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An exact solution for the unsteady flow of a micropolar fluid due to eccentrically rotating disks

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Abstract

A class of exact solutions is obtained for the unsteady flow of a micropolar fluid confined between two non-coaxially rotating parallel disks which are subjected to elliptic harmonic oscillations. These infinite one-parameter family of solutions reduce to a single unique solution when one prescribes the pressure gradient similar to a Poiseuille flow. The expressions for the component of forces and couples acting on the plate are obtained.

Key words: Micropolar fluids, rotating disks.

1. Introduction

The axisymmetric flows above an infinite single rotating disk and between two infinite coaxial rotating parallel disks have become the subject matter of many research workers, after the pioneering work of Von Kármán[1]. Recently, Berker[2] has established the existence of asymmetric solutions for a flow confined between two coaxially or non-coaxially rotating disks which are very much different from the axisymmetric solutions discussed earlier. The flow between two non-coaxially rotating disks is directly useful in elucidating the principles behind an instrument orthogonal rheometer which is used to determine the complex viscosity of viscoelastic fluids. The literature concerning these flows and other viscoelastic fluid flows has been reviewed excellently in a recent article by Huilgol and Phan-Thien[3].

Theories including couple stresses in the fluid medium have been developed to model the behaviour of fluids such as blood, liquid crystals and polymeric suspensions. Eringen[4] has proposed the theory of micropolar fluids in which the motion is described by a local microrotation vector in addition to the velocity vector. A class of exact solutions for the steady flow of a micropolar fluid between two non-coaxially rotating disks has been presented by Ramachandra Rao and Kasiviswanathan[5]. Smith[6] has obtained exact unsteady solutions of the Navier-Stokes equations due to eccentrically rotating disks. However, Ramachandra Rao and Kasiviswanathan[7–9] have presented the exact solutions of the unsteady flows in a variety of situations with the same geometrical configuration.

The unsteady flow of a micropolar fluid confined between two non-coaxially rotating disks in which each point of the disk is subjected to non-torsional elliptic harmonic oscillations is investigated in this paper. A class of exact solutions depending upon the arbitrary pressure gradient similar to Poiseuille flow is obtained. These solutions reduce to a unique symmetric solution different from the usual axisymmetric solution or an asymmetric solution according as the modified pressure gradient in the planes parallel to the disks is zero or different from zero. The expression for the components of the forces and couples acting on the plates is also obtained.

2. Formulation

Consider the unsteady flow of an incompressible micropolar fluid confined between two infinite parallel disks performing elliptic harmonic oscillations in their own planes and rotating with the same angular velocity Ω about two distinct axes perpendicular to the disks. In Cartesian coordinate system, let the upper and lower disks $z = \pm h$ rotate about the axes parallel to z-axis passing through the points $P_1(x_1, y_1, h)$ and $P_2(-x_1, -y_1, -h)$, respectively, where the midpoint of P_1P_2 is taken as origin. The equations governing the flow of a micropolar fluid in the absence of body forces and body couples are given by

$$\rho\dot{\bar{v}} = -\nabla p + (\mu+\kappa)\nabla^2\bar{v} + \kappa\nabla x\bar{\nu}, \quad \nabla_0\bar{v} = 0, \tag{1}$$

$$\rho j\dot{\bar{\nu}} = (\alpha+\beta+\gamma)\nabla(\nabla\cdot\bar{\nu}) - \gamma(\nabla x\nabla x\bar{\nu}) + k\nabla x\bar{v} - 2\kappa\bar{\nu}, \tag{2}$$

where $\bar{v} = (u, v, w)$ is the velocity vector, $\bar{\nu} = (\xi, \eta, \zeta)$ the microrotation vector, p the thermodynamic pressure, ρ the density, j the microinertia, μ, κ, α, β and γ the material constants and the dot signifies the material differentiation. Further, the material constants have to satisfy the following inequalities:

$$2\mu + k \geqslant 0, \quad \kappa \geqslant 0, \quad 3\alpha + \beta + \gamma \geqslant 0, \quad |\beta| \leqslant \gamma. \tag{3}$$

