

Ring source potential outside a vertical co-axial circular cylinder immersed in a liquid with an inertial surface

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Abstract

The velocity potential due to a submerged horizontal circular ring source of time-dependent strength outside an immersed vertical coaxial circular cylinder is derived for a liquid with an inertial surface. A reduction procedure is used to obtain the transformed potential after using Laplace transform in time. For time-harmonic source strengths, the steady-state development demonstrates that the progressive waves originating from the ring source cannot propagate if the inertial surface is too heavy.

Key words: Water waves, inertial surface, ring source, velocity potential, Laplace transform, reduction method.

1. Introduction

Within the framework of the linearised theory of water waves, the problem of generation of infinitesimal gravity waves in a motionless liquid of infinite depth with an inertial surface composed of uniformly distributed floating particles due to a two-dimensional line source which begins to operate in a time-dependent manner at a given instant has been considered by Rhodes–Robinson¹. Later, Mandal and Kundu^{2,3} have extended this problem to the case of a liquid of finite depth and other types of sources such as two- and three-dimensional multipoles of time-dependent strengths, the effect of surface tension at the inertial surface being neglected by Mandal and Kundu² while this being included by Mandal and Kundu³. After using Laplace transform in time, the transformed potentials in these problems have been obtained in a manner analogous to the usual time-harmonic problems in a liquid with a free surface. Laplace inversion then produces the required potentials.

The study of different problems of generation of water waves in the presence of a vertical body of revolution having a common vertical axis of symmetry with the fluid motion requires the consideration of velocity potentials due to submerged circular rings of wave sources as the problems can be formulated in terms of a suitable distribution of rings of wave sources around the body (*cf.* Fenton⁴ and Hulme⁵). This motivates the consideration of various problems of generation of water waves due to a submerged horizontal ring of wave sources. Recently, Kundu⁶ obtained the velocity potential due to

a horizontal ring of wave sources submerged in a liquid of finite depth with an inertial surface, while Mandal and Kundu⁷ considered the same problem for a liquid of infinite depth in the presence of surface tension at the inertial surface. The mathematical technique used to solve this problem was similar to that used by Mandal and Kundu^{2,3}.

Construction of offshore structures for oil prospecting in high seas requires modelling of water wave diffraction problems in a sea in the presence of cylindrical columns of circular cross-sections. Hence consideration of velocity potentials due to submerged circular rings of wave sources (arising in diffraction problems due to submerged or partially immersed circular structure around the column) outside a coaxial immersed cylinder is of some importance. Rhodes-Robinson⁹ obtained the ring source potentials outside such a coaxial circular cylinder while considering a class of time-harmonic surface wave problems involving immersed vertical boundaries.

The present paper is concerned with deriving the potential due to a uniform distribution of point sources of the same time-dependent strength around a horizontal ring submerged in a liquid of infinite depth with an inertial surface, outside an immersed coaxial circular cylinder. This may be regarded as an extension of the problem of ring source outside a coaxial cylinder in water with a free surface considered by Rhodes-Robinson⁹ to a liquid with an inertial surface. After taking Laplace transform in time, a reduction procedure is used to obtain the transformed potential. This procedure was used first by Williams⁸ while considering a general scattering problem due to a submerged time-harmonic point source in deep water with a free surface. Later, Rhodes-Robinson⁹ applied the same method to an entire class of problems for time-harmonic surface waves in water involving immersed vertical boundaries allowing for the influence of surface tension at the free surface. Recently, Mandal¹⁰ also applied this procedure to obtain the potential due to a ring of wave sources submerged in a liquid of infinite depth with an inertial surface.

When the surface density of the floating materials in the results of the present paper is made equal to zero, known results for a liquid with a free surface are recovered.

