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On the spectral resolution of a differential operator—IV

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Abstract

In the present paper we investigate the asymptotic formulae involving $\chi(x, y, \lambda + a)$ and $\chi(x, y, a)$ as a tends to infinity, where $\chi(x, y, \lambda)$ stands for $D^p(H(x, y, \lambda) - \mathscr{H}(x, y, \lambda))$, $H(\cdot)$, $\mathscr{H}(\cdot)$ being the resolution matrices associated with two different second-order differential systems with the same boundary conditions at two arbitrary points a and b. Replacing $\mathscr{H}(x, y, \lambda)$ by the resolution matrix $H^F(x, y, \lambda)$ of the Fourier system and then making $x \to y$ we derive some special asymptotic formulae. A modified form of a Tauberian theorem due to Wiener plays a key role in the investigation that follows.

Key words: Spectral resolution, resolution matrix, majorizing a matrix, Wiener-Tauberian theorem.

1. Introduction

Consider the differential system

$$MU = \lambda U$$

where

$$M \equiv \begin{pmatrix} -D^2 + p(x) & r(x) \\ r(x) & -D^2 + q(x) \end{pmatrix}, \quad D = \mathrm{d}/\mathrm{d}x, \quad U = (u, v)^T \tag{1.1}$$

and λ, an eigenvalue parameter, real or complex;

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$$

is differentiable, the pth derivative $Q^p(x)(p \geq 1)$ being absolutely continuous over any finite interval $(a, b) \subset (-\infty, \infty)$.

Let ϕ_l, ϕ_j be the boundary condition vectors associated with the system (1.1), the boundary conditions at $x = a$ and $x = b$ being given by

$$[U, \phi_l]_a = 0 = [U, \phi_j]_b, \quad l = 1, 2; \quad j = 3, 4 \tag{1.2}$$

with $[\phi_1, \phi_2] = 0 = [\phi_3, \phi_4]$, $[\cdot]$, the bilinear concomitant[1] of the vectors U and ϕ.

Let $\phi_r(x,\lambda)=(u_r,v_r)^T$, $r=1,2$, be the solutions of (1.1) satisfying the initial conditions $(u_j,v_j,u'_j,v'_j)=\varepsilon_j$, $j=1,2$, where ε_j is the jth unit vector which ε R^4. If $\theta_r(x,\lambda)=(x_r,y_r)^T$, $r=1,2$ are two other solutions of (1.1) connected with ϕ_r by means of the relations $[\phi_r,\theta_k]=\delta_{rk}$; $[\theta_1,\theta_2]=0$, $r,k=1,2$, δ_{rk}, the Kronecker delta, then $\phi_1,\phi_2,\theta_1,\theta_2$ are linearly independent. We had introduced[2] the resolution matrix $H(x,y,\lambda)$, λ real, as the matrix

$$\begin{aligned} H(x,y,\lambda) &= \lim_{\nu\to 0}\int_0^{\lambda} \operatorname{im} G(x,y,\sigma+i\nu)\,\mathrm{d}\sigma, \lambda>0 \\ &= \lim_{\nu\to 0}\int_{\lambda}^{0} \operatorname{im} G(x,y,\sigma+i\nu)\,\mathrm{d}\sigma, \lambda<0 \\ &= 0, \qquad\qquad\qquad\qquad \lambda=0 \end{aligned}$$

where $G(\cdot)$ is the Green's matrix for the system (1.1). For an explicit form of $H(x,y,\lambda)$ involving matrices

$$\phi=\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}, \quad \theta=\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

and the matrices ξ,η,ζ see Chakravarty and Roy Paladhi[2]. The resolution matrix $H(x,y,\lambda)$ generates the resolution of the identity E of the differential operator T generated by the differential operation M; T and E are connected[2] by the relation $T=\int_{-\infty}^{\infty}\lambda\,\mathrm{d}E(\lambda)$.

A number of properties of $H(x,y,\lambda)$ and its derivatives were investigated in the previous papers[1–3]. These are theorems like the spectral representation theorem, the generalized Parseval relation, equiconvergence theorem, the Riesz summability theorems and certain asymptotic relations involving $H(x,y,\lambda)$ and its derivatives.

Let M_1 be the differential operation corresponding to M in (1.1) with p,q,r replaced by p_1,q_1,r_1:

$$M_1U=\lambda U. \tag{1.3}$$

(1.3) with boundary conditions (1.2) gives rise to a self-adjoint eigenvalue problem similar to that of (1.1) with (1.2). Let $\mathscr{H}(x,y,\lambda)$, λ real, be the resolution matrix for this system. Put $\chi(x,y,\lambda)=D^p(H(x,y,\lambda)-\mathscr{H}(x,y,\lambda))$, $\quad D^p=\partial^{s+t}/\partial x^s\partial y^t$, $\quad p=s+t, s,t=0,1,2,\ldots$.

Our object in the present paper is to study the asymptotic relations which exist between $\chi(x,y,\lambda+a)$ and $\chi(x,y,a)$, as a tends to infinity, when λ is fixed and x,y vary in a bounded domain. We shall confine our discussion to the cases $s=0$, $t=0$; $s=1$, $t=1$ and $s=0$, $t=1$; $s=1$, $t=0$ only. Extensions to higher derivatives follow readily.

Put $\lambda=\mu^2$, $H(x,y,\lambda)=H_1(x,y,\mu)$, $\chi(x,y,\lambda)=\chi_1(x,y,\mu)$ and for fixed x,y,H_1, χ_1 are continued to negative μ as matrices whose elements are odd functions of μ.

