty} P_m I_m(\rho\gamma)\cos m\theta, \tag{34}$$

where $I_m(\rho\gamma)$ are modified Bessel functions of the first kind and order m. The coefficients A_m and B_m are determined using the boundary conditions (32) and are given by

$$A_m = \left[\sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}\right] \varepsilon_m \frac{I_m(\beta)}{I_m(\gamma)}, \quad m = 0, 1, 2, \ldots, \tag{35}$$

$$B_m = (-1)^m A_m, \quad m = 0, 1, 2, \ldots, \tag{36}$$

where

$$\begin{aligned} \varepsilon_m &= 1, \ m = 0 \\ &= 2, \ m > 0. \end{aligned} \tag{37}$$

Using (33) and (34) in (27) and (28), we obtain ϕ and ψ and in turn from (23) and (24), we get

$$\bar{v} + \frac{M}{R}\bar{h} = e^{-\beta\rho\cos\theta}\left[\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}\right)\cdot \varepsilon_m \frac{I_m(\beta)}{I_m(\gamma)} \cdot I_m(\rho\gamma)\cos m\theta - \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}\right], \tag{38}$$

and

$$\bar{v} - \frac{M}{R}\bar{h} = e^{\beta\rho\cos\theta}\left[\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}\right) \varepsilon_m \frac{I_m(\beta)}{I_m(\gamma)} I_m(\rho\gamma) \cdot \cos m\theta - \sum_{n=0}^{\infty} \frac{k_n}{\alpha^2 s(s-in)}\right]. \tag{39}$$

Evaluating the integrals of the inverse Laplace transforms of (38) and (39) by contour integration methods, we get

$$v+\frac{M}{R}h=-\sum_{n=0}^{\infty}\frac{ik_n}{\alpha^2 n}\ (1-e^{int})+e^{-\beta\rho\cos\theta}$$

$$\cdot\left[\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{ik_n}{\alpha^2 n}\left(1-\frac{I_m[\rho(\beta^2+i\alpha^2 n)^{1/2}]}{2I_m[(\beta^2+i\alpha^2 n)^{1/2}]}\right)e^{int}\right.\right.$$

$$\left.-\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}\frac{2k_n\xi_{mr}J_m(\xi_{mr})\exp[-(\xi_{mr}^2+\beta^2)/\alpha^2]t}{J'(\xi_{mr})(\beta^2+\xi_{mr}^2)(\beta^2+\xi_{mr}^2+in)}\right\}$$

$$\left.\cdot\varepsilon_m I_m(\beta)]\cos m\theta\right],\tag{40}$$

and

$$v-\frac{M}{R}h=-\sum_{n=0}^{\infty}\frac{ik_n}{\alpha^2 n}(1-e^{int})+e^{\beta\rho\cos\theta}$$

$$\cdot\left[\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{ik_n}{\alpha^2 n}\left(1-\frac{I_m[\rho(\beta^2+i\alpha^2 n)^{1/2}]}{2I_m[(\beta^2+i\alpha^2 n)^{1/2}]}\right)e^{int}\right.\right.$$

$$\left.\cdot-\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}\frac{2k_nJ_m(\xi_{mr})}{J'_m(\xi_{mr})}\ \frac{\exp[-(\beta^2+\xi_{mr}^2)/\alpha^2]t}{(\beta^2+\xi_{mr}^2)(\beta^2+\xi_{mr}^2+in)}\right\}$$

$$\left.\cdot(-1)^m\varepsilon_m I_m(\beta)\cos m\theta\right],\tag{41}$$

where ξ_{mr} are the roots of $I_m(\gamma)=0$. Now using the results

$$\begin{aligned}&\sum_{m=0}^{\infty}\ \varepsilon_m I_m(\rho\beta)\cos m\theta=e^{\beta\rho\cos\theta},\\&\sum_{m=0}^{\infty}\ (-1)^m\varepsilon_m I_m(\rho\beta)\cos m\theta=e^{-\beta\rho\cos\theta},\end{aligned}\tag{42}$$

and solving (40) and (41) for v and h, we obtain

$$v=\sum_{n=0}^{\infty}\frac{ik_n}{\alpha^2 n}e^{int}-\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{ik_nI_m[\rho(\beta^2+i\alpha^2 n)^{1/2}]}{2\alpha^3 n\,I_m[(\beta^2+i\alpha^2 n)^{1/2}]}e^{int}\right.$$

$$\left.-\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}\frac{2k_n\xi_{mr}J_m(\xi_{mr})\exp[-(\beta^2+\xi_{mr}^2)/\alpha^2]t}{J'_m(\xi_{mr})(\beta^2+\xi_{mr}^2)(\beta^2+\xi_{mr}^2+in)}\right\}$$

$$\cdot\varepsilon_m I_m(\beta)\cos m\theta\,[\exp(-\beta\rho\cos\theta)+(-1)^m\exp(\beta\rho\cos m\theta)]\tag{43}$$

and

$$h=\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{ik_nR}{4\alpha^2n\beta}\frac{I_m[\rho(\beta^2+i\alpha^2n)^{1/2}]}{I_m[(\beta^2+i\alpha^2n)^{1/2}]}e^{int}\right.$$
$$\left.-\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}\frac{k_nR\,\xi_{mr}J_m(\xi_{mr})\exp[-(\beta^2+\xi_{mr}^2)/\alpha^2]t}{2\beta J_m'(\xi_{mr})(\beta^2+\xi_{mr}^2)(\beta^2+\xi_{mr}^2+in)}\right\}$$
$$\cdot\varepsilon_mI_m(\beta)\cos m\theta\,[\exp(-\beta\rho\cos\theta)-(-1)^m\exp(\beta\rho\cos m\theta)]. \tag{44}$$

For large times, *i.e.*, $t\to\infty$, we obtain the solution due to steady-oscillating flow and it is given by

$$v=-\frac{k_0}{4\beta}\sum_{m=0}^{\infty}\varepsilon_m\frac{I_m'(\beta)}{I_m(\beta)}I_m(\rho\beta)\cos m\theta\,[\exp(-\beta\rho\cos\theta)$$
$$+(-1)^m\exp(\beta\rho\cos\theta)]+\sum_{n=1}^{\infty}\frac{ik_n}{\alpha^2n}e^{int}$$
$$\left\{1-\sum_{m=0}^{\infty}\frac{I_m[\rho(\beta^2+i\alpha^2n)^{1/2}]}{2I_m[(\beta^2+i\alpha^2n)^{1/2}]}\varepsilon_mI_m(\beta)\cos m\theta\right.$$
$$\left.\cdot[\exp(-\beta\rho\cos\theta)+(-1)^m\exp(\beta\rho\cos\theta)]\right\}. \tag{45}$$

$$h=-\frac{k_0R}{8\beta^2}\sum_{m=0}^{\infty}\left\{\varepsilon_m\frac{I_m'(\beta)}{I_m(\beta)}I_m(\beta\rho)\cos m\theta\right.$$
$$\left.\cdot[\exp(-\beta\rho\cos\theta)-(-1)^m\exp(\beta\rho\cos\theta)]\right\}-2\rho\cos\theta$$
$$+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{ik_nR}{4\alpha^2n\beta}\frac{I_m[\rho(\beta^2+i\alpha^2n)^{1/2}}{I_m[(\beta^2+i\alpha^2n)^{1/2}]}\varepsilon_m\right.$$
$$\cdot I_m(\beta)\cos m\theta\,[\exp(-\beta\rho\cos\theta)-(-1)^m\exp(\beta\rho\cos\theta)]. \tag{46}$$

Taking the limit $n\to0$ in (45) and (46), we obtain the results given by Gold[10]. The results for the non-magnetic case are obtained by taking the limit $\beta\to0$ in (45). In this limit (45) reduces to

$$v=\frac{k_0}{4}(1-\rho^2)-\sum_{n=1}^{\infty}\frac{ik_n}{\alpha^2n}\left[1-\frac{J_0(\alpha n^{1/2}i^{3/2})}{J_0'(\alpha n^{1/2}i^{3/2})}\right]e^{int}. \tag{47}$$

The velocity given in (47) corresponds to the flow due to an oscillatory pressure gradient discussed by Uchida[2].