2. Formulation of the problem

We consider the motion under gravity in an ideal liquid of density ρ covered by an inertial surface composed of a thin uniform distribution of floating particles of area density $\rho\varepsilon$. $\varepsilon=0$ corresponds to a liquid with a free surface. A horizontal ring with radius S of uniformly distributed sources, each of the same strength $m(t)$, is present in the liquid at a depth f below the mean position of the inertial surface outside an immersed coaxial circular cylinder of radius $a (< S)$. Using a cylindrical co-ordinate system with the axis of the ring as y -axis, $y=0$ as the position of inertial surface at rest, the ring source then has the position $R=S (> a)$, $y=f (> 0)$. The point sources on the ring start operating in a time-dependent manner at a given instant simultaneously. Since the motion starts from rest it is irrotational and can be described by a potential function $\phi(R, y; t)$ satisfying the Laplace's equation in the liquid region except at points on the ring. Within the

framework of linearised theory, φ satisfies, the inertial surface condition

$$\frac{\partial^2}{\partial t^2}(\varphi - \varepsilon\varphi_y) - g\varphi_y = 0 \text{ on } y = 0$$

where g is the gravity, the condition for no motion at infinite depth is given by

$$\varphi \rightarrow 0 \text{ as } y \rightarrow \infty,$$

the initial conditions

$$\frac{\partial}{\partial t}(\varphi - \varepsilon\varphi_y) = \varphi - \varepsilon\varphi_y = 0 \text{ on } y = 0 \text{ at } t = 0.$$

$$\text{Let } \bar{\varphi}(R, y; p) = \int_0^{\infty} \exp(-pt)\varphi(R, y; t) dt, (p > 0), \quad (2.1)$$

then $\bar{\varphi}$ is the solution of the BVP described by

$$\nabla^2 \bar{\varphi} = 0, y > 0 \text{ except at points on the ring,} \quad (2.2)$$

$$p^2 \bar{\varphi} - (g + \varepsilon p^2) \bar{\varphi}_y = 0 \text{ on } y = 0, \quad (2.3)$$

$$\frac{\partial \bar{\varphi}}{\partial R} = 0 \text{ on } R = a \quad (2.4)$$

$$\bar{\varphi} \rightarrow 0 \text{ as } y \rightarrow \infty. \quad (2.5)$$

3. Solution of the problem

In view of (2.3) and following Williams⁸, we introduce a function $\Psi(R, k; p)$ ($k > 0$) such that

$$p^2 \bar{\varphi} - (g + \varepsilon p^2) \bar{\varphi}_y = -\frac{2}{\pi} \int_0^{\infty} k \Psi \sin ky dk, y \geq 0. \quad (3.1)$$

Solving (3.1) and using (2.4) we obtain

$$\bar{\varphi} = -\frac{2}{\pi} \frac{1}{g + \varepsilon p^2} \int_0^{\infty} k \Psi \frac{k \cos ky + q \sin ky}{k^2 + q^2} dk \quad (3.2)$$

$$\text{where } q = p^2 / (g + \varepsilon p^2). \quad (3.3)$$

This reduces the BVP described by (2.2) to (2.5) by one dimension to another BVP in Ψ given in (3.4) below.

$$\Psi'' + \frac{1}{R} \Psi' - k^2 \Psi = 0, R \neq a,$$

$$\Psi' = 0 \text{ on } R = a,$$

$$\Psi'(S+0) - \Psi'(S-0) = -\frac{\bar{m}(p)}{kS} A(k, f, p),$$

$$\Psi(S+0) - \Psi(S-0) = 0 \text{ at } R = S,$$

$$\text{and } \Psi \rightarrow 0 \text{ as } R \rightarrow \infty \quad (3.4)$$

where $A(k, f, p) = k(g + cp^2) \cos kf + p^2 \sin kf$ and Ψ', Ψ'' denote respectively the first and second derivatives with respect to R .