The Fourier system corresponding to (1.1) is the system (1.1) with $p=q=r=0$ and similarly for (1.3). Thus the Fourier systems corresponding to (1.1) and (1.3) are the same. Therefore, for the Fourier systems corresponding to (1.1) and (1.3) we obtain the same resolution matrix $H^F(x,y,\lambda)=\mathscr{H}^F(x,y,\lambda)$.

Spectral theory of differential operators forms an important subject of study in the present-day mathematics and intensive work on the self-adjoint/non-self-adjoint differential operators is being carried out. Levitan and Sargsjan[4] have presented a volume dealing with certain basic topics in the modern spectral theory of ordinary self-adjoint differential operators of Sturm-Liouville type and of Dirac-type first-order differential systems. They have further given an introduction to the spectral theory of the nth order ordinary differential equations. Among other workers dealing with the spectral theory of differential operators, Coddington, E. A., Bennewitz, C., Dijksma, A., Pleijel, A. K., Langer, H., Textorious, B. and Naimark, M. A. are prominent. However, spectral problems associated with the system $LY = \lambda MY$, a system consisting of m equations each of order n, are yet to be fully investigated. The system (1.1), a special case of this system with $m = n = 2$, finds application in the theory of deuterons. A comprehensive study for the spectral properties of the system (1.1) is therefore called for.

The ideas involved in the present investigation are similar to those of Levitan[5] used for the discussion of an asymptotic problem involving the spectral functions for a Sturm-Liouville operator. However, there are certain differences. The basic formula of Levitan stems from the solutions of the scalar Cauchy problem

$$\partial^2 u/\partial x^2 - q(x)u = \partial^2 u/\partial t^2, \quad u|_{t=0} \neq 0, \quad \partial u/\partial t|_{t=0} = 0.$$

He uses the Fourier cosine transform theory in the sequel, the formulation of the problem being such that the Fourier sine transform theory cannot be used. We utilise the Cauchy problem for vector-valued functions, *viz.*,

$$\partial^2 U/\partial x^2 - Q(x)U = \partial^2 U/\partial t^2, \quad U|_{t=0} \neq 0, \quad \partial U/\partial t|_{t=0} \neq 0. \tag{A}$$

$U = (u, v)^T$, $Q(x)$ is the matrix which occurs in the system (1.1), and the Fourier sine transforms, the Fourier cosine transform theory being inapplicable in our case. More over, Levitan did not consider similar problems for the derivatives of the spectral functions. We shall therefore emphasise those parts of our theory where we considerably differ. It may be noted that in a recent paper[6], the senior author (N.K.C.) has developed a theory giving the asymptotic formulae for the spectral matrix $\rho(\lambda)$ associated with the system (1.1) over the interval $[0, \infty)$. The method adopted there is however the method of integral equations, entirely different from that adopted in the present analysis.

2. Certain auxiliary lemmas

It is well known[7] that if $T(x, t, s)$ is one of the Riemann matrices which occurs in the solution of the system (A), then $T(x, t, s)$ satisfies the inequalities

$$|T(x, t, s)| \leqslant \int_{x-t}^{x+t} |Q(\sigma)|\,\mathrm{d}\sigma \exp\left(\tfrac{1}{2}t \int_{x-t}^{x+t} |Q(\sigma)|\,\mathrm{d}\sigma\right). \tag{2.1}$$

Also

$$T(x, t, s) = \sum_{r=1}^{\infty} (-)^r T_r(x, t, s),$$

where

$$T_r(x,t,s)=\tfrac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}Q(y)\,T_{r-1}(y,\tau,s)\,\mathrm{d}\tau\,\mathrm{d}y$$

and

$$T_1(x,t,s)=\tfrac{1}{2}\left(\int_s^{\frac{1}{2}(x+t+s)}+\int_s^{\frac{1}{2}(x-t+s)}\right)Q(\sigma)\,\mathrm{d}\sigma.$$

Then, as detailed in ref. 3, it easily follows by mathematical induction that

$$|\partial/\partial x\,T(x,t,s)|\leqslant\int_{x-t}^{x+t}(|Q'(\sigma)|+|Q(\sigma)|)\,\mathrm{d}\sigma\exp\left(\tfrac{1}{2}t\int_{x-t}^{x+t}(|Q'(\sigma)|+|Q(\sigma)|\right)\mathrm{d}\sigma. \tag{2.2}$$

Let $\Omega(x,t,s)$ be the indefinite integral $\int^s T(x,t,y)\mathrm{d}y$ and $K(x,t,s)=\Omega(x,t,s)-A\Omega(x,t,x+t)-B\Omega(x,t,x-t)$, where the constants A, B are defined as follows.

$$A=1,\quad \text{if } s\in(0,x+\varepsilon) \qquad\qquad B=1,\quad \text{if } s\in(0,x-\varepsilon)$$
$$=0,\quad \text{otherwise} \qquad \text{and} \qquad =0,\quad \text{otherwise.}$$

Then, we have

Lemma 2.1. For all $|t|\geqslant 0$, there exists a monotonically increasing function $\phi(t)>0$ for which

$$|K(x,t,s)|,\quad |\partial/\partial x\,K(x,t,s)|,\quad |\partial^2/\partial x\,\partial s\,K(x,t,s)|\leqslant\phi(t) \tag{2.3}$$

where $x\in(x_0,x_1)$, a given finite interval. The function $\phi(t)$ has an exponential growth *i.e.* as t tends to infinity, $\phi(t)>\exp(\alpha|t|)$ for some constant $\alpha>0$.