4. Results and conclusions

Evaluating the expressions for the velocity and magnetic fields presented in (45) and (46) numerically is quite complicated and difficult as they contain double infinite series and modified Bessel functions of different orders with complex-valued arguments. We consider only up to two harmonics in all our numerical calculations as the contribution due to the third and higher harmonics is negligible compared to the first and second harmonics for the small values of α and β, the parameters governing the system. The corresponding pressure gradient takes the form

$$S_2 = P_0 + P_{c1}\cos t + P_{s1}\sin t + P_{c2}\cos 2t + P_{s2}\sin 2t, \tag{48}$$

and is represented in fig. 11. The Ps in (48) are calculated using (A4)−(A6).

The oscillatory part of the velocity field in (45) is computed numerically for $\theta = 0$, $\alpha = 0{\cdot}25$, $\beta = 0{\cdot}8$ and for different ρ is plotted as a function of t in fig. 1. The dotted lines in all the figures correspond to $\beta = 0$, *i.e.*, the non-magnetic case. It is observed that the velocity field as a function of time is similar to the prescribed pressure gradient curve S_2 shown in fig. 11 with a shift in the axis. The effect of magnetic field is to reduce the magnitude of velocity for all times compared with the corresponding non-magnetic case. Figure 2 shows the velocity field for $\theta = 0$, $\alpha = 0{\cdot}25$, $\beta = 0{\cdot}8$ for different values of t as a function of ρ and it is seen that the reduction in velocity due to magnetic effect is more pronounced in the central region of the pipe. The velocity field for $\theta = 0$, $\alpha = 1$, for different values of $\beta = 0$, 2, 3, 4 and for two values of t is depicted in fig. 3. Similar behaviour is observed for other values of t and is not shown in the figure. It is observed that for a fixed Womersley parameter α, the velocity decreases remarkably and the

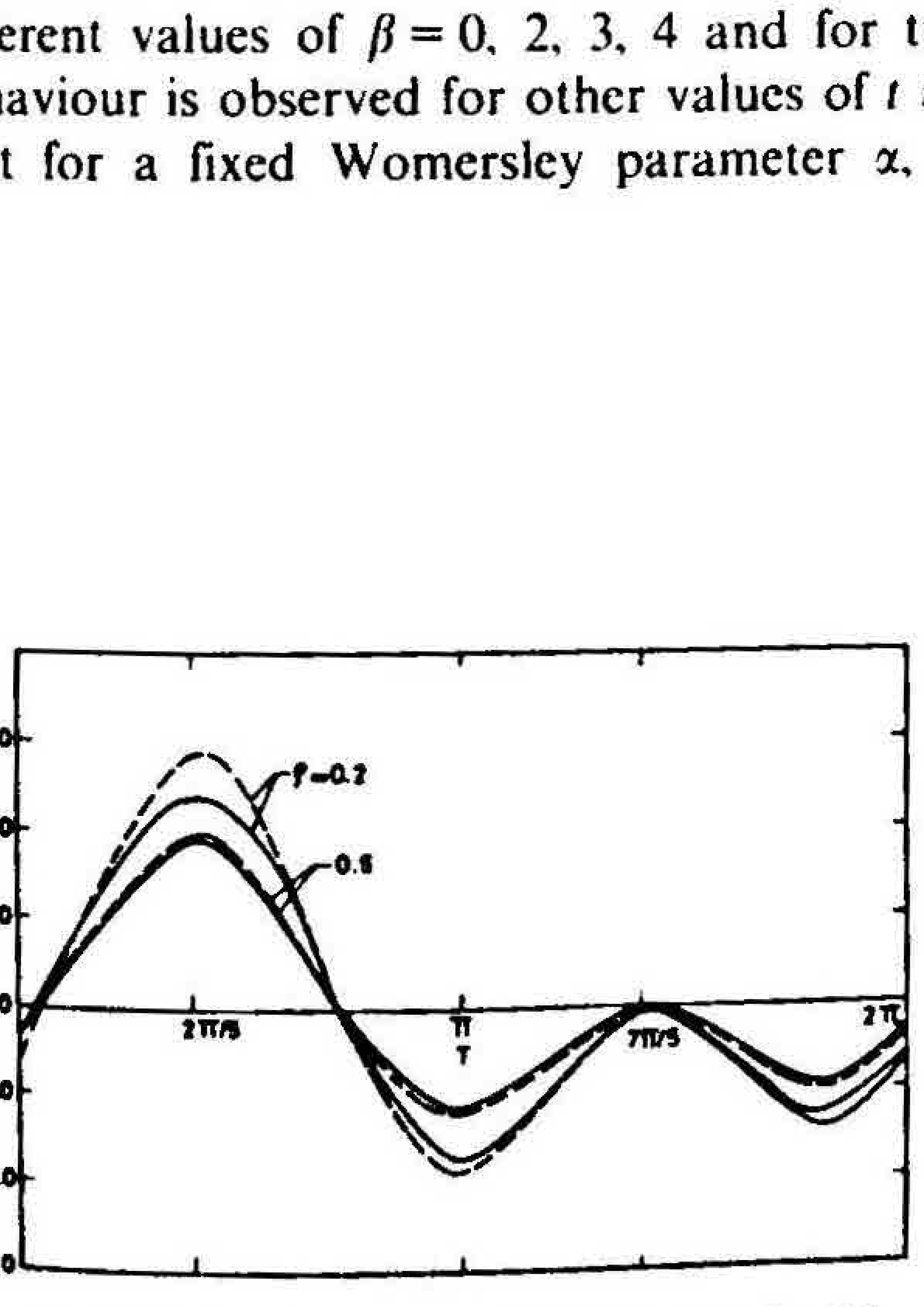

FIG. 1. Velocity field for $\alpha = 0{\cdot}25$, $\beta = 0{\cdot}8$, $\theta = 0{\cdot}0$ for different values of t.

