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# Unsteady flow of a dusty elastico-viscous liquid in the Ekman layer

SRIKUMAR MUKHERJEE*, GOPAL CHANDRA MANDAL** AND SWAPNA MUKHERJEE†
*Department of Mathematics, Malda College, Malda 732 101, West Bengal, India.
**Calcutta Port Trust, Berhampore 742 101, West Bengal, India.
†Department of Mathematics, Raiganj University College, Raiganj 733 134, West Bengal, India.

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**Abstract**

The transient flow in the Ekman layer of a dusty elastico-viscous liquid near a flat plate is discussed. Initially the dusty fluid and the plate were rotating together and the plate then suddenly starts moving with a uniform velocity in its own plane relative to the rotating frame of reference. The effect of rotation manifests itself through inertial oscillations which decays exponentially with time. It is shown that the frequency of inertial oscillations decreases with increase in either mass concentration or elastic element in the liquid.

Keywords: Unsteady flow, dusty fluid flow, elastico-viscous liquid, inertial oscillations, Ekman layer.

## 1. Introduction

Multiphase flow problems are of current interest in fluid dynamics. In particular, fluid mechanical problems involving gas-particle mixtures arise in various fields of engineering. Based on the theoretical model proposed by Saffman[1], many authors investigated a number of dusty-gas flow problems in various geometries. Regarding the plate problems, Michael and Miller[2], Liu[3,4], Healy and Yang[5] investigated a number of dusty-gas flow problems. But little attention is paid to the flow of a dusty gas in a rotating system although this has some bearing on the pollution problem as well as on the motion of aerosol over the rotating earth. Gupta and Pop[6] investigated the unsteady boundary layer flow in a rotating viscous liquid bounded by an infinite flat plate when there is a suspension of dust particles in the liquid. Jana *et al*[7] investigated the unsteady flow in the Ekman layer of an elastico-viscous liquid.

In the present investigation, we extend the analysis of Gupta and Pop[6] to cover a wider class of elastico-viscous liquid, *viz.* Walters' liquid B′ (with short memory) and, in particular, to observe (qualitatively) the effects of elastic element and mass concentration on the flow field.

*For correspondence.

## 2. Mathematical formulation and asymptotic analysis

We consider an infinite plate coinciding with the plane $z=0$ and rotating in unison with a dusty elastico-viscous liquid occupying the region $z>0$ with a uniform angular velocity $\Omega$ about the $z$-axis for time $t \leqslant 0$. At time $t>0$, the plate starts moving with a uniform velocity $U$ in its own plane relative to the rotating frame of reference. The horizontal homogeneity of the problem demands that the physical quantities depend on $z$ and $t$ only. The equation of continuity of the liquid then gives $w \equiv 0$ everywhere in the flow, where $(u, v, w)$ are components of the liquid velocity at a point.

The equation of continuity of dust particles is given as

$$\frac{\partial N}{\partial t}+(N v_i')_{,i}=0,$$

where $N$ is the number density of dust particle and $v_i'$ is the velocity of dust particle. Since the distribution of dust particles is uniform, the number density of the particles $N=N_0$, a constant throughout the motion.

Following Saffman[1] and Walters[8] we get the equations of motion for the liquid and dust particle in a rotating frame of reference as

$$\frac{\partial u}{\partial t}-2\Omega v=\nu\left(1-\frac{K_0}{\rho\nu}\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial z^2}+\frac{KN_0}{\rho}(u'-u), \tag{1}$$

$$\frac{\partial v}{\partial t}+2\Omega u=\nu\left(1-\frac{K_0}{\rho\nu}\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2}+\frac{KN_0}{\rho}(v'-v), \tag{2}$$