From the definition of $\Omega(x,t,s)$ and the inequality (2.1) it follows that

$$|K(x,t,s)|\leqslant 3\int_0^{x+t}|T(x,t,y)|\,\mathrm{d}y\leqslant 3(x_1+t)\left(\int_{x_0-t}^{x_1+t}|Q(\sigma)|\,\mathrm{d}\sigma\right.$$
$$\left.\times\exp\left(\tfrac{1}{2}t\int_{x_0-t}^{x_1+t}|Q(\sigma)|\mathrm{d}\sigma\right)\right). \tag{2.4}$$

Similarly, using the inequality (2.2), we obtain, in view of $0<x-t<s<x+t$,

$$|\partial/\partial x\,K(x,t,s)|\leqslant 3(x_1+t)\int_{x_0-t}^{x_1+t}(|Q'|+|Q|)\,\mathrm{d}\sigma\exp\left(\tfrac{1}{2}t\int_{x_0-t}^{x_1+t}(|Q'|+|Q|)\,\mathrm{d}\sigma\right) \tag{2.5}$$

$$|\partial^2/\partial x\,\partial s\,K(x,t,s)|\leqslant|\partial/\partial x\,T(x,t,s)|$$
$$\leqslant\int_{x_0-t}^{x_1+t}(|Q'|+|Q|)\,\mathrm{d}\sigma\exp\left(\tfrac{1}{2}t\int_{x_0-t}^{x_1+t}(|Q'|+|Q|)\,\mathrm{d}\sigma\right). \tag{2.6}$$

The lemma follows by choosing $\phi(t)=\max$ (right-hand expressions in (2.4)–(2.6)).

Let $w(t)$ which does not vanish for any real value of t, be defined by

$$1/w(t)=(1+|t|)(2+\phi(t))^2\quad\text{for all } t.$$

Then since $\phi(t) > \exp(\alpha|t|)$, where α is a positive constant, it follows that

$$\text{(i)}\ |w(t)| \leqslant 1/|t| \exp(-2\alpha|t|), \quad |t| > 0, \quad \alpha > 0, \tag{2.7}$$

and

$$\text{(ii)}\ w(t) \in L(-\infty, \infty);$$

$w(t)$ tends to zero as t tends to infinity.

Put $w(t,a) = w(t)\cos at$, where a is an arbitrary real number. Also put

$$h(\lambda^{\frac{1}{2}}, a) = \int_0^\infty \sin \lambda^{\frac{1}{2}} t\, w(t) \cos at\, \mathrm{d}t \tag{2.8}$$

the existence of which is obvious when $\lambda > 0$. When λ is negative, the existence of (2.8) is ensured by choosing, for example, $\phi_1(t) = \exp(\alpha t^2) > \phi(t) > \exp(\alpha|t|)$ where $\phi_1(t)$ evidently satisfies the conditions of lemma 2.1.

Define the matrix

$$P(x,s,a) = (P_{ij}(x,s,a)) = \int_{|x-s|}^\infty (I + K(x,t,s)) w(t,a)\, \mathrm{d}t \tag{2.9}$$

and

$$Q(x,s,a) = \int_{|x-s|}^\infty K(x,t,s) w(t,a)\, \mathrm{d}t. \tag{2.10}$$

Then from the inequality $|K(x,t,s)| \leqslant Ct^{a+1}$, $a > 0$ and C, constant (ref. 1, p. 136) and the inequality (2.7) it follows that $P(x,s,a)$ and $Q(x,s,a)$ are finite.

The following lemma is obtained next.

Lemma 2.2. For fixed x and a,

(i) $w(x-s,a) \in L_2(-\infty,\infty)$;

(ii) $P(x,s,a)$; $Q(x,s,a) \in L_2(-\infty,\infty)$.

The first part of the lemma follows from the definition of $w(x-s,a)$ and the inequality (2.7). To prove the second part we introduce the notation that for any $n \times n$ matrix $A = (a_{ij})$, $\|A\| = \max_{1 \leqslant i,j \leqslant n |a_{ij}|}$.

Then since $\|I + K(x,t,s)\| \leqslant 2 + \phi(t)$, by lemma 2.1, it follows that

$$\begin{aligned}
\int_{-\infty}^\infty \|P(x,t,s)\|^2 \mathrm{d}s &\leqslant \int_{-\infty}^\infty \mathrm{d}s \left(\int_{|x-s|}^\infty \|I + K(x,t,s)\| |w(t,a)|\, \mathrm{d}t \right)^2 \\
&\leqslant \int_{-\infty}^\infty \mathrm{d}s \left(\int_{|x-s|}^\infty (2 + \phi(t))|w(t)||\cos at|\, \mathrm{d}t \right)^2 \\
&\leqslant 1/\delta^2 \int_{-\infty}^\infty \mathrm{d}s \left(\int_{|x-s|}^\infty \exp(-\alpha t)\, \mathrm{d}t \right)^2 \\
&= 1/\delta^2\alpha^2 \int_{-\infty}^\infty \exp(-2\alpha|x-s|)\, \mathrm{d}s,
\end{aligned}$$

where we have utilised the inequality $w(t)(2+\phi(t)) \leqslant 1/\delta \exp(-\alpha t), t \geqslant \delta > 0$, which easily follows from the definition of $w(t)$ and $\phi(t)$. Thus $P(x,t,s) \in L_2(-\infty, \infty)$. Similarly for $Q(x,s,a)$ and the lemma follows.