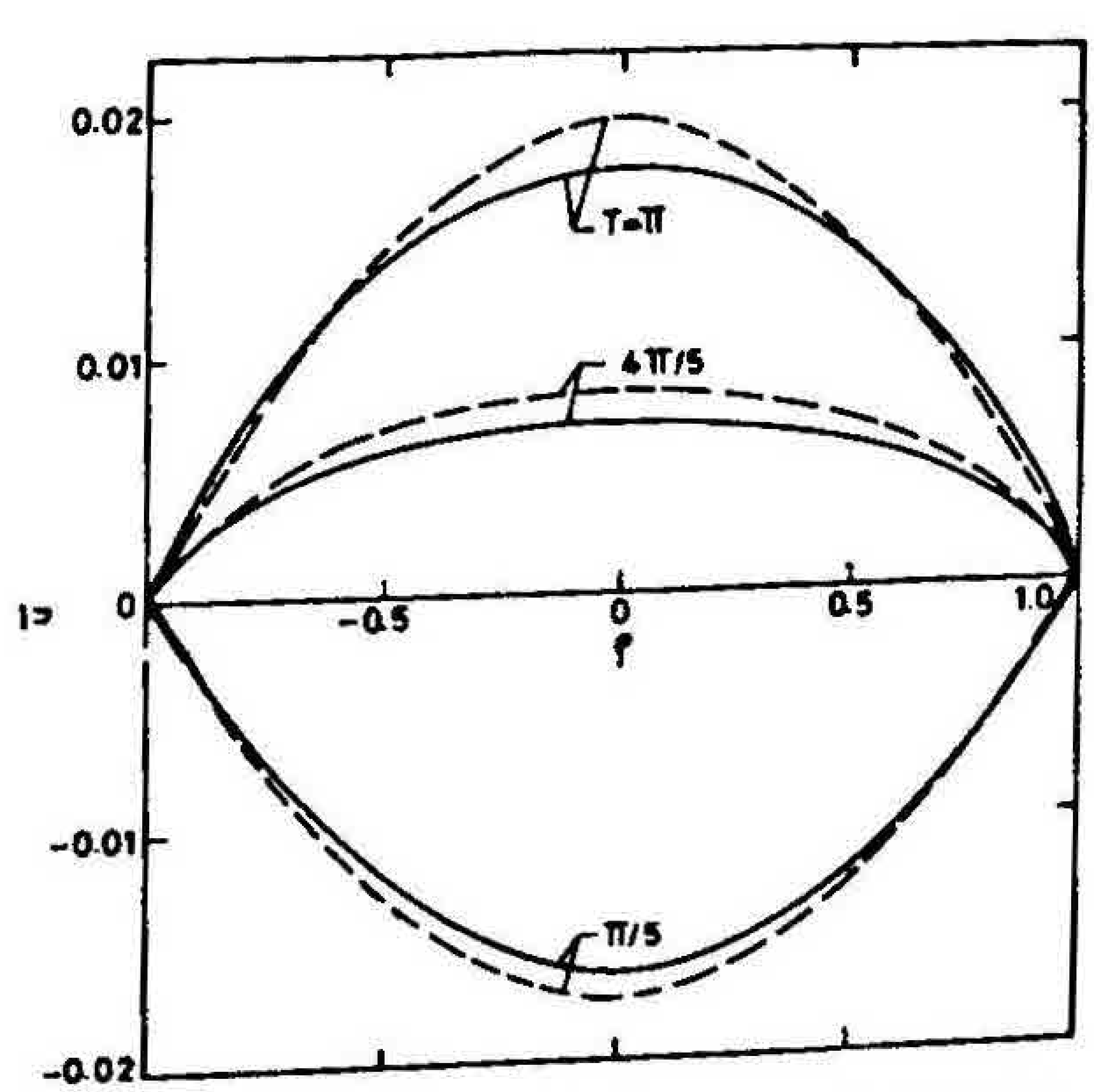

FIG. 2. Velocity field for $\alpha = 1$, $\beta = 1$, $\theta = 0$ for different values of t.

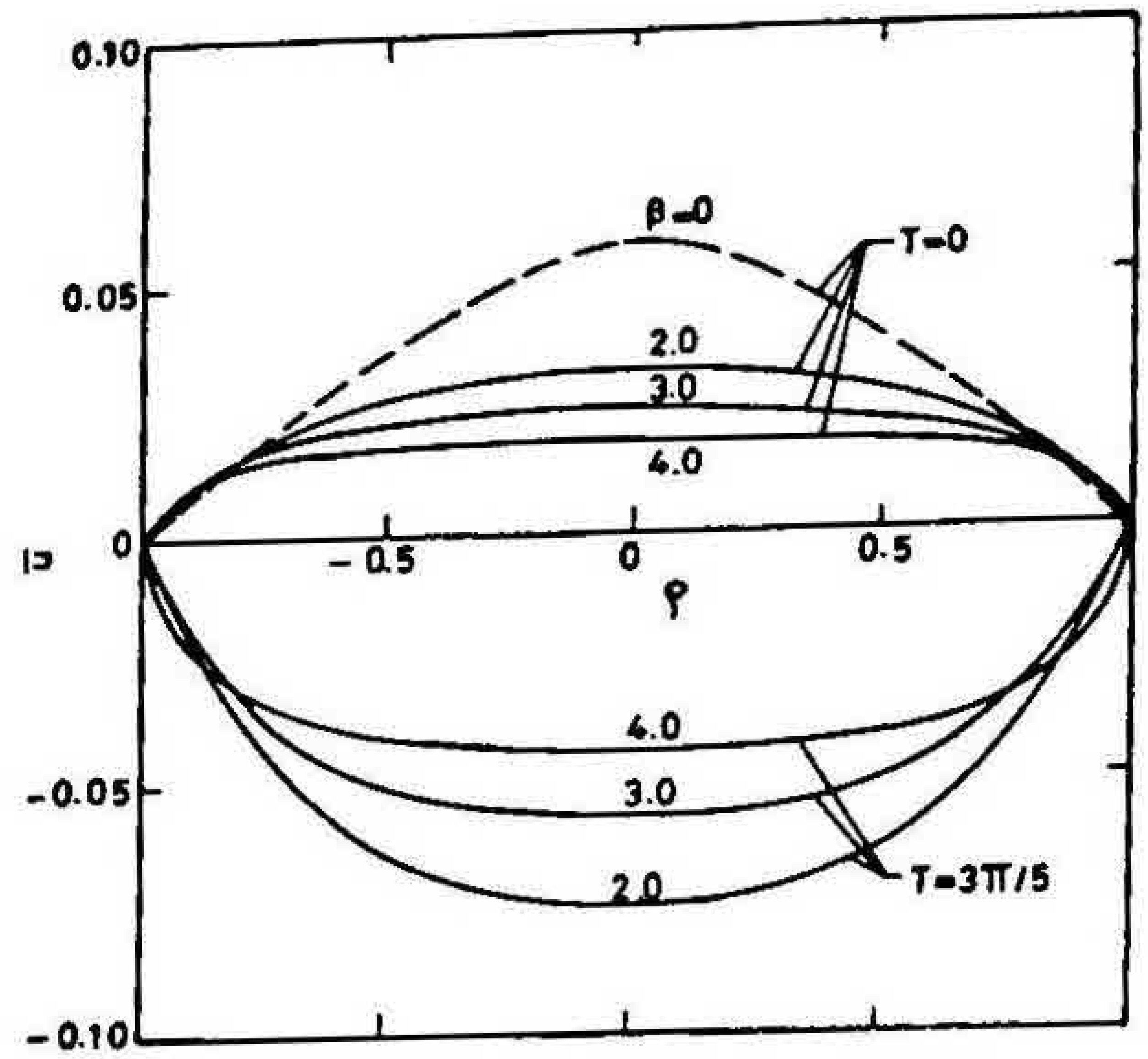

Fig. 3. Velocity field for $\alpha = 1$, $\theta = 0$, $\beta = 0, 2, 3, 4$ for different values of t.