$$\frac{\partial u'}{\partial t}-2\Omega v'=\frac{K}{m}(u-u'), \tag{3}$$

$$\frac{\partial v'}{\partial t}+2\Omega u'=\frac{K}{m}(v-v'), \tag{4}$$

where $m$ is the mass of a dust particle, $K$ the Stokes resistance coefficient, $K_0$ the elastic coefficient. $\nu=\eta/\rho$, $\eta$ the limiting viscosity at small rates of shear and $\rho$ the density Equations (1) and (2), and (3) and (4) are combined as

$$\frac{\partial q}{\partial T}-\left(1-K_1\frac{\partial}{\partial T}\right)\frac{\partial^2 q}{\partial z^2}+(2\,i\omega+f)\,q-fq'=0, \tag{5}$$

$$\frac{\partial q'}{\partial T}+(2i\omega+1)\,q'-q=0 \tag{6}$$

where

$$q=\frac{u+iv}{U}(=u_1+iv_1),\ q'=\frac{u'+iv'}{U}(=u_1'+iv_1'),$$

$$T = \frac{t}{\tau},\ Z = \frac{z}{(\nu\tau)^{1/2}},\ \tau = \frac{m}{K},\ \Omega\tau = \omega,\ K_1 = \frac{K_0}{\rho\nu\tau}$$

and $f = mN_0/\rho$ is the mass concentration of dust particles and $\tau$ is the relaxation time of dust particles.

The initial and boundary conditions are

$$q = q' = 0 \text{ for } T \leqslant 0, \tag{7}$$

$$q = 1 \text{ at } Z = 0 \text{ for } T > 0, \tag{8}$$

$$q \longrightarrow 0 \text{ as } Z \longrightarrow \infty. \tag{9}$$

Taking Laplace transforms of (5) and (6) and using (7), we get

$$(1 - K_1 p)\frac{d^2\bar{q}}{dZ^2} - (2\,i\omega + f + p)\bar{q} + f\bar{q}' = 0, \tag{10}$$

$$(p + 2\,i\omega + 1)\bar{q}' = \bar{q}, \tag{11}$$

where

$$\bar{q}(Z, p) = \int_0^t q(Z, T)\, e^{-pT}\, dT,$$

$$\bar{q}'(Z, p) = \int_0^t q'(Z, T)\, e^{-pT}\, dT.$$

Eliminating $\bar{q}'$ from (10) and (11) and then solving for $\bar{q}$ with the help of transformed boundary condition, we have

$$\bar{q}(Z, p) = \frac{1}{p}\, e^{-MZ}, \tag{12}$$

where

$$M = \frac{1}{(1 - K_1 p)^{1/2}}\left[p + 2\,i\omega + f - \frac{f}{1 + p + 2\,i\omega}\right]^{1/2}$$

with the proviso that the real part of $M$ is taken positive.

The dust velocity at the plate $Z = 0$ is given by (11) and (12) as

$$q'(O, T) = \mathscr{L}^{-1}\left[\frac{1}{p(p + 1 + 2\,i\omega)}\right]. \tag{13}$$

Taking inverse transformation of (13) and separating real and imaginary parts, we have

$$u_1'(O, T) = \frac{1}{1 + 4\,\omega^2}\,[1 - e^{-T}\cos 2\omega T + 2\omega e^{-T}\sin 2\omega T], \tag{14}$$

$$v_1'(O, T) = \frac{1}{1 + 4\,\omega^2}\,[e^{-T}\sin 2\omega T - 2\omega(1 - e^{-T}\cos 2\omega T)]. \tag{15}$$

It is interesting to note from (14) and (15) that the dust particles do not stick to the plate but move relative to it. However, such a velocity slip is compatible with the assumption that the bulk concentration of the dust particles is small since we can then imagine that the particles nearest to the plate will in general be several particle diameters away from it. It is also remarked that the velocity of dust particles on the plate is unaffected by the presence of elastic parameter of the fluid.