3. The basic formula and certain consequences

It is easy to deduce from (A) that

$$\lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t\, \phi_j(x,\lambda) = \tfrac{1}{2} \int_{x-t}^{x+t} (I + \Omega(x,t,s)) \phi_j(s,\lambda)\, \mathrm{d}s$$
$$- \tfrac{1}{2}(\Omega(x,t,s)\Big|_{s=x+t} \int_0^{x+t} \phi_j(y,\lambda)\, \mathrm{d}y - \Omega(x,t,s)\Big|_{s=x-t} \int_0^{x-t} \phi_j(y,\lambda)\, \mathrm{d}y \tag{3.1}$$

(see ref. 1, p. 131).

Multiplying both sides of (3.1) by $w(t,a)$, we integrate over $(0,T)$, T arbitrary. Then adopting the usual mean-convergence analysis, we obtain, in view of (2.8), on changing the order of integration

$$\lambda^{-\frac{1}{2}} h(\lambda^{\frac{1}{2}}, a) \phi_j(x,\lambda) = \tfrac{1}{2} \int_{-\infty}^{\infty} P(x,s,a) \phi_j(s,\lambda)\, \mathrm{d}s \tag{3.2}$$

where $P(x,s,a)$ is given by (2.9). Similarly,

$$\lambda^{-\frac{1}{2}} h(\lambda^{\frac{1}{2}}, a) \theta_j(x,\lambda) = \tfrac{1}{2} \int_{-\infty}^{\infty} P(x,s,a) \theta_j(s,\lambda)\, \mathrm{d}s. \tag{3.3}$$

Using (3.2) and (3.3) and the results obtained from them by changing x to y, we obtain from the generalised Parseval relation (see ref. 2, p. 151) applied to different row vectors of $P(x,s,a)$, $P(y,s,a)$ that

$$\int_{-\infty}^{\infty} 1/\lambda\, h^2(\lambda^{\frac{1}{2}}, a)\, d_\lambda H(x,y,\lambda) = \tfrac{1}{4} \int_{-\infty}^{\infty} P(x,s,a) P^T(y,s,a)\, \mathrm{d}s \tag{3.4}$$

the right-hand side being finite by lemma 2.2 (ii).

For the Fourier system (for which $p = q = r = 0$) the corresponding formula is

$$\int_{-\infty}^{\infty} 1/\lambda\, h^2(\lambda^{\frac{1}{2}}, a)\, \mathrm{d}_\lambda H^F(x,y,\lambda) = \tfrac{1}{4} \int_{-\infty}^{\infty} P_F(x,y,a) P_F^T(y,s,a)\, \mathrm{d}s \tag{3.5}$$

where

$$P_F(x,s,a) = I \int_{|x-s|}^{\infty} w(t,a)\, \mathrm{d}t \tag{3.6}$$

the P-matrix for the Fourier case. It follows from (2.9), (2.10) and (3.6) that

$$P(x,s,a)P^T(y,s,a) - P_F(y,s,a)P_F^T(y,s,a)$$
$$= P_F(x,s,a)Q^T(y,s,a) + Q(x,s,a)P^T(y,s,a).$$

Then, from (3.4) and (3.5)

$$\int_{-\infty}^{\infty} 1/\lambda h^2(\lambda^{\frac{1}{2}}, a)\, \mathrm{d}_\lambda(H(x,y,\lambda) - H^F(x,y,\lambda))$$
$$= \tfrac{1}{4}\int_{-\infty}^{\infty} (P_F(x,s,a)Q^T(y,s,a) + Q(x,s,a)P^T(y,s,a))\, \mathrm{d}s, \tag{3.7}$$

where the existence of the right-hand integral follows from lemma 2.2 (ii). Differentiating (3.1) with respect to x, multiply both sides of the result so obtained by $w(t,a)$ and proceed as before so as to obtain finally

$$\int_{-\infty}^{\infty} 1/\lambda h^2(\lambda^{\frac{1}{2}}, a)\, \mathrm{d}_\lambda(\partial^2/\partial x\, \partial y(H(x,y,\lambda) - H^F(x,y,\lambda))$$
$$= \tfrac{1}{4}I\int_{-\infty}^{\infty} (w(x-s,a)Q^T(y,s,a) + w(y-s,a)Q(x,s,a))\, \mathrm{d}s$$
$$+ \tfrac{1}{4}\int_{-\infty}^{\infty} Q(x,s,a)Q^T(y,s,a)\, \mathrm{d}s \tag{3.8}$$

and

$$\int_{-\infty}^{\infty} 1/\lambda h^2(\lambda^{\frac{1}{2}}, a)\, \mathrm{d}_\lambda(\partial/\partial x(H(x,y,\lambda) - H^F(x,y,\lambda))$$
$$= \tfrac{1}{4}I\int_{-\infty}^{\infty} w(x-s,a)Q^T(y,s,a)\, \mathrm{d}s + \tfrac{1}{4}\int_{-\infty}^{\infty} Q(x,s,a)P^T(y,s,a)\, \mathrm{d}s. \tag{3.9}$$

The convergence of the integrals on the right of (3.8) and (3.9) follows from lemma 2.2. Similar results hold when $H(\cdot)$ is replaced by $\mathscr{H}(\cdot)$. Results (3.7)–(3.9) are basic in the investigations that follow.