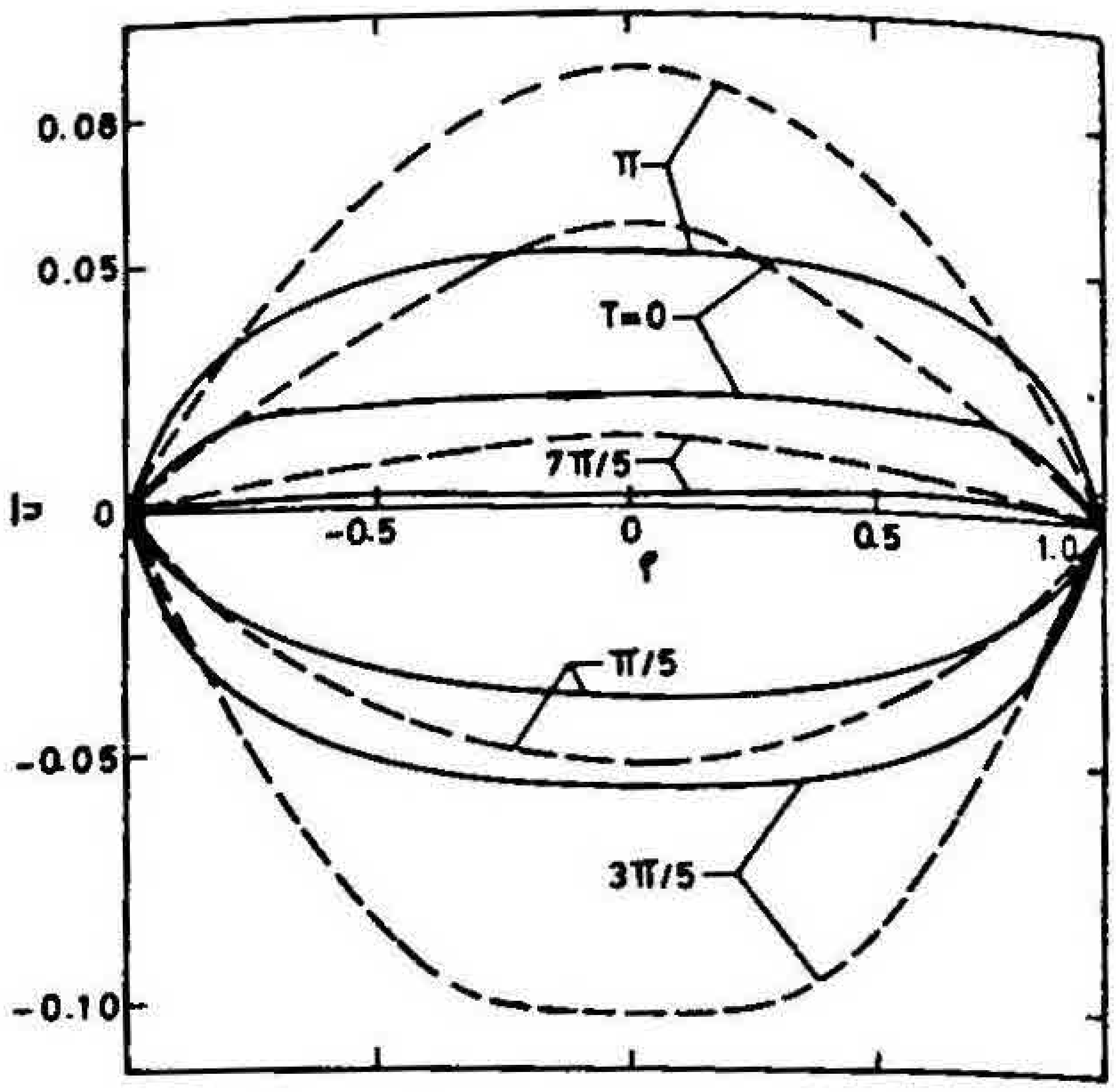

Fig. 4. Velocity field for $\beta = 3$, $\theta = 0$, $\alpha = 1$ for different values of t.

velocity profile becomes more and more flat as the magnetic parameter β increases, as expected, Shercliff[16]. The velocity field for fixed $\beta = 3$, $\theta = 0$, for different values of t is plotted in figs 4 and 5, respectively, for $\alpha = 1$ and $\alpha = 3$ and the flattening of the velocity field is seen as α increases as predicted by Uchida[2]. In order to compare our results with those of non-magnetic case, the velocity field for $\theta = \pi/2$ is depicted in fig. 6 with the other parameters remaining the same as in fig. 4. There is no significant difference in the velocity fields for these values of θ. The induced-magnetic field for $\theta = 0$, $\alpha = 1$ and $\beta = 3$, for different values of t is depicted in fig. 7. It is observed that the induced-magnetic field is more significant near the walls than at the centre.

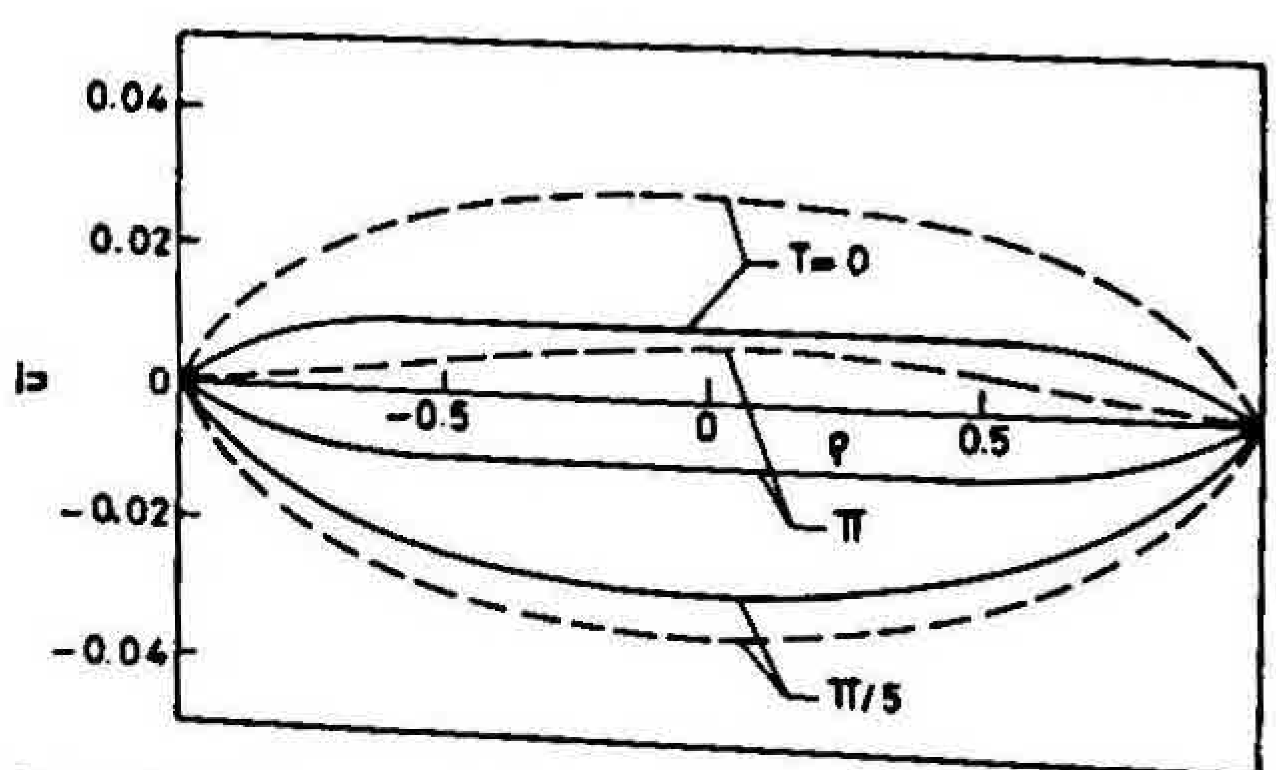

Fig. 5. Velocity field for $\beta = 3$, $\theta = 0$, $\alpha = 3$ for different values of t.