The solution of the BVP (3.4) is

$$\Psi(R, k; p) = \frac{\bar{m}(p)}{k} A(k, f, p) \begin{cases} I_1(ka)K_0(kS) \left\{ \frac{I_0(kR)}{I_1(ka)} + \frac{K_0(kR)}{K_1(ka)} \right\} & R < S, \\ I_1(ka)K_0(kR) \left\{ \frac{I_0(kS)}{I_1(ka)} + \frac{K_0(kS)}{K_1(ka)} \right\} & R \geq S. \end{cases}$$

Hence $\bar{\varphi}(R, y; p)$

$$= -\frac{2\bar{m}(p)}{\pi} \int_0^\infty \frac{A(k, f, p) A(k, y, p)}{p^4 + k^2(g + cp^2)^2} dk \begin{cases} I_1(ka)K_0(kS) \left\{ \frac{I_0(kR)}{I_1(ka)} + \frac{K_0(kR)}{K_1(ka)} \right\} & R < S, \\ I_1(ka)K_0(kR) \left\{ \frac{I_0(kS)}{I_1(ka)} + \frac{K_0(kS)}{K_1(ka)} \right\} & R \geq S. \end{cases} \quad (3.5)$$

To obtain the Laplace inversion of $\bar{\varphi}$ we now simplify (3.5) for $R \geq S$. The case $R < S$ can be dealt with similarly. For $R \geq S$, from (3.5) we can write

$$\bar{\varphi}(R, y; p) = -\frac{2\bar{m}(p)}{\pi} \left[\int_0^\infty \sin ky \sin kf B(k) dk + \frac{\pi}{2} \int_0^\infty \frac{k}{k+q} \exp\{-k(y+f)\} X(k) dk \right] \quad (3.6)$$

$$\text{where } B(k) = I_1(ka)K_0(kR) \left\{ \frac{I_0(kS)}{I_1(ka)} + \frac{K_0(kS)}{K_1(ka)} \right\} \quad (3.7)$$

$$\text{and } X(k) = J_0(kR)J_0(kS) - J_1(ka) \operatorname{Re} \left(\frac{H_0^{(2)}(kR)H_0^{(2)}(kS)}{H_1^{(2)}(ka)} \right). \quad (3.8)$$

$\bar{\varphi}$ in (3.6) can be further simplified to

$$\bar{\varphi} = -\frac{2\bar{m}(p)}{m} \left[\int_0^{\varepsilon} \sin ky \sin kf B(k) dk + \frac{\pi}{2} \int_0^{\varepsilon} \frac{ke}{1+ke} \exp\{-k(y+f)\} X(k) dk \right. \\ \left. + \frac{\pi}{2} \int_0^{\varepsilon} \frac{\exp\{-k(y+f)\}}{1+ke} X(k) \frac{\mu^2}{\mu^2+p^2} dk \right] \quad (3.9)$$

where $\mu^2 = ak/(1+ke)$. The Laplace inversion of (3.9) is now obvious and thus we obtain

$$\varphi(R, y; t) = m(t) U(R, y) + \int_0^{\varepsilon} \frac{\mu \exp\{-k(y+f)\}}{1+ke} X(k) \int_0^t m(\tau) \sin \mu(t-\tau) d\tau dk, \quad (3.10)$$

$$\text{where } U(R, y) = -\frac{2}{\pi} \left[\int_0^{\varepsilon} \sin ky \sin kf B(k) dk + \frac{\pi}{2} \varepsilon \int_0^{\varepsilon} \frac{k}{1+ke} \exp\{-k(y+f)\} X(k) dk \right] \quad (3.11)$$

(3.10) is the general result for any arbitrary time-dependent strength $m(t)$ of the sources.

4. The time-harmonic ring source

To compare our results with the known potential due to a time-harmonic ring source submerged in water with a free surface, we now consider the special case where the strength of the sources varies harmonically with time. Let us put $m(t) = \sin \sigma t$ in (3.10). Then

$$\varphi(R, y; t) = \sin \sigma t U(R, y) - \int_0^{\varepsilon} \frac{\mu \exp\{-k(y+f)\}}{1+ke} X(k) \frac{\mu \sin \sigma t - \sigma \sin \mu t}{\mu^2 - \sigma^2} dk. \quad (4.1)$$