Lemma 3.1. For fixed x, y (or, if x, y vary uniformly in a bounded domain),

$$\lim_{a\to\infty}\int_{-\infty}^{0} 1/\lambda h^2(\lambda^{\frac{1}{2}}, a)\, \mathrm{d}_\lambda\, Y(x,y,\lambda) = 0,$$

where $Y(x,y,\lambda)$ is equal to either $H(x,y,\lambda)$ or $\partial/\partial x\, H(x,y,\lambda)$ or $\partial^2/\partial x\,\partial y\, H(x,y,\lambda)$. The lemma is also true when $H(\cdot)$ is replaced by $\mathscr{H}(\cdot)$.

We establish the case when $Y(x,y,\lambda) = H(x,y,\lambda)$. The other cases follow similarly.

Let t_0 be an arbitrary positive number. Then it follows from (3.1) that

$$\lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t_0\, \phi_j(x,y,\lambda) = \tfrac{1}{2}\int_{x-t_0}^{x+t_0} R(x,s,t_0)\phi_j(s,\lambda)\, \mathrm{d}s \tag{3.10}$$

with a similar expression for $\lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t_0 \theta_j(x,\lambda)$, where

$$R(x,s,t) = I + K(x,s,t). \tag{3.11}$$

Similar results also hold when one replaces x by y.

Hence by the generalised Parseval relation[2] (p. 151)

$$\int_{-\infty}^{\infty} 1/\lambda \sin^2 \lambda^{\frac{1}{2}} t_0 \, \mathrm{d}_\lambda H(x, y, \lambda) = \tfrac{1}{4} \int_{\Delta xy} R(x, s, t_0) R^T(y, s, t_0) \, \mathrm{d}s \tag{3.12}$$

where

$$\Delta_{xy} = (x - t_0, x + t_0) \bigcap (y - t_0, y + t_0).$$

Put $\lambda = \mu^2$ so that $H(x, y, \lambda) = H_1(x, y, \mu)$; $H_1(x, y, \mu)$ is continued to negative values of μ as a matrix whose elements are odd functions of μ. Then from (3.12) and the relations $\| R(x, s, t_0) \|, \| R(y, s, t_0) \| \leqslant 2 + \phi(t_0)$, it follows that

$$\int_{-\infty}^{0} \mu^{-2} \sin^2 \mu t_0 \, \mathrm{d}_\mu H_1(x, y, \mu) \ll \tfrac{1}{4} \cdot 1/w(t_0) I \tag{3.13}$$

where the symbol $\ll$ means that the matrix on the right-hand side majorises that on the left[8] (p. 328). Now

$$\begin{aligned} I_1 &= \int_{-\infty}^{0} 1/\mu^2 h^2(\mu, a) \, \mathrm{d}_\mu H_1(x, y, \mu) \\ &= \int_{-\infty}^{0} \left(\int_0^\infty \sin \mu t/\mu \, w(t) \cos at \, \mathrm{d}t \right)^2 \mathrm{d}_\mu H_1(x, y, \mu) \\ &= \int_{-\infty}^{0} \int_0^\infty \sin \mu t/\mu \, w(t) \cos at \, \mathrm{d}t \int_0^\infty \sin \mu s/\mu \, w(s) \cos as \, \mathrm{d}s \, \mathrm{d}_\mu H_1(x, y, \mu) \\ &= \int_0^\infty \int_0^\infty w(t) w(s) \cos at \cos as \, \mathrm{d}t \, \mathrm{d}s F(x, y, s, t), \end{aligned}$$

by an easily verifiable change in the order of integration, where

$$F(x, y, s, t) = \int_{-\infty}^{0} \sin \mu t/\mu \sin \mu s/\mu \, \mathrm{d}_\mu H_1(x, y, \mu).$$

Then making use of the formula $\| \int C(u) \, \mathrm{d}u \| \leqslant \int \| C(u) \| \, \mathrm{d}u$, for a continuous $n \times n$ matrix[8] (p. 343), and the Schwarz inequality, we obtain, in view of (3.13),

$$\| F(x, y, s, t) \| \leqslant \tfrac{1}{4} (w(t) w(s))^{-\frac{1}{2}}. \tag{3.14}$$

The analysis now proceeds as in Levitan[5] (pp. 236–237) and the lemma is proved for $Y(x, y, \lambda) = H(x, y, \lambda)$. The other parts of the lemma involving the derivatives of $H(x, y, \lambda)$ and the lemma with $H(\cdot)$ replaced by $\mathscr{H}(\cdot)$ follow similarly.

The following lemma involves limits as a tends to infinity of expressions containing $P(x, s, a)$ and $Q(x, s, a)$ defined in (2.8) and (2.9), respectively.

Lemma 3.2.

(i) $\lim_{a \to \infty} \int_{-\infty}^{\infty} \| P(x, s, a) \|^2 \, \mathrm{d}s = 0;$

$$\text{(ii)} \lim_{a\to\infty} \int_{-\infty}^{\infty} \| Q(x,s,a)\|^2 \, \mathrm{d}s = 0,$$

uniformly for x in a bounded interval.