To investigate the asymptotic nature of the solution for large time, we assume $P \ll 1$. Equation (12) can then be approximated by

$$\bar{q}(Z,p) = \frac{1}{p} e^{-\sqrt{A}(p+B/A)^{1/2} Z}, \tag{16}$$

where

$$A = \alpha + i\beta, \tag{17}$$

$$B = 2i\omega\left(1 + \frac{f}{1+2i\omega}\right), \tag{18}$$

so that

$$\alpha = (1 + 4\omega^2 + 4\omega^2 f K_1)/(1+4\omega^2),$$
$$\beta = 2\omega K_1 (1 + 4\omega^2 + f)/(1+4\omega^2).$$

Equation (16) gives, on using the table of the inverse Laplace transform due to Campbell and Foster[9],

$$q(z,T) = \frac{1}{2}\left[e^{\sqrt{B}Z} \cdot \operatorname{erfc}\left(\frac{\sqrt{A}}{2\sqrt{T}} Z + \sqrt{\frac{BT}{A}}\right) + e^{-\sqrt{B}Z} \right.$$
$$\left. \times \operatorname{erfc}\left(\frac{\sqrt{A}}{2\sqrt{T}} Z - \sqrt{\frac{BT}{A}}\right)\right]. \tag{19}$$

It is known that

$$\operatorname{erfc}(Z) \approx Z^{-1} \pi^{-1/2} \exp(-Z^2) \text{ as } |Z| \longrightarrow \infty,$$
$$\operatorname{erfc}(-Z) = 2 - \operatorname{erfc}(Z). \tag{20}$$

Using (19) and (20) and separating $q(Z,T)$ into real and imaginary parts, we get the asymptotic expressions for $u_1(Z,T)$ and $V_1(Z,T)$ for large $T$ as

$$u_1(Z,T) = e^{-Z\alpha_1} \cos\beta_1 Z - \frac{Z}{2\sqrt{\pi T}} \cdot \frac{1}{(\alpha_5^2 + \beta_5^2)} \cdot \{(\alpha_2\alpha_5 + \beta_2\beta_5)\cos\alpha_4$$

$$+(\beta_2\alpha_5-\alpha_2\beta_5)\sin\alpha_4\}\exp\left\{-\left(\frac{\alpha Z^2}{4T}+(\alpha_3^2-\beta_3^2)T\right)\right\}, \tag{21}$$

$$V_1(Z,T)=e^{-Z\alpha_1}\sin\beta_1 Z-\frac{Z}{2\sqrt{\pi T}}\cdot\frac{1}{(\alpha_5^2+\beta_5^2)}\cdot\{(\beta_2\alpha_5-\alpha_2\beta_5)\cos\alpha_4$$

$$-(\alpha_2\alpha_5+\beta_2\beta_5)\sin\alpha_4\}\cdot\exp\left\{-\left(\frac{\alpha Z^2}{4T}+(\alpha_3^2-\beta_3^2)T\right)\right\}, \tag{22}$$

where

$$\alpha_1,\beta_1=\left(\frac{\omega}{1+4\omega^2}\right)^{1/2}[\{(1+4\omega^2+f)^2+4\omega^2 f^2\}^{1/2}\pm 2\omega f]^{1/2},$$

$$\alpha_2,\beta_2=\frac{1}{(2(1+4\omega^2))^{1/2}}[\{(1+4\omega^2+4\omega^2 fK_1)^2+4\omega^2K_1^2$$

$$\times(1+f+4\omega^2)^2\}^{1/2}\pm(1+4\omega^2+4\omega^2 fK_1)]^{1/2},$$

$$\alpha_3=(\alpha_1\alpha_2+\beta_1\beta_2)/(\alpha_2^2+\beta_2^2),$$

$$\beta_3=(\alpha_2\beta_1-\alpha_1\beta_2)/(\alpha_2^2+\beta_2^2),$$

$$\alpha_4=(\beta+2\,\alpha_3\beta_3 T),$$

$$\beta_4=0,$$

$$\alpha_5=(\alpha_3^2-\beta_3^2)T-\alpha Z^2/4T,$$

$$\beta_5=2\,\alpha_3\beta_3 T-\beta Z^2/4T.$$

The dimensionless skin-friction can be calculated from equation (19) as

$$-\left.\frac{\partial q}{\partial Z}\right|_{Z=0}=(\alpha_1+i\beta_1)\operatorname{erf}\left((\alpha_3+i\beta_3)\sqrt{T}\right)+\frac{\alpha_2+i\beta_2}{\sqrt{\pi T}}e^{-BT/A}. \tag{23}$$