To find the steady-state development in φ as $t \rightarrow \infty$, we note that the transient term, if any, occurs only in the integral in (4.1) involving $\sin \mu t$. Now $\mu^2 - \sigma^2$ vanishes at $k = k_0^*$ in the range of integration $k = 0$ only when $0 \leq \varepsilon K < 1$ where $k_0^* = K(1 - \varepsilon K)^{-1}$ with $K = \sigma^2/g$. Hence for $0 \leq \varepsilon K < 1$, introducing a Cauchy principal value at $k = k_0^*$ in this integral, we can write the term involving $\sin \mu t$ as

$$2\sigma \int_0^{(y+f)} \left[\frac{\mu \exp\{-\mu(y+f)\} X(\mu)}{\mu(\mu + \sigma)} \right]_{k_0^*}^k \frac{\sin \mu t d\mu}{\mu - \sigma} \\ + k_0^* \exp\{-k_0^*(y+f)\} \lambda(k_0^*) \int_0^{(y+f)} \frac{\sin \mu t}{\mu - \sigma} d\mu. \quad (4.2)$$

By Riemann-Lebesgue lemma the first integral in (4.2) tends to zero and the second integral tends to $\pi \cot \sigma t$ as $t \rightarrow \infty$. Thus we obtain for large t ,

$$\varphi \sim -\frac{2 \sin \sigma t}{\pi} \left[\int_0^y \sin ky \sin kf B(k) dk + \frac{\pi}{2} \int_0^{\frac{y}{k_0^*}} \frac{k}{k - k_0^*} \exp \{-k(y+f)\} X(k) dk \right] + k_0^* \exp \{-k_0^*(y+f)\} X(k_0^*) \cos \sigma t. \quad (4.3)$$

It can be shown that (4.3) represents an outgoing progressive wave as $R \rightarrow \infty$.

When $\varepsilon K \geq 1$, there is no zero of $\mu^2 - \sigma^2$ for $k > 0$ and thus the integral involving $\sin \mu t$ is wholly transient. Hence in this case.

$$\varphi \sim -\frac{2 \sin \sigma t}{\pi} \left[\int_0^y \sin ky \sin kf B(k) dk + (K\varepsilon - 1) \int_0^{\frac{y}{k(K\varepsilon - 1) + K}} \frac{k \exp \{-k(y+f)\}}{k(K\varepsilon - 1) + K} X(k) dk \right] \quad (4.4)$$

as $t \rightarrow \infty$. It can be shown that now there is no progressive wave as $R \rightarrow \infty$.

For a time-harmonic ring source with circular frequency σ , the steady-state development to the potential is given by (4.3) for $\sigma < (g/\varepsilon)^{1/2}$ i.e. $\varepsilon < g/\sigma^2$ and by (4.4) for $\sigma \geq (g/\varepsilon)^{1/2}$ i.e. $\varepsilon \geq g/\sigma^2$. The former is interpreted physically as the inertial surface to be heavy while the latter is interpreted as the inertial surface to be light. Rhodes-Robinson's¹⁰ result can be obtained from (4.3) by putting $\varepsilon = 0$. The effect of inertial surface on the potential function thus appears to be straight forward enough to visualize, namely, the time-harmonic waves (produced at the ring source) whose circular frequency is less than $\sigma_0 = (g/\varepsilon)^{1/2}$ will propagate to large distances from the ring source while those with frequency exceeding σ_0 , will die out at large distances from the ring source. This phenomenon is in conformity with the fact that infinitesimal time-harmonic progressive gravity wave can propagate in an ideal liquid with an inertial surface only if the inertial surface is not too heavy (cf. Rhodes-Robinson¹, Peters¹¹).

5. Conclusion

Potential function due to a ring source of time-dependent strength submerged outside an immersed coaxial circular cylinder is obtained by a reduction procedure. The boundary value problem concerning the transformed potential is reduced to another boundary value problem from whose solution the potential function is obtained. The problem may be extended to a liquid of finite depth and also to include the effect of surface tension at the inertial surface.

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