We prove result (i); result (ii) follows similarly. Let N be an arbitrary positive number. Then

$$I = \int_{-\infty}^{\infty} \| P(x,s,a)\|^2 \, \mathrm{d}s \leqslant 2 \int_{|s|\leqslant N} \mathrm{d}s \left\| \int_{|x-s|}^{\infty} (I + K(x,t,s))w(t,a)\,\mathrm{d}t \right\|^2$$
$$+ 2 \int_{|s|>N} \mathrm{d}s \int_{|x-s|}^{\infty} \|(I + K(x,t,s))w(t,a)\,\mathrm{d}t\|^2 = I_{11} + I_{12}, \quad \text{say.}$$

Now

$$I_{12} \leqslant 2 \int_{|s|>N} \mathrm{d}s \left(\int_{|x-s|}^{\infty} \|I + K(x,t,s)\| |w(t)| |\cos at| \,\mathrm{d}t \right)^2$$
$$\leqslant 2 \int_{|s|>N} \mathrm{d}s \left(\int_{|x-s|}^{\infty} (2 + \phi(t)) w(t)\,\mathrm{d}t \right)^2$$
$$\leqslant 2/\delta^2 \int_{|s|>N} \exp(-2\alpha|x-s|)\,\mathrm{d}s = o(1),$$

if n is large enough. Having so chosen N, we have

$$I_{11} = 2 \int_{|s|\leqslant N} \mathrm{d}s \left\| \left(\int_{|x-s|}^{X} + \int_{X}^{\infty} \right) (I + K(x,t,s))w(t) \cos at \,\mathrm{d}t \right\|^2$$
$$\leqslant 4 \int_{|s|\leqslant N} \mathrm{d}s \left\| \int_{|x-s|}^{X} (I + K(x,t,s))w(t) \cos at \,\mathrm{d}t \right\|^2$$
$$+ 4 \int_{|s|\leqslant N} \mathrm{d}s \left\| \int_{X}^{\infty} (I + K(x,t,s))w(t) \cos at \,\mathrm{d}t \right\|^2 = J_1 + J_2, \text{ say.}$$

For fixed N, X, J_1 tends to zero as a tends to infinity, by the Riemann-Lebesgue lemma. Also by arguments as before,

$$J_2 \leqslant 4/\delta^2 \int_{|s|\leqslant N} \exp(-2\alpha X)\,\mathrm{d}s = o(1),$$

as X tends to infinity. Similarly, for

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} \| Q(x,s,a)\|^2 \, \mathrm{d}s.$$

The lemma therefore follows.

4. The theorem

The theorem as proposed in section 1, connecting $\chi_1(x,y,\mu+a)$ and $\chi_1(x,y,a)$ as a tends to infinity, is stated as follows.

Theorem 4.1. Let $x, y, \mu_0 > 0$, be fixed or let $\mu_0 > 0$ be fixed and x, y lie in a bounded domain D. Then uniformly in D,

$$\lim_{a\to\infty} (\chi_1(x, y, \mu_0 + a) - \chi_1(x, y, a)) = 0.$$

Put

$$\Phi_1 = D^P(H_1(x, y, \mu) - H_1^F(x, y, \mu)) \quad \text{and} \quad \Phi_2 = D^P(\mathscr{H}_1(x, y, \mu) - \mathscr{H}_1^F(x, y, \mu))$$

where

$$D^P = D^{s+t} = \partial^{s+t}/\partial x^s\, \partial y^t.$$

Then, since $\chi_1(x, y, \mu) = \Phi_1(x, y, \mu) - \Phi_2(x, y, \mu)$, the theorem follows by showing that

$$\lim_{a\to\infty} (\Phi_j(x, y, \mu_0 + a) - \Phi_j(x, y, a)) = 0, \quad j = 1, 2. \tag{4.1}$$

We shall prove (4.1) for the cases $s = 0,\ t = 0$; $s = 1,\ t = 0$; $s = 0,\ t = 1$ and $s = 1,\ t = 1$. Extension to higher order derivatives follows easily. The proof of (4.1) and therefore that of the theorem depends upon the following lemmas.

Lemma 4.1. If $k(\mu) = \int_0^\infty \sin \mu t w(t)\, \mathrm{d}t$, where $w(t)$ is defined as in section 2, then x, y lying in any fixed interval,

$$\lim_{a\to\infty} \int_0^\infty k(\mu + a)k(\mu - a)/\mu^2\, \mathrm{d}_\mu Y(x, y, \mu) = 0,$$

where $Y(x, y, \mu)$ is either $H_1(x, y, \mu)$ or $\partial/\partial x H_1(x, y, \mu)$ or $\partial^2/\partial x\, \partial y H_1(x, y, \mu)$ or similar expressions with H_1 replaced by $\mathscr{H}_1$.

We prove the result for $Y(x, y, \mu) = H_1(x, y, \mu)$; the other cases follow similarly. Obviously $k(\mu)$ is an odd function of μ; so that $k(\mu + a)k(\mu - a)$ is even. Put

$$\begin{aligned} J &= \int_{-\infty}^{\infty} 1/\mu^2 k(\mu + a)k(\mu - a)\, \mathrm{d}_\mu H_1(x, y, \mu) \\ &= \left(\int_{-\infty}^{0} + \int_0^\infty\right) 1/\mu^2 k(\mu + a)k(\mu - a)\, \mathrm{d}_\mu H_1(x, y, \mu) = J_1 + J_2, \text{say.} \end{aligned}$$

Now

$$\begin{aligned} \| J_2 \| &\leqslant \int_0^\infty 1/\mu^2 k(\mu + a)k(\mu - a) \| \mathrm{d}_\mu H_1(x, y, \mu) \| \\ &\leqslant \left(\int_0^\infty 1/\mu^2 k^2(\mu + a) \| \mathrm{d}_\mu H_1(x, y, \mu) \| \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^\infty 1/\mu^2 k^2(\mu - a) \| \mathrm{d}_\mu H_1(x, y, \mu) \| \right)^{\frac{1}{2}} \end{aligned} \tag{4.2}$$

by the Schwarz inequality. By integration by parts

$k(\mu) = 0(1/\mu)$ and $\sup_{-\infty<\mu<\infty} V_\mu^{\mu+1} H_1(x, y, \mu) \ll C$, where the constant C depnds on x_0, x_1; $x_0 < x,\ y < x_1$, the interval (x_0, x_1) is arbitrary but fixed[1] (p. 135).