The unsteady motion in the above geometrical configuration (orthogonal rheometer) depends only on z and t in addition to rigid rotation. Following Ramachandra Rao and Kasiviswanathan[5], it can be shown that the velocity and microrotation for this flow are given by

$$u = -\Omega[y - g(z,t)], \quad u = \Omega[x - f(z,t)], \quad w = 0, \tag{4}$$

$$\xi = \xi(z,t), \quad \eta = \eta(z,t), \quad \zeta = \Omega. \tag{5}$$

The boundary conditions are the no-slip for the velocity and no relative spin for the microrotation and they are given by

$$f = x_1 + (a_1 e^{i\omega t} + \text{c.c.}), \quad g = y_1 + (b_1 e^{i\omega t} + \text{c.c.}), \quad \xi = \eta = 0 \quad \text{on} \quad z = h, \tag{6}$$

$$f = -x_1 + (a_2 e^{i\omega t} + \text{c.c.}), \quad g = -y_1 + (b_2 e^{i\omega t} + \text{c.c.}), \quad \xi = \eta = 0 \quad \text{on} \quad z = -h, \tag{7}$$

where ω is the frequency of non-torsional oscillations of the plates, a_1, b_1, a_2, b_2 are real constants giving the amplitude of oscillations and c.c. denotes the complex conjugate. For circular harmonic motions one has $a_1 = b_1$ and $a_2 = b_2$.

The equations governing the flow (1) and (2) in view of (4) and (5) reduce to

$$(\mu + k)\Omega g_{zz} - \rho\Omega g_t - \rho\Omega^2 f = \rho P_x + \kappa\eta_z, \tag{8}$$

$$(\mu + k)\Omega f_{zz} - \rho\Omega f_t + \rho\Omega^2 g = -\rho P_y + \kappa\xi_z, \tag{9}$$

$$P_z = 0, \tag{10}$$

$$\rho j\xi_t = \gamma\xi_{zz} + \kappa\Omega f_z - 2\kappa\xi, \tag{11}$$

$$\rho j\eta_t = \gamma\eta_{zz} + k\Omega g_z - 2\kappa\eta, \tag{12}$$

where $P = -P/\rho + \Omega^2(x^2 + y^2)/2$ and subscripts denote the partial differential with respect to that variable. Eliminating P from (8) and (9) by differentiating with respect to z (in view of (10)), introducing $F = f + ig$, $G = \xi + i\eta$, the equations (8)–(12) are rewritten as

$$(\mu + \kappa)F_{zzz} - \rho F_{zt} - i\Omega\rho F_z = (\kappa|\Omega)G_{zz}, \tag{13}$$

$$\gamma G_{zz} - \rho j G_t - 2\kappa G = -\kappa\Omega F_z. \tag{14}$$

The corresponding boundary conditions are

$$F = x_1 + iy_1 + \{(a_1 + ib_1)e^{i\omega t} + \text{c.c.}\}, \quad G = 0 \quad \text{on} \quad z = h, \tag{15}$$

$$F = -(x_1 + iy_1) + \{(a_2 + ib_2)e^{i\omega t} + \text{c.c.}\}, \quad G = 0 \quad \text{on} \quad z = -h. \tag{16}$$

As the governing differential equation for F given in (13) is of the third order and as there are only two boundary conditions, (15) and (16), we need one more on F to solve the problem and it is prescribed arbitrarily by

$$F = x_p + iy_p + \{(a + ib)e^{i\omega t} + \text{c.c.}\} \quad \text{on} \quad z = 0. \tag{17}$$

The condition (17) implies that the space curve Γ given by $x = f(z, t)$, $y = g(z, t)$ passes through the arbitrary point (x_p, y_p) in the middle plane (for more details see Ramachandra Rao and Kasiviswanathan[7]) and it is also subjected to elliptic harmonic oscillations through the arbitrary amplitudes, a and b.