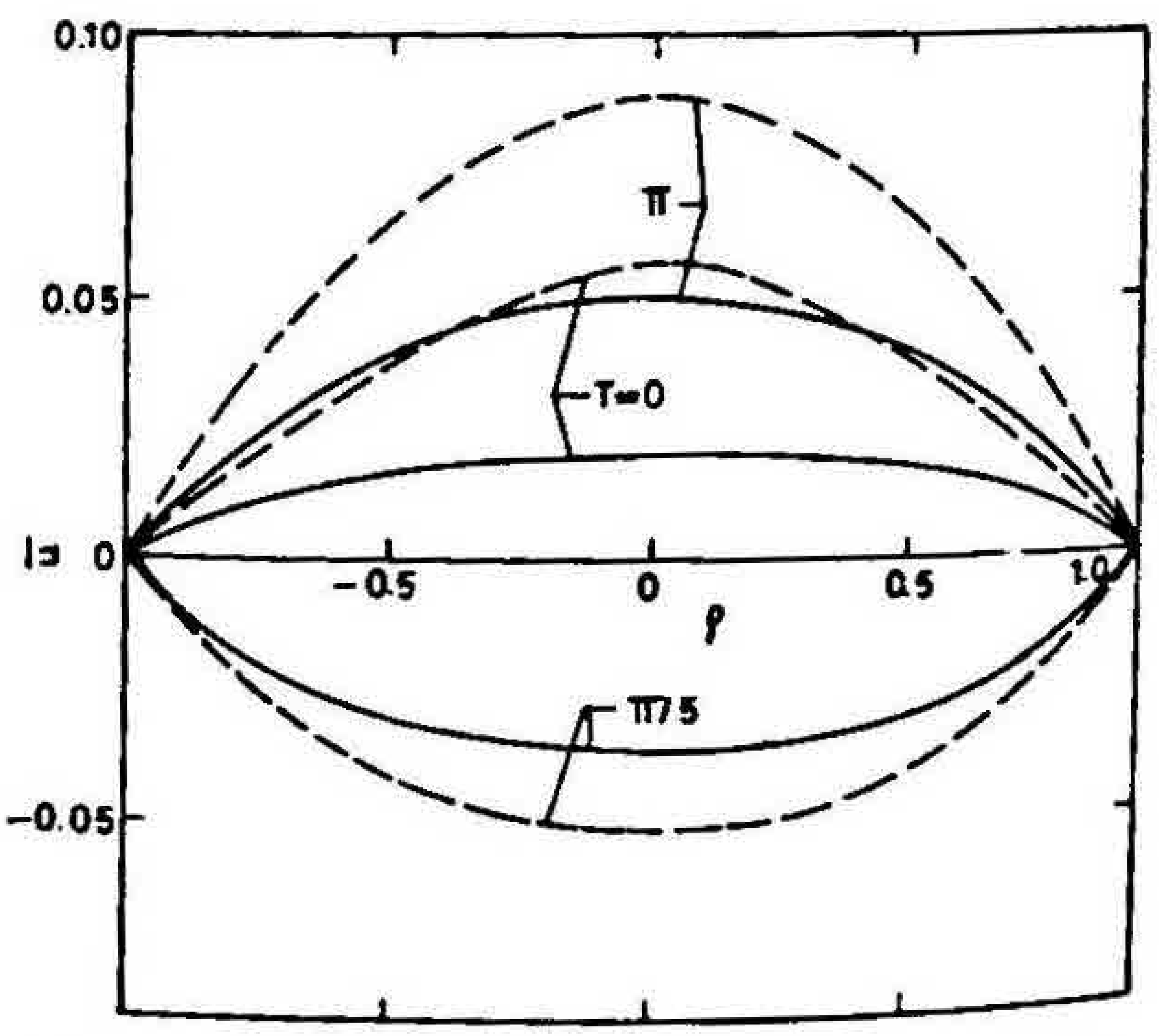

Fig. 6. Velocity field for $\alpha = 1$, $\beta = 3$, $\theta = \pi/2$ for different values of t.