The notation $V_\alpha^\beta(\cdot)$ means variation of $(\cdot)$ on the interval (α, β). Hence for all $\mu \geqslant \delta > 0$,

$$\int_\delta^\infty 1/\mu^2 k^2(\mu + a)\, \mathrm{d}_\mu H_1(x, y, \mu) = O\left(\int_0^\infty (\mu + a)^{-2}\, \mathrm{d}_\mu H_1(x, y, \mu)\right)$$

$$= O\left(\sum_0^\infty (\mu + a)^{-2}\right) = O\left(\sum_0^N (\mu + a)^{-2} + \sum_N^\infty \mu^{-2}\right) = O(1),$$

as a tends to infinity. Therefore, by a well-known theorem on Stieltjes integral[9] (p. 437),

$$\int_0^\infty k^2(\mu + a)/\mu^2\, \mathrm{d}_\mu H_1(x, y, \mu)$$

exists and is equal to $O(1)$, as a tends to infinity uniformly for x, y lying in a fixed interval. Along with this we observe that

$$\int_\delta^\infty k^2(\mu - a)/\mu^2\, \mathrm{d}_\mu H_1(x, y, \mu) = O\left(\int_0^\infty k^2(\mu - a)\, \| \mathrm{d}_\mu H_1(x, y, \mu) \|\right)$$

$$= O\left(\int_{-\infty}^{-1} \mu^{-2} \| \mathrm{d}_\mu H_1(x, y, \mu) \| + \int_{-1}^{1} (\mu - a)^{-2} \| \mathrm{d}_\mu H_1(x, y, \mu) \|\right.$$

$$\left. + \int_1^\infty \mu^{-2} \| \mathrm{d}_\mu H_1(x, y, \mu) \|\right)$$

$$= O\left(\sum_{-\infty}^{-1} \mu^{-2} + \sum_1^\infty \mu^{-2}\right) = O(1),$$

for all $\mu \geqslant \delta > 0$; a is large enough. Hence it follows that

$$\int_0^\infty \mu^{-2} k^2(\mu - a)\, \mathrm{d}_\mu H_1(x, y, \mu) \tag{4.3}$$

is finite. Therefore, from considerations made before, J_2 tends to zero as a tends to infinity uniformly for x, y in a finite interval. Similarly for J_1. The lemma thus follows.

The following lemma is now established.

Lemma 4.2. In any fixed interval $x_0 < x,\ y < x_1$,

$$\lim_{a \to \infty} \int_{-\infty}^\infty \mu^{-2} k^2(\mu - a)\, \mathrm{d}_\mu \Phi_j(x, y, \mu) = 0,$$

where $\Phi_j(\cdot)$, $j = 1, 2$, are those which occur in (4.1) and $\Phi_j(x, y, \mu)$ are continued to the negative values of μ as matrices whose elements are odd functions of μ. The result holds uniformly for x, y in the given interval.

We establish the case when $\Phi_1(x, y, \mu) = H_1(x, y, \mu) - H_1^F(x, y, \mu)$. The other cases follow similarly.

It easily follows that

$$\mu^{-2} h^2(\mu, a) = \tfrac{1}{4}\mu^{-2}(k(\mu + a) + k(\mu - a))^2 \tag{4.3a}$$

In (3.7) put

$$\lambda = \mu^2 \quad \text{and} \quad H(x,y,\lambda) = H_1(x,y,\mu), H^F(x,y,\lambda) = H_1^F(x,y,\mu).$$

Then

$$\left\| \int_{-\infty}^{\infty} \mu^{-2} h^2(\mu,a)\, \mathrm{d}_\mu \Phi_1(x,y,\mu) \right\| \leqslant \frac{1}{4} \int_{-\infty}^{\infty} (\|P_F(x,s,a)\| \, \|Q^T(y,s,a)\| + \|Q(x,s,a)\| \, \|P^T(y,s,a)\|)\, \mathrm{d}s.$$

Substituting for $h(\mu, a)$ by the relation (4.3a), then applying the Schwarz inequality and lemma 3.2 followed by lemmas 3.1 and 4.1, it follows that

$$\lim_{a\to\infty} \int_0^{\infty} \mu^{-2}(k^2(\mu+a) + k^2(\mu-a))\, \mathrm{d}_\mu \Phi_1(x,y,\mu) = 0 \tag{4.4}$$

uniformly for x, y in the given finite interval. Since

$$\int_0^{\infty} \mu^{-2} k^2(\mu+a)\, \mathrm{d}_\mu \Phi_1(x,y,\mu) = \int_{-\infty}^{0} \mu^{-2} k^2(\mu-a)\, \mathrm{d}_\mu \Phi_1(x,y,\mu)$$

the lemma follows from (4.4).

We have the following Tauberian theorem due to Wiener as modified by Levitan[5] (p. 241).