3. Exact solution

The coupled equations in (13) and (14) are solved subjected to the boundary conditions (15)–(17) for an oscillatory flow with non-vanishing mean, by taking

$$(F, G) = (F_0, G_0) + [(F_1, G_1)e^{i\omega t} + \text{c.c.}]. \tag{18}$$

The solutions are obtained by a straight forward but lengthy procedure and are given by

$$F_0 = (x_1 + iy_1)\phi_1(z)/\phi_1(h) + (x_p + iy_p)[\phi_2(z) - \phi_2(h)]/[\phi_2(0) - \phi_2(h)], \tag{19}$$

$$G_0 = (x_1 + iy_1)\phi_3(z)/\phi_3(h) + (x_p + iy_p)\phi_4(z)/[\phi_2(0) - \phi_2(h)], \tag{20}$$

$$F_1 = \tfrac{1}{2}[(a_1 - a_2) + i(b_1 - b_2)]\psi_1(z)/\psi_1(h) + \tfrac{1}{2}[(a_1 + a_2 - 2a) + i(b_1 + b_2 - 2b)][\psi_2(z) - \psi_2(0)]/[\psi_2(h) - \psi_2(0)], \tag{21}$$

$$G_1 = \tfrac{1}{2}[(a_1 - a_2) + i(b_1 - b_2)]\psi_3(z)/\psi_3(h) + \tfrac{1}{2}[(a_1 + a_2 - 2a) + i(b_1 + b_2 - 2b)]\psi_4(z)/[\psi_2(h) - \psi_2(0)], \tag{22}$$

where

$$\begin{aligned}
\phi_1(z) &= l_1 \cosh m_2 h \sinh m_1 z - l_2 \cosh m_1 h \sinh m_2 z,\\
\phi_2(z) &= l_1 \sinh m_2 h \cosh m_1 z - l_2 \sinh m_1 h \cosh m_2 z,\\
\phi_3(z) &= \cosh m_2 h \cosh m_1 z - \cosh m_1 h \cosh m_2 z,\\
\phi_4(z) &= \sinh m_2 h \sinh m_1 z - \sinh m_1 h \sinh m_2 z,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\psi_1(z) &= q_1 \cosh r_2 h \sinh r_1 z - q_2 \cosh r_1 h \sinh r_2 z,\\
\psi_2(z) &= q_1 \sinh r_2 h \cosh r_1 z - q_2 \sinh r_1 h \cosh r_2 z,\\
\psi_3(z) &= \cosh r_2 h \cosh r_1 z - \cosh r_1 h \cosh r_2 z,\\
\psi_4(z) &= \sinh r_2 h \sinh r_1 z - \sinh r_1 h \sinh r_2 z,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&l_1 = (2\kappa_2 - m_1^2)/\Omega\kappa_2 m_1, \quad l_2 = (2\kappa_2 - m_2^2)/\Omega\kappa_2 m_2,\\
&q_1 = (2\kappa_2 - r_1^2)/\Omega\kappa_2 m_1(1 + i\kappa_4), \quad q_2 = (2\kappa_2 - r_2^2)/\Omega\kappa_2 m_2(1 + i\kappa_4),
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\kappa_1 = \frac{\Omega\rho}{2\kappa(\mu+\kappa)}, \quad \kappa_2 = \frac{\kappa}{\gamma}, \quad \kappa_3 = \frac{\kappa + 2\mu}{2(\kappa+\mu)}, \quad \kappa_4 = \frac{\rho j \omega}{2\kappa},\\
&m_1^2 = \kappa_6 + [\kappa_6^2 - 4\kappa_7]^{\frac{1}{2}}, \quad m_2^2 = \kappa_6 - [\kappa_6^2 - 4\kappa_7]^{\frac{1}{2}},\\
&r_1^2 = \kappa_8 + [\kappa_8^2 - 4\kappa_9]^{\frac{1}{2}}, \quad r_2^2 = \kappa_8 - [\kappa_8^2 - 4\kappa_9]^{\frac{1}{2}},\\
&\kappa_5 = \omega/\Omega, \quad \kappa_6 = \kappa_2\kappa_3 + i\kappa_1, \quad \kappa_7 = i\kappa_1\kappa_2,\\
&\kappa_8 = \kappa_6 + i(\kappa_2\kappa_4 + \kappa_1\kappa_5) \quad \text{and} \quad \kappa_9 = \kappa_7(1 + \kappa_5)(1 + i\kappa_4).
\end{aligned} \tag{26}$$

These results reduce to those for an incompressible viscous liquid[7] in the limit that the material constants for the micropolar fluid given by κ, α, β, γ go to zero. The solutions for a steady flow discussed[5] correspond to F_0 and G_0 given in (19) and (20). Equations (19)–(22) contain the arbitrary constants x_p, y_p, a and b and corresponding to each of these constants one will have one parameter family of solutions.