## 3. Discussion

The first terms in (21) and (22) represent the velocity components for the Ekman boundary layer (modified by the presence of dust particles) on the plate, which is established in the final steady state. The boundary layer thickness is clearly of order $(\alpha_1)^{-1}$ and gradually it becomes thinner with the increase in $f$, the mass concentration. This distribution is independent of $K_1$. Another distinctive feature of the above asymptotic solution is that the second terms in (21) and (22) confirm the existence of inertial oscillations which decay exponentially with time. The effect of rotation manifests itself through these oscillations with frequency, $(2\alpha_3\beta_3)$. The effects of elastic element,

**Table I**

**(a) Effect of elastic element on frequency of inertial oscillations when f = 0·1, $\omega$ = 1·0**

| $K_1$ | 0·5 | 1·0 | 1·5 | 2·0 | 2·5 |
|---|---|---|---|---|---|
| $2\alpha_3\beta_3$ | 1·4004176 | 0·8837880 | 0·6260497 | 0·4809395 | 0·4193873 |

**(b) Effect of mass concentration on frequency of inertial oscillations when $K_1$ = 0·5, $\omega$ = 1·0**

| $f$ | 0·10 | 0·20 | 0·30 | 0·35 | 0·40 |
|---|---|---|---|---|---|
| $2\alpha_3\beta_3$ | 1·4004176 | 1·3872846 | 1·3747694 | 1·3699121 | 1·3609122 |

mass concentration on the frequency of inertial oscillations are shown in Table I. It reveals that the frequency decreases with increase in either mass concentration or elastic element in the liquid.

As $T \to \infty$, (23) gives the steady-state skin friction as $(\alpha_1 + i\beta_1)$. In the absence of dust parameter, it reduces to $(1+i)\omega^{1/2}$, which agrees with the classical result for Ekman spiral near a plate in a rotating frame. It is of interest to have an estimate of the time which elapses from the start of the plate in the rotating frame till the steady state is reached. It is clear from (23) that the steady state is reached after a time $T_0$ where

$$\operatorname{erf}((\alpha_3 + i\beta_3)T_0) \approx 1.$$

Since erf $(x) \approx 1$ when $|x| \simeq 2$, it follows that

$$T_0 \simeq \frac{2}{(\alpha_3^2 + \beta_3^2)^{1/2}}.$$

The effects of mass concentration and elastic parameter on $T_0$ can be seen from Table II. It reveals that the effect of mass concentration is to decrease the time to reach the steady state while it increases as the elastic element increases.

**Table II**

**(a) Effect of elastic element on the time ($T_0$) to reach the steady state when f = 0·1, $\omega$ = 1·0**

| $K_1$ | 0·5 | 1·0 | 1·5 | 2·0 | 2·5 |
|---|---|---|---|---|---|
| $T_0$ | 1·5906671 | 1·9306756 | 2·2767133 | 2·6301002 | 3·0001219 |

**(b) Effect of mass concentration on the time ($T_0$) to reach the steady state when $K_1$ = 0·5, $\omega$ = 1·0**

| $f$ | 0·10 | 0·15 | 0·20 | 0·25 | 0·30 |
|---|---|---|---|---|---|
| $T_0$ | 1·5906671 | 1·5851219 | 1·5768748 | 1·5751921 | 1·5728127 |

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