Tauberian theorem

Let (i) $f(\mu)$, $g(\mu)$ be two bounded and measurable functions which satisfy $f(\mu), g(\mu) = O(1/\mu^2)$;
(ii) $\theta(\mu)$ be a function which satisfies $\sup_{-\infty<\mu<\infty} V_\mu^{\mu+1} \theta(\mu) < \infty$;
(iii) the Fourier transform of f does never vanish. Then

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} f(\mu-a)\, \mathrm{d}\theta(\mu) = 0$$

implies

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} g(\mu-a)\, \mathrm{d}\theta(\mu) \doteq 0.$$

To prove theorem 4.1 we choose, in the Tauberian theorem quoted above, $f(\mu) = \mu^{-2} k^2(\mu)$; so that $f(\mu) = O(1/\mu^4) = O(1/\mu^2)$ and $\theta(\mu)$ elements of $\Phi_j(x, y, \mu)$. Then closely following Levitan[5] (p. 241) we obtain from lemma 4.2,

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} g(\mu-a)\, \mathrm{d}_\mu \Phi_j(x,y,\mu) = \lim_{a\to\infty} \int_a^{\mu_0+a} \mathrm{d}_\mu \Phi_j(x,y,\mu) = 0,$$

where $g(\mu) = 1$ for $0 \leqslant \mu \leqslant \mu_0$, but $g(\mu) = 0$ otherwise. This proves (4.1) and hence the theorem 4.1 follows.

5. Some special asymptotic formulae

The following asymptotic formulae are derived from theorem 4.1.

(i) $\lim_{\lambda\to\infty} H_1(x,x,\lambda)/\lambda = I/\pi$, where I is the 2×2 unit matrix.

Since $\lim_{x \to y} H_1^F(x, y, \mu) = I/\pi \quad \lim_{x \to y} \sin \mu(x - y)/(x - y) = I\mu/\pi$,

for a fixed μ (see ref. 3 for the explicit expression for $H_1^F(x, y, \mu)$), it follows from theorem 4.1 that

$$\lim_{a \to \infty} (H_1(x, x, \mu_0 + a) - H_1(x, x, a)) = I\mu_0/\pi.$$

Changing a to $a - \lambda$, μ_0 to 2 so that $\mu_0 + a$ changes to $a + \lambda$, we have

$$\lim_{a \to \infty} (H_1(x, x, a + \lambda) - H_1(x, x, a - \lambda)) = 2I/\pi. \tag{5.1}$$

Put $a + \lambda = N + 2k + 1, a - \lambda = N + 2k - 1$ so that $\lambda = 1$, where $k \geqslant 0$. If N is a fixed positive integer as large as we please, then from (5.1) it follows that for an arbitrary $\varepsilon > 0$, there exists an integer N such that

$$H_1(x, x, N + 2k + 1) - H_1(x, x, N + 2k - 1) = 2/\pi \cdot I + \varepsilon_k,$$

where $|\varepsilon_k| < \varepsilon$ for all $k \geqslant 0$. Putting $k = 0, 1, 2, \ldots, n$ and then summing, we have

$$H_1(x, x, N + 2n + 1) - H_1(x, x, N - 1) = 2(n + 1)/\pi + \eta_\pi(n + 1),$$

where

$$|\eta_n| = |(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_n)/(n + 1)| < \varepsilon.$$

Thus

$$|H_1(x, x, N + 2n + 1)/(N + 2n + 1) - I/\pi| < \varepsilon,$$

if $n \geqslant \delta_1$, where δ_1 is large enough. Hence the result.

In particular, putting $x = 0$ and noting that $H_1(0, 0, \lambda)$ is the matrix $\rho(\lambda)$[1] (p. 144), we obtain

$$\lim_{\lambda \to \infty} \rho(\lambda)/\lambda = I/\pi,$$

which tallies with the result obtained in ref. 1, p. 144 by a different consideration.

$$\text{(ii)} \quad \lim_{a \to \infty} a^{-2}(h_1(x, x, \mu_0 + a) - h_1(x, x, a)) = I/\pi\mu_0 \tag{5.2}$$

where

$$h_1(y, y, \mu) = \lim_{x \to y} (x - y)^{-1} \partial/\partial x \, H_1(x, y, \mu).$$

Since

$$\lim_{x \to y} (x - y)^{-1} \partial/\partial x \sin \mu(x - y)/(x - y) = -1/3 \, \mu^3$$

therefore

$$\lim_{a \to \infty} a^{-2} \lim_{x \to y} (x - y)^{-1} \partial/\partial x (H_1^F(x, y, \mu_0 + a) - H_1^F(x, y, a)) = I/\pi\mu_0.$$

Hence the result follows from theorem 4.1.

$$\text{(iii)} \quad \lim_{a \to \infty} a^{-2}(h_2(x, x, \mu_0 + a) - h_2(x, x, a)) = I/\pi\mu_0 \tag{5.3}$$

where

$$h_2(y, y, \mu) = \lim_{x \to y} \partial^2/\partial x \, \partial y H_1(x, y, \mu).$$

Since

$$\lim_{x \to y} \partial^2/\partial x\, \partial y \sin \mu(x-y)/(x-y) = 1/3\mu^3,$$

therefore

$$\lim_{a \to \infty} a^{-2} \lim_{x \to y} \partial^2/\partial x\, \partial y (H_1^F(x, y, \mu_0 + a) - H_1^F(x, y, a)) = I/\pi \mu_0.$$

The result therefore follows from theorem 4.1.

It is to be noted that (5.1)–(5.3) are valid uniformly for all x lying in a fixed interval.

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