4. Discussions

From (8) and (9), the modified pressure gradient in the plane parallel to the disks is expressed in terms of F and G, and is given by

$$i\nabla P = i[P_x + iP_y] = \Omega F_t + i\Omega^2 F - KG_z/\rho - (\mu + \kappa)\Omega F_{zz}/\rho. \tag{27}$$

Substituting the expressions for F and G given by (19)–(22) in (27) and writing

$$i\nabla P = \nabla P_0 + (\nabla P_1 e^{i\omega t} + \text{c.c.}), \tag{28}$$

the expressions for ∇P_0 and ∇P_1 are obtained as

$$\nabla P_0 = (x_p + iy_p)\Omega^2\phi_2(h)/[\phi_2(h) - \phi_2(0)], \tag{29}$$

$$\nabla P_1 = \Omega(\Omega + \omega)[(a_1 + a_2 - 2a) + i(b_1 + b_2 - 2b)]\psi_2(h)/2[\psi_2(h) - \psi_2(0)]. \tag{30}$$

A solution is defined as a symmetric solution with respect to the origin 0, if the velocity field satisfies the condition $\bar{v}(-x, -y, -z) = -\bar{v}(x, y, z)$, which is different from the usual axisymmetric solution. In our problem this condition reduces to $F(-z) = -F(z)$ and therefore symmetric solution contains only odd functions of z. From (19) and (21), for a symmetric solution, one must satisfy:

$$x_p = y_p = 0, \tag{31}$$

$$a_1 + a_2 = 2a, \quad b_1 + b_2 = 2b_0 \tag{32}$$

or

$$a_1 + a_2 = 0 = b_1 + b_2 = a = b. \tag{33}$$

Condition (31) implies that the space curve Γ passes through origin whereas (32) gives that the amplitude of oscillation of the middle plane is equal to the average of the amplitude of oscillations of the upper and lower disks. Condition (33) states that if the amplitudes of oscillations of the upper and lower disks are of the opposite sign then the amplitude of oscillations of the middle plane is zero.

When the conditions (31)–(33) are satisfied we can easily see that the pressure gradient given in (29) and (30) vanishes and *vice versa*. Thus we can state, the necessary and sufficient condition for the solution to be symmetric is that the pressure gradient should vanish. When the pressure gradient in the planes parallel to disks is different from zero, there is a possibility of the existence of infinite number of solutions (because of the presence of arbitrary constants x_p, y_p, a and b) similar to Poiseuille flow in a channel or pipe. On the other hand, if we prescribe the pressure gradient then a unique asymmetric solution is possible. Interestingly similar results hold good for steady, unsteady viscous fluid flows, steady micropolar fluid flow or for any other fluid flow in this geometric configuration.

From the expressions of F and G given in (19)–(22), one can obtain f, g, ξ and η using the following relations:

$$f = \tfrac{1}{2}(F + \text{c.c.}), \quad g = \frac{1}{2i}(F - \text{c.c.}), \tag{34}$$

$$\xi = \tfrac{1}{2}(G + \text{c.c.}), \quad \eta = \frac{1}{2i}(G - \text{c.c.}). \tag{35}$$

The x and y components of the traction $\hat{t}$ on the top plate are given by

$$\hat{t}_x = -(\mu + \kappa)\Omega g_z(h) + \kappa\eta(h), \quad \hat{t}_y = (\mu + \kappa)\Omega f_z(h) - \kappa\xi(h). \tag{36}$$

The components of couple-stress tensor m_{ij} are given by

$$m_{xz} = \beta\xi_z, \quad m_{yz} = \beta\eta_z, \quad m_{zx} = \gamma\xi_z, \quad m_{zy} = -\gamma\eta_z, \tag{37}$$

as the other components are zero.

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