J Indian Inst. Sci, Sept.–Oct 1989, 69, 361–366.

Propagation of elastic disturbance in a hollow cone of nonhomogeneous material under a longitudinal impact

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Received on November 25, 1988; Revised on April 2, 1989

Abstract

Solution for the problem of elastic disturbance in a nonhomogeneous hollow cone under a longitudinal impact has been reduced to one of a Volterra integral equation of the first kind. A numerical method for the solution of the integral equation has been given and the results illustrated graphically.

Key words: Nonhomogeneous material, longitudinal impact, elastic disturbance, Volterra integral equation, Laplace transform.

1. Introduction

Propagation of elastic disturbance in a semi-infinite hollow cone with slightly truncated apex has been considered by a number of investigators[1–3]. In some of the papers, uniaxial theory was followed and the solution was obtained in the form of an integral equation.

In this paper we shall discuss the propagation of elastic disturbance in a truncated hollow cone due to the application of time-dependent normal force at one end. We have solved a similar problem[4] on assuming that the Laplace transform parameter is large. The solution of the problem obtained by this method is valid only for small values of time. This difficulty is removed here by adopting entirely different technique by which the solution of the problem is reduced to the solution of Volterra integral equation of the first kind. Following Kromm[5], a numerical method for the solution of such equation has been suggested. In deriving the solution, no asymptotic expansion has been required and hence stresses and displacements may be computed for all values of time and for all positions. Numerical evaluations of the strain component have been done for particular values of the nonhomogeneity parameter and the results have been shown graphically.

2. Formulation of the problem

Choosing the x-axis along the axis of the hollow cone and the origin at the apex of the outside surface of the cone, the one-dimensional equation of the elastic wave propagation, when the effects of lateral inertia and radial shear are neglected, becomes

$$\frac{\partial\sigma}{\partial x}+\frac{\sigma}{x-\frac{h}{2}\operatorname{cosec}\beta_0}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where the axial stress $\sigma=\sigma(x,t)=E(x)(\partial u/\partial x)(x,t)$ is measured at an axial distance x and at time t. In (1), h represents the wall thickness, β_0 the semi-vertical angle, ρ the material density of the cone and $E(x)$ the variable Young's modulus of the material. Assuming the material nonhomogeneity in the following form

$$E(\xi)=E_0\xi^m, \quad \rho(\xi)=\rho_0\xi^{s_0} \tag{2}$$

and introducing nondimensional variables ξ and τ, (1) can be written as

$$\xi^2\frac{\partial^2 u}{\partial\xi^2}(\xi,\tau)+(1+m)\xi\frac{\partial u}{\partial\xi}(\xi,\tau)=\xi^{n+2}\frac{\partial^2 u}{\partial\tau^2}(\xi,\tau). \tag{3}$$

In (2), m and s_0 are any real numbers, and in (3),

$$\xi=(x-h/2\operatorname{cosec}\beta_0)/d,$$

$$\tau=c_0 t/d,$$

$$c_0=(E_0/\rho_0)^{1/2}$$

and

$$n=s_0-m, \tag{4}$$

d being the distance between apex of the middle surface and truncated end of the cone.

The initial and the boundary conditions of the problem are

$$u(\xi,0)=\frac{\partial u(\xi,0)}{\partial\tau}=0 \tag{5}$$

$$\varepsilon(1,\tau)=\left.\frac{\partial u(\xi,\tau)}{\partial\xi}\right|_{\xi=1}=\begin{cases}\varepsilon_0\sin^2(\pi t/\tau_1), & 0\leqslant\tau\leqslant\tau_1\\ 0, & \tau_1<\tau\end{cases}. \tag{6}$$

Hence the problem concerned reduces to find the solution of equation (3) subject to the initial conditions (5), the boundary condition (6) and also the regularity condition that the displacement $u(\xi,\tau)$ and strain $\varepsilon(\xi,\tau)$ should tend to zero as $\xi\to\infty$.

3. Solution of the problem

Making use of the initial conditions (5), the Laplace transforms of equation (3) and the boundary condition (6) yield

$$\xi^2\frac{d^2\bar{u}}{d\xi^2}(\xi,p)+(1+m)\xi\frac{d\bar{u}}{d\xi}(\xi,p)-p^2\xi^{n+2}\bar{u}(\xi,p)=0, \tag{7}$$

and

$$\bar{\varepsilon}(1,p)=\frac{2\pi^2\varepsilon_0}{p(p^2\tau_1^2+4\pi^2)}[1-\exp(-p\tau_1)]. \tag{8}$$

For $n \neq -2$, equation (7) may be transformed into a modified Bessel's equation, whose solution is known. Hence the general solution of (7), when $n \neq -2$, may be written as

$$\bar{u}(\xi, p) = A\xi^{-m/2} I_\beta(p\xi^s/s) + B\xi^{-m/2} K_\beta(p\xi^s/s) \tag{9}$$

where

$$\beta = m/(n+2), \; s = 1 + (n/2).$$

Equation (7) assumes simple form for $n = -2$. We shall consider this special case later.