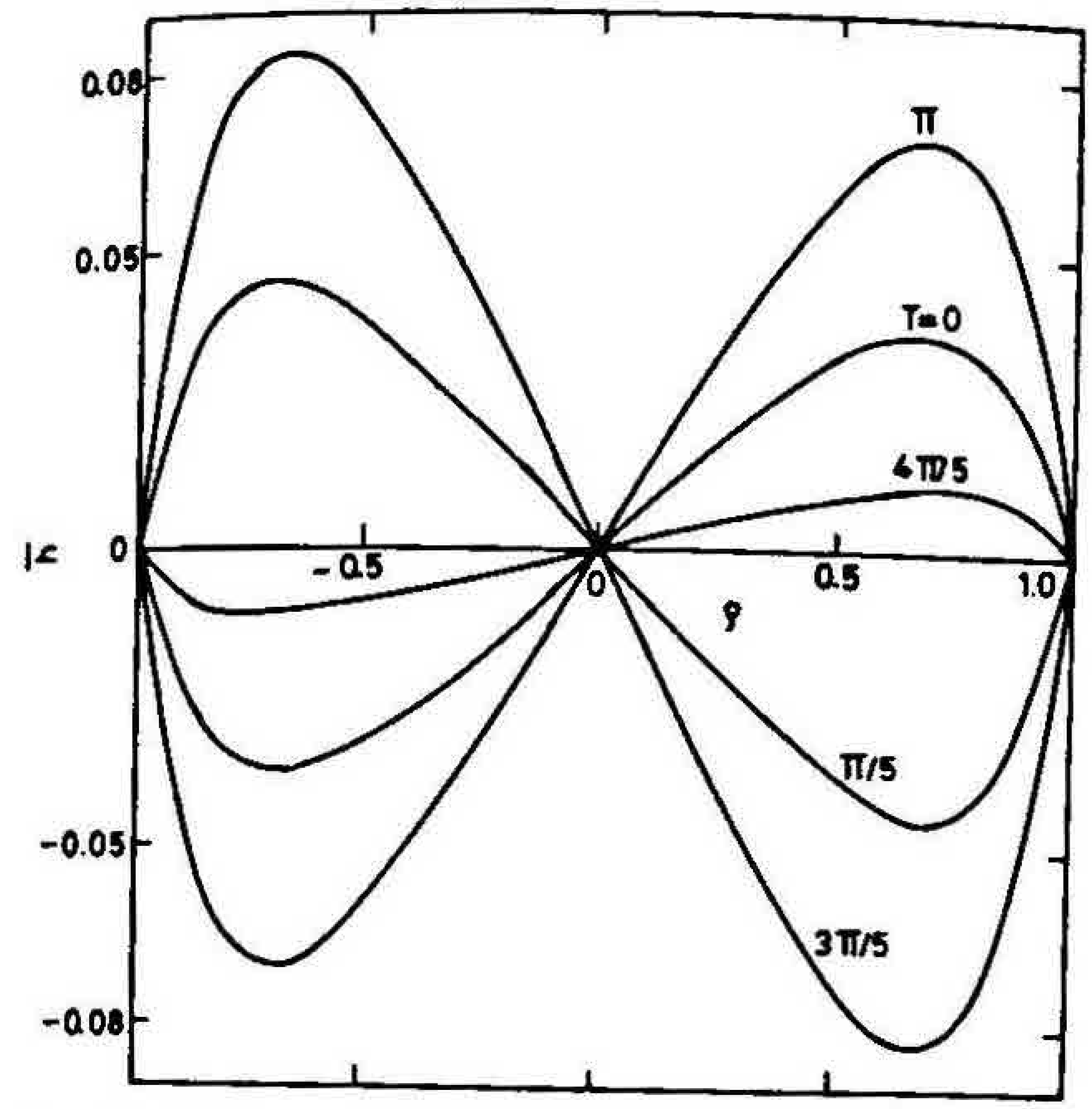

FIG. 7. Induced magnetic field for $\alpha = 1$, $\theta = 0$, $\beta = 3$ for different values of t.

The sectional mean velocity v_m for the flow is given by

$$v_m = \frac{1}{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} v\rho \, d\rho \, d\theta. \tag{49}$$

Using (45) in (49) and simplifying, we obtain

$$v_m = -\frac{k_0}{4\beta} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m \frac{I'_m(\beta)}{I_m(\beta)} \left[\frac{m^2}{\beta^2} I_m^2(\beta) - I_m'^2(\beta) \right] + \sum_{n=1}^{\infty} \frac{ik_n}{\alpha^2 n} e^{int} \left\{ 1 - \sum_{m=0}^{\infty} \frac{2(-1)^m K_m I_m(\beta) I_m(\beta, n)}{I_m[(\beta^2 + i\alpha^2 n)^{1/2}]} \right\}, \tag{50}$$

where

$$I_m(\beta, n) = \int_0^1 I_m[\rho(\beta^2 + i\alpha^2 n)^{1/2}] \, I_m(\rho\beta) \, \rho \, d\rho. \tag{51}$$

If one prescribes the volume flux by a Fourier series instead of pressure gradient, the unknown coefficients of the Fourier series in the pressure gradient could be determined by comparing with the coefficients of the Fourier series representing the volume flux.

For a periodic pressure gradient given in terms of a cosine series ($k_{sn} = 0$ in (A.3)), we have the sectional mean velocity as

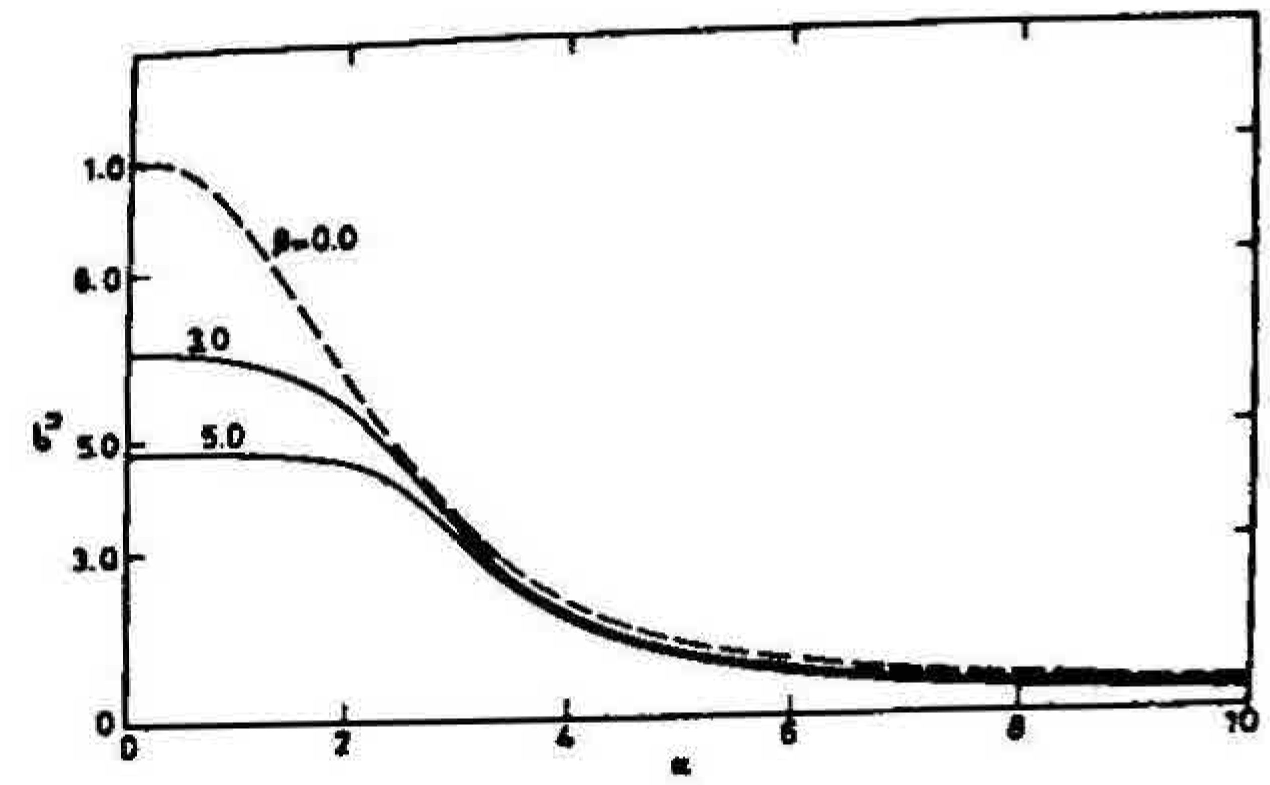

FIG. 8. Coefficient of amplitudes of mean velocity σ_u.