To get the appropriate solution for our problem for $n \neq -2$ we have to consider the following two cases:

Case 1. $n > -2$. In this case $s > 0$ and the argument of the Bessel functions in (9) is positive for $p > 0$ and tends to infinity as $\xi \to \infty$. Imposing the regularity condition $\bar{u}(\infty, p) = 0$, the appropriate solution in this case will be

$$\bar{u}(\xi, p) = B\xi^{-m/2} K_\beta(p\xi^s/s). \tag{10}$$

Case 2. $n < -2$. In this case $s < 0$, consequently the argument of the modified Bessel function I_β and K_β in (9) is negative for $p > 0$, so neither $I_\beta(z)$ nor $K_\beta(z)$ is real valued. Moreover, the argument now tends to zero as $\xi \to \infty$. Since both $I_\beta(z)$ and $K_\beta(z)$ have linearly independent singularities at $z = 0$, the general solution (9) cannot satisfy the regularity requirement. This means that $u(\xi, \tau)$ no longer tends to zero at all times as $\xi \to \infty$. Thus, we are forced to abandon the regularity condition, but then the single boundary condition (6) will be insufficient to determine both A and B in (9). Under the circumstances, it has been pointed out by Sternberg and Chakravorty[6] that for the possibility of a diverging wave in the physical domain, the Bessel function of first kind $I_\beta(z)$ is inadmissible. Thus setting $A = 0$, we get the solution of (7) in this case as

$$\bar{u}(\xi, p) = B\xi^{-m/2} K_\beta(p\xi^s/s).$$

From (10) we find $\bar{\varepsilon}(\xi, p)$, then using (8) we obtain

$$\bar{\varepsilon}(\xi, p) = \xi^{(n-m)/2} \bar{\varepsilon}(1, p) \frac{K_{\mu^*}(\hat{\xi} p)}{K_{\mu^*}(\hat{d} p)} \tag{11}$$

where

$$\mu^* = (s_0 + 2)/(n+2), \; \hat{d} = 2/(n+2), \; \hat{\xi} = \hat{d}\xi^{1/\hat{d}}. \tag{12}$$

Assuming $\bar{\varphi}(\hat{\xi}, p) = K_{\mu^*}(\hat{\xi} p)/p$, the use of convolution theorem for the inverse Laplace transform in the inversion of (11), yields

$$\int_0^\tau \varepsilon(\xi, \lambda) \varphi(\hat{d}, \tau - \lambda)\, \mathrm{d}\lambda = \xi^{(n-m)/2} \int_0^\tau \varepsilon(1, \lambda) \varphi(\hat{\xi}, \tau - \lambda)\, \mathrm{d}\lambda \tag{13}$$

which gives

$$\varphi(\hat{\xi}, \tau) = L^{-1}[\bar{\varphi}(\hat{\xi}, p)] = H(\tau - \hat{\xi}) \varphi_1(\hat{\xi}, \tau), \tag{14}$$

where

$$\varphi_1(\hat{\xi}, \tau) = \frac{1}{\mu^*} \sinh[\mu^* \cosh^{-1}(\tau/\hat{\xi})].$$

With the help of (6) and (14), equation (13) becomes

$$H(\tau - \hat{d}) \int_0^{\tau - \hat{d}} \varepsilon(\xi, \lambda)\varphi_1(\hat{d}, \tau - \lambda)\,\mathrm{d}\lambda$$

$$= \varepsilon_0 H(\tau - \hat{\xi})\xi^{(n-m)/2} \int_0^{\tau - \hat{\xi}} \sin^2[\pi(\tau - \hat{\xi} - \eta)/\tau_1]$$

$$\times H(\tau_1 + \hat{\xi} - \tau - \eta)\varphi_1(\hat{\xi}, \eta + \hat{\xi})\,\mathrm{d}\eta. \qquad (15)$$

Now, introducing the notations

$$\alpha = \hat{\xi} - \hat{d}$$

and

$$\tau_2 = \tau - \hat{d}$$

(15) may be rewritten as

$$H(\tau_2) \int_\alpha^{\tau_2} \varepsilon(\xi, \lambda)\varphi_1(\hat{d}, \tau - \lambda)\,\mathrm{d}\lambda$$

$$= \varepsilon_0 \xi^{(n-m)/2} H(\tau_2 - \alpha) \int_0^{\tau_2 - \alpha} \sin^2\{\pi(\tau_2 - \alpha - \eta)/\tau_1\}$$

$$\times H(\tau_1 + \eta - \tau_2 + \alpha)\varphi_1(\hat{\xi}, \eta + \hat{\xi})\,\mathrm{d}\eta.$$

Since the right-hand side of this equation contains $H(\tau_2 - \alpha)$, it follows that there will be no disturbance in the medium at the position ξ before $\tau_2 < \alpha$ and the disturbance reaches the position at time $\hat{d} + \alpha = \hat{\xi}$.

Also, for $n \neq -2$, $\tau_2 > \alpha \Rightarrow \tau_2 > 0$. Thus, for $n \neq -2$, $\tau_2 > \alpha$, we have

$$\int_\alpha^{\tau_2} \varepsilon(\xi, \lambda)\varphi_1(\hat{d}, \tau - \lambda)\,\mathrm{d}\lambda$$

$$= \varepsilon_0 \xi^{(n-m)/2} \int_0^{\tau_2 - \alpha} \sin^2\left\{\frac{\pi}{\tau_1}(\tau_2 - \alpha - \eta)\right\}$$

$$\times H(\tau_1 + \eta - \tau_2 + \alpha)\varphi_1(\hat{\xi}, \eta + \hat{\xi})\,\mathrm{d}\eta. \qquad (16)$$

The value of the strain when the disturbance arrives at ξ may be computed as

$$\varepsilon(\xi, \tau_2)|_{\tau_2 \to \alpha} = 0.$$

To evaluate the strain $\varepsilon(\xi, \tau)$ for any $\xi > 1$ and $\tau > \hat{\xi}$, numerical procedure has to be followed. We may adopt the method of Kromm[5] in which the integral is replaced by a system of linear algebraic equations in the unknowns ε, assuming that the strain ε is sectionally constant over the range of integration. Thus we may write

$$\sum_{M=1}^{N} \int_{\delta_{M-1}}^{\delta_M} \varepsilon(\xi, \lambda)\varphi_1(\hat{d}, \delta_N + \hat{d} - \lambda)\,\mathrm{d}\lambda$$

$$= \varepsilon_0 \xi^{(n-m)/2} \int_0^{\tau_2 - \alpha} \sin^2\{\pi(\tau_2 - \alpha - \eta)/\tau_1\} H(\tau_1 + \eta - \tau_2 + \alpha)\varphi_1(\hat{\xi}, \eta + \hat{\xi})\,\mathrm{d}\eta$$

where $\delta_0 = \alpha$, $\delta_1 = \delta_0 + h^*$, $\delta_2 = \delta_0 + 2h^*, \ldots, \delta_N = \tau - \hat{d} = \tau_2$.

For $n = -2$, the appropriate solution is

$$\bar{u}(\xi, p) = B\xi^{-[m+(m^2+4p^2)^{1/2}]/2}$$

Using (8), we may obtain

$$\bar{\varepsilon}(\xi, p) = \bar{\varepsilon}(1, p)\xi^{-[\{m+(m^2+4p^2)^{1/2}\}/2+1]}.$$

Taking Laplace inversion we get

$$\varepsilon(\xi, \tau) = \xi^{-(m/2+1)}\left[\int_0^\tau \varepsilon(1, \xi) g(\tau - \xi)\, \mathrm{d}\xi\right]$$

where

$$g(\tau) = L^{-1}[F(p^2 + m^2/4)^{1/2}]$$
$$= f(\tau) - (m/2)\int_0^\tau f[(\tau^2 - u^2)^{1/2}] J_1(mu/2)\, \mathrm{d}u,$$
$$f(\tau) = L^{-1}[F(p)]$$
$$= \delta(\tau - b),$$
$$b = \ln \xi$$

and

$$F(p) = \exp(-bp).$$

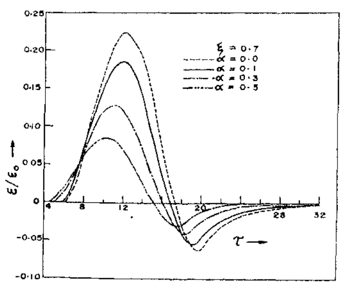

FIG. 1.

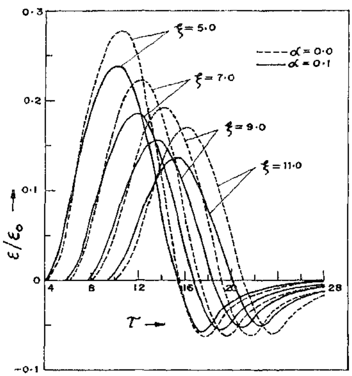

FIG. 2.

FIGS 1 and 2. Variation of strain with time.

4. Numerical calculation

To study the effect of nonhomogeneity on the magnitude of strain, some numerical computations have been done. In our computation, we have assumed $\tau_1 = 14.51$. The variations of the strain field with time for fixed position have been represented in figs 1–2. The corresponding results of associated homogeneous case are shown by broken lines in each figure.

Acknowledgement

The authors wish to thank the referee for valuable suggestions to have the paper in its present form.

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