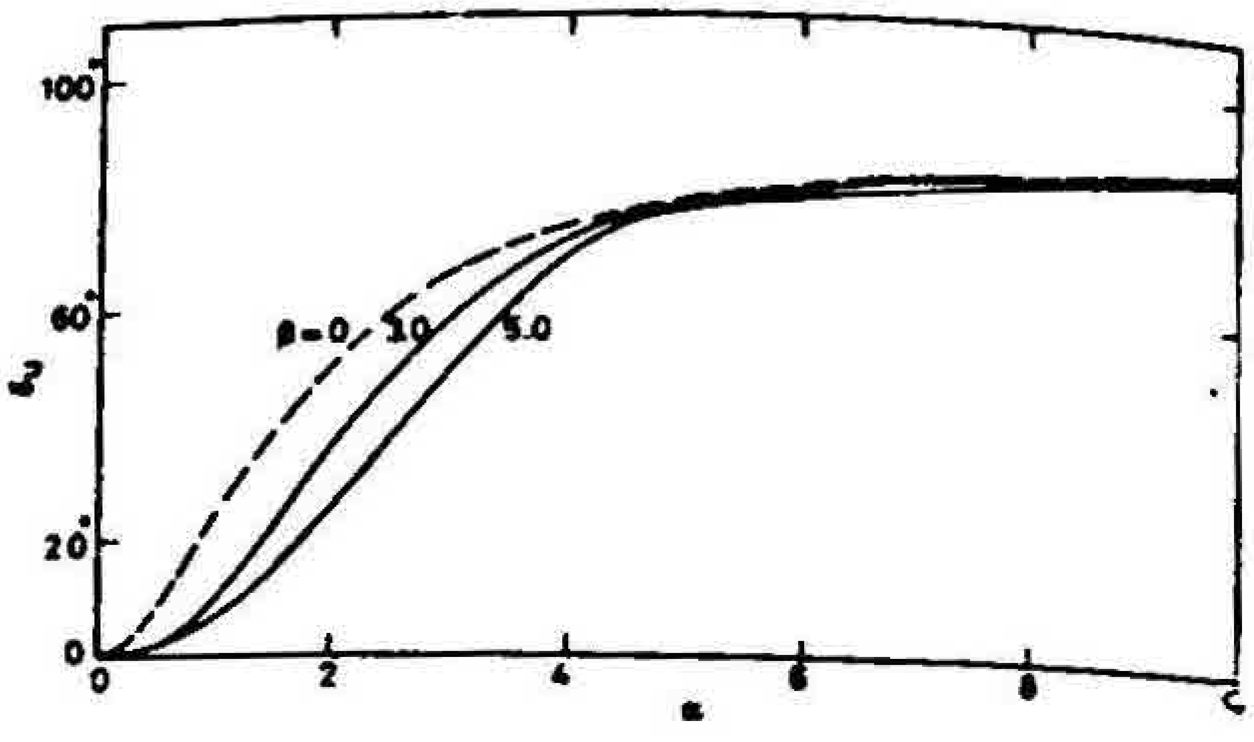

FIG. 9. Coefficient of phase lag of mean velocity δ_u.

$$v_m = -\frac{k_0}{4\beta}\sum_{m=0}^{\infty}(-1)^m \varepsilon_m \frac{I'_m(\beta)}{I_m(\beta)}\left[\frac{m^2}{\beta^2} I_m^2(\beta) - I_m'^2(\beta)\right]$$

$$+\sum_{n=1}^{\infty}\frac{k_{cn}}{\alpha^2 n}\left\{\left(\sum_{m=0}^{\infty} D_m\right)\cos nt + \left(1 - \sum_{m=0}^{\infty} C_m\right)\sin nt\right\}, \tag{52}$$

where C_m and D_m are the real and imaginary parts of

$$\frac{2(-1)^m \varepsilon_m I_m(\beta)\, I_m(\beta, n)}{I_m[(\beta^2 + i\alpha^2 n)^{1/2}]}. \tag{53}$$

Following Uchida[2], the coefficients of the amplitude and the phase lag of the mean velocity in (52), are given by

$$\sigma_u = \left[\left(1 - \sum_{m=0}^{\infty} C_m\right)^2 + \left(\sum_{m=0}^{\infty} D_m\right)^2\right]^{1/2} \Big/ \alpha^2 n, \tag{54}$$

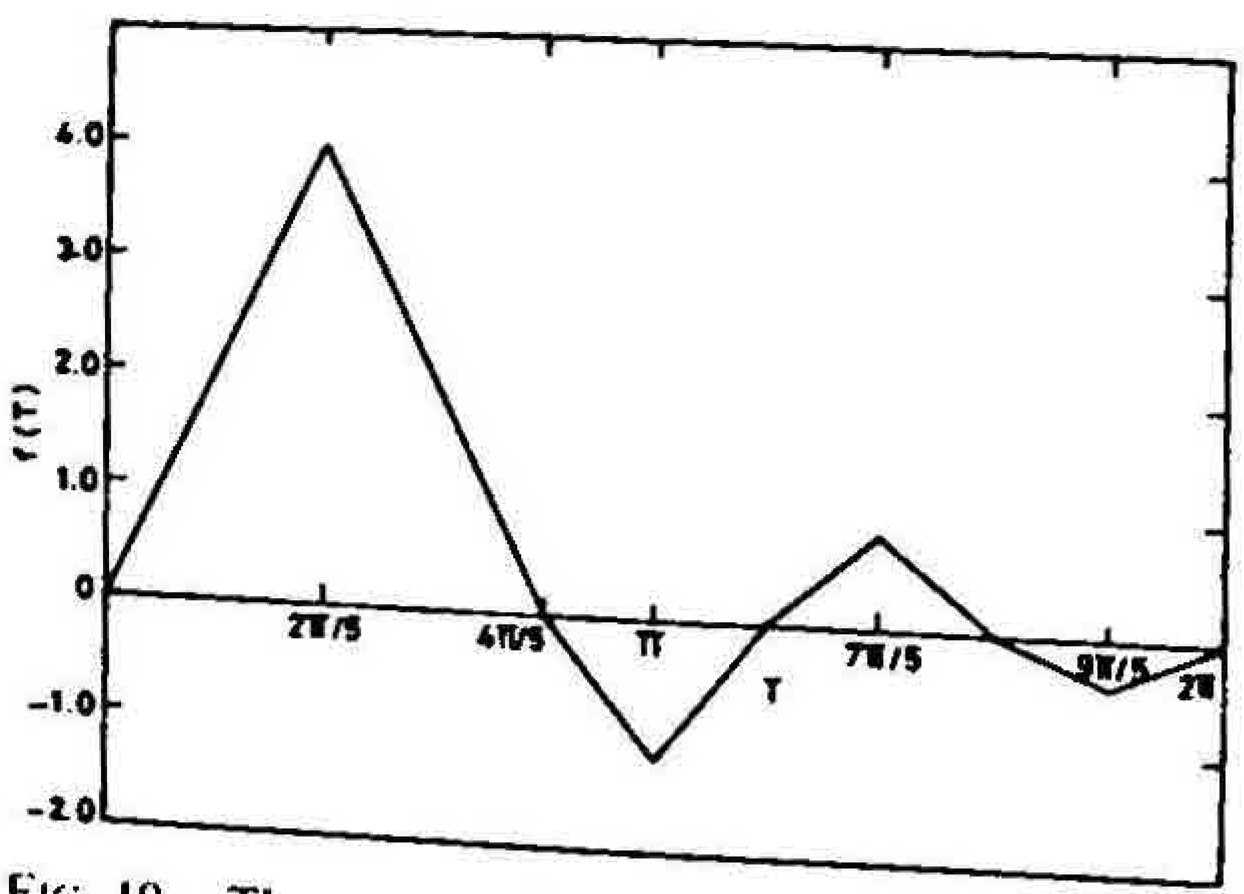

FIG. 10. The pressure gradient curve as defined by $F(t)$ in (A2).

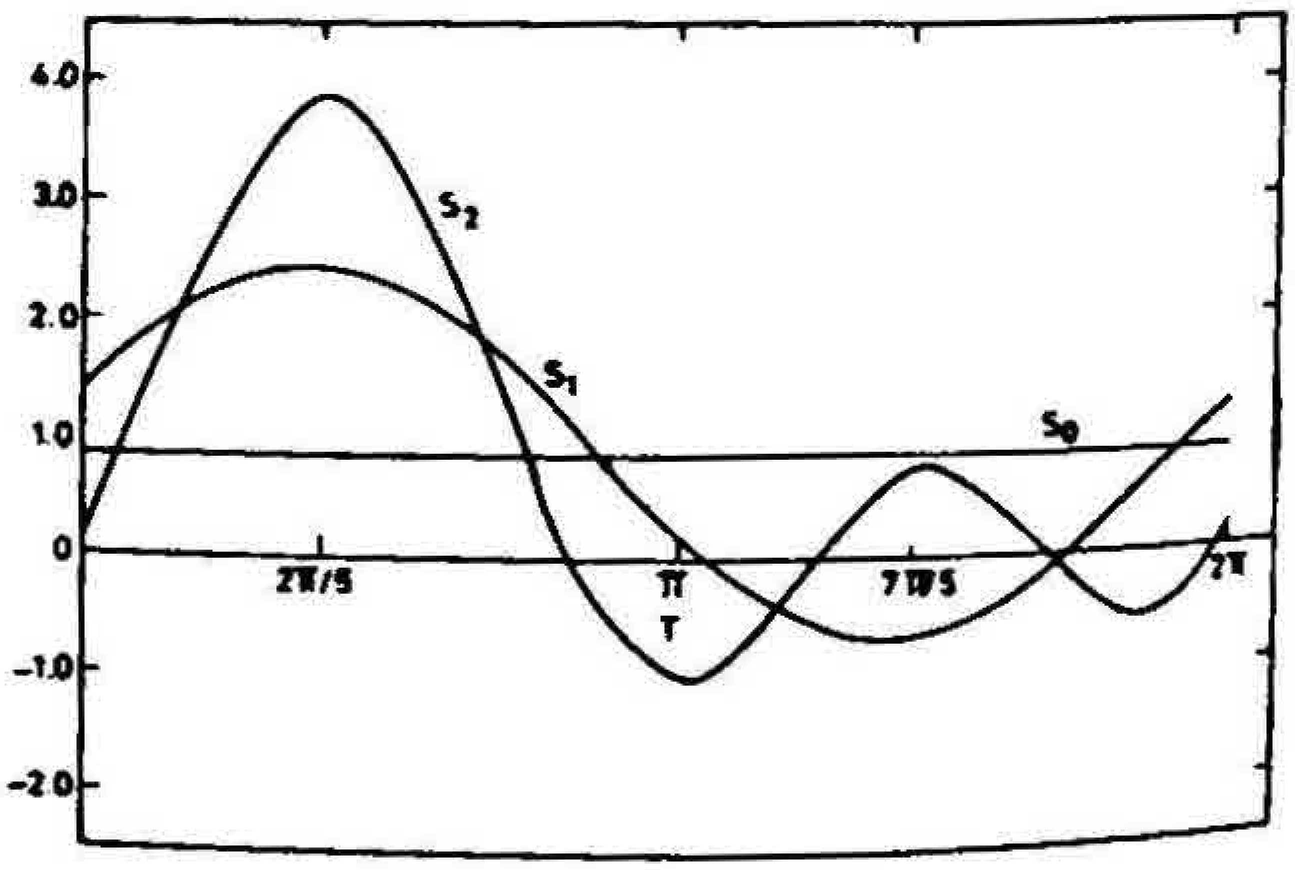

FIG. 11. The partial sums S_0, S_1, S_2 of the Fourier series of $F(t)$.

$$\delta_u = \tan^{-1}\left\{\left(1-\left(\sum_{m=0}^{\infty} C_m\right)\Big/\left(\sum_{m=0}^{\infty} D_m\right)\right)\right\}. \tag{55}$$

The values of σ_u and δ_u are computed numerically from (54) and (55) and are plotted in figs 8 and 9 for various values of α and β. From these figures we observe that the curves for the coefficients of amplitude and phase lag in the magnetic case have features similar to those in the non-magnetic case presented by Uchida[2], and Womersley[1]. Further it is seen that both σ_u and δ_u reduce significantly for small values of α only, in the presence of a magnetic field.

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Appendix I

For a physiological flow the pressure gradient $\partial P'/\partial z'$ in (10) is taken to be

$$\partial P'/\partial z' = F(t), \tag{A1}$$

where $F(t)$ has the form given in fig. 10 and is defined by

$$\begin{aligned}
F(t) &= m_1 t, && \text{for} && 0 \leqslant t \leqslant 2\pi/5,\\
&= m_1(4\pi/5 - t), && \text{,,} && 2\pi/5 \leqslant t \leqslant 4\pi/5,\\
&= m_2(t - 4\pi/5), && \text{,,} && 4\pi/5 \leqslant t \leqslant \pi,\\
&= m_2(6\pi/5 - t), && \text{,,} && \pi \leqslant t \leqslant 6\pi/5,\\
&= m_3(t - 6\pi/5), && \text{,,} && 6\pi/5 \leqslant t \leqslant 7\pi/5,\\
&= m_3(8\pi/5 - t), && \text{,,} && 7\pi/5 \leqslant t \leqslant 8\pi/5,\\
&= m_4(t - 8\pi/5), && \text{,,} && 8\pi/5 \leqslant t \leqslant 9\pi/5,\\
&= m_4(2\pi - t), && \text{,,} && 9\pi/5 \leqslant t \leqslant 2\pi,
\end{aligned} \tag{A2}$$

with $m_1 = (\sqrt{3}+1)/(\sqrt{3}-1)$, $m_2 = -3m_1/5$, $m_3 = 2m_1/5$ and $m_4 = -m_1/5$. The pressure gradient of this type is chosen in such a way that the amplitude reduces in each interval of time for a cycle of oscillation. Expanding $F(t)$ in (A2) as a Fourier series, we get

$$F(t) = \sum_{n=0}^{\infty} P_n e^{int} = P_0 + \sum_{n=1}^{\infty} P_{cn} \cos nt + \sum_{n=1}^{\infty} P_{sm} \sin nt, \tag{A3}$$

where the coefficients are given by

$$P_0 = (9\pi/25)\, m_1, \tag{A4}$$

$$\begin{aligned}
P_{cn} = (m_1/5\pi n^2)\,[&10\cos(2n\pi/5) - 2\cos(4n\pi/5) - 6\cos n\pi + \cos(6n\pi/5)\\
&+ 4\cos(7n\pi/5) - \cos(8n\pi/5) - 2\cos(9n\pi/5) - 4],
\end{aligned} \tag{A5}$$

$$\begin{aligned}
P_{sn} = (m_1/5\pi n)\,[&10\sin(2n\pi/5) - 2\sin(4n\pi/5) + \sin(6n\pi/5) + 4\sin(7n\pi/5)\\
&- \sin(8n\pi/5) - 2\sin(9n\pi/5)].
\end{aligned} \tag{A6}$$

The graphs of the partial sums S_n, $n = 0, 1, 2$, of the Fourier series (A3) are shown in fig. 11. The curve for S_2 resembles the curve for volume flux given by Siouffi *et al*[6]. and that for pressure gradient in the femoral artery of a dog presented in Womersley[1].