J. Indian Inst. Sci., Nov.–Dec. 1989, 69, 421–427.

Decomposition theorem for weak almost periodic functions

M. V. Deshpande and M. N. Parekh

Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400076, India.

Received on January 19, 1989; Revised on September 18, 1989.

Abstract

Let S be a group and a topological semigroup. It is proved that a continuous complex weak almost periodic function on S can be represented uniquely as a sum of a continuous almost periodic function and a continuous weak almost periodic function such that the weak closure of its orbit contains zero. The proof depends mainly on the weak almost periodic compactification. To obtain this compactification, we use topological concepts like Ruppert (*Compact semitopological semigroup: an intrinsic theory*, 1984, Springer-Verlag) rather than the operator theoretic techniques of Deleeuw and Glicksberg (*Acta Math.*, 1961, **105**, 63–97).

Key words: Weak almost periodic functions, weak almost periodic compactification of a semigroup.

1. Introduction

In order to describe the results obtained we shall first introduce the notation. Let S be a topological semigroup and $[W(S)]$ $A(S)$, the space of continuous complex [weak] almost periodic functions on S. Let $C(S)$ denote the space of bounded continuous scalar-valued functions on S endowed with the topology of uniform convergence on S. For f in $C(S)$, let the orbit of f, $O_R(f) = \{f_s : s \in S\}$, where f_s, the right translate of f is defined by $f_s(t) = f(ts)$, $t \in S$. Denote by $W(S)_0$ the set of those f in $W(S)$ for which zero belongs to the weak closure of $O_R(f)$. Let S^w be the weak operator closure of $\{R_s : s \in S\}$, where R_s is the right translation operator on $W(S)$. Then according to Deleeuw and Glicksberg[1] S^w is the weak almost periodic compactification of S. Let $\tilde{R}$ be the isometric isomorphism of $C(S^w)$ on to $W(S)$ (theorem 5.3 of Deleeuw and Glicksberg[1]) and K the kernel of S^w. If $W(S)$ has two-sided invariant mean and E is the identity of K let $C(S^w)_p$ be the set of right E translates of functions in $C(S^w)$, and $W(S)_p = \tilde{R}(C(S^w)_p)$. Burckel[2] proved that if $W(S)$ has two-sided invariant mean then $W(S) = W(S)_0 \oplus W(S)_p$ and, in fact, when S is a group and a topological semigroup, $W(S) = W(S)_0 \oplus A(S)$. The proofs of these theorems essentially depend on the structure of S^w.

We construct S^w by using uniform spaces which leads to the simple proofs of the above decomposition theorems. We also obtain alternative but simpler proofs of other related results from Burckel[2].

2. Preliminaries

The following definitions are from Burckel[2].

Definition 2.1. A set S is called a topological semigroup if S is a semigroup with identity e and if S has a Hausdorff topology such that the multiplication on S is separately continuous. That is, for each t in S, the maps $s \to st$ and $s \to ts$ are continuous functions.

Definition 2.2. A function f in $C(S)$ is right weak almost periodic if $O_R(f)$, the right orbit of f, is relatively compact in the weak topology on $C(S)$.

Left weak almsot periodic functions can be defined similarly.

For any topological semigroup, the set of right weak almost periodic functions coincides with the set of left weak almost periodic functions (Corollary 1.12 of Burckel[2]). We denote this common set by $W(S)$ and call its functions weak almost periodic. For the proof of the following lemma, we refer to lemma 1.6 of Burckel[2].

Lemma 2.3. The space $W(S)$ is translation invariant norm closed linear subspace of $C(S)$.

To obtain the results mentioned in the introduction we first construct the weak almost periodic compactification of a topological semigroup by using uniform spaces.

Construction of S^w: Let S be a topological semigroup. Denote by $\tilde{S}$ the quotient structure of S determined by the equivalence relation, 's is related to t if and only if $f_s = f_t$ for all f in $W(S)$. Then $\tilde{S}$ is a semigroup with multiplication $\tilde{s}\cdot\tilde{t} = \widetilde{st}$. For $\varepsilon > 0$ and $f \in W(S)$ and $f' \in W(S)'$, the dual space of $W(S)$, define $U(\varepsilon, f, f') = \{(\tilde{s}, \tilde{t}) \varepsilon \tilde{S} \times \tilde{S} : |f'(f_s - t_t)| < \varepsilon, s \in \tilde{s}, t \in \tilde{t}\}$. Then the family of sets of the form $U(\varepsilon, f, f')$ forms a subbase for a uniformity, say $\tilde{U}$, on $\tilde{S}$. The proof of the following proposition uses the arguments similar to those of theorem 2.10 (Ch. III) of Ruppert[3] and hence omitted.

Proposition 2.4. The uniform space $(\tilde{S}, \tilde{U})$ is a totally bounded Hausdorff topological semigroup and its completion S^w is a compact topological semigroup.

Remark 1. The space S^w obtained above coincides with the weak almost periodic compactification of S[1,3].

Theorem 2.5. Let S be a topological semigroup. The homomorphism $\phi: S \to (\tilde{S}, \tilde{U}) \subset S^w$ defined by $\phi(s) = \tilde{s}$, is continuous with $\phi(S)$ dense in S^w. The induced map $\tilde{\phi}: C(S^w) \to C(S)$ given by $\tilde{\phi}(\hat{f}) = \hat{f} \circ \phi$ is an isometric isomorphism of $C(S^w)$ on to $W(S)$.

Proof. For each f in $W(S)$, the map $s \to f_s$ is weak continuous from S into $W(S)$ (theorem 1.7 (iii), Burckel[2]). Therefore, it follows that ϕ is continuous. Also since $\phi(S) = (\tilde{S}, \tilde{U})$, a dense sub-space of S^w, $\tilde{\phi}$ is one–one and an isometry. Moreover, $\tilde{\phi}$ preserves the ordinary multiplication of functions, so $W(S)$ is an algebra and $\tilde{\phi}$ is an algebra isomorphism. Therefore, it remains to prove that $\tilde{\phi}$ maps $C(S^w)$ on to $W(S)$. $\tilde{\phi}[C(S^w)] \subset W(S)$ (theorem 1.8 (i), Burckel[2]). Let $f \in W(S)$. Define $\tilde{f}:(\tilde{S}, \tilde{\mathscr{U}}) \to \mathbb{C}$ by $\tilde{f}(\tilde{s}) = f(s)$. It is then easy to see that $\tilde{f}$ is uniformly continuous. Hence $\tilde{f}$ extends to $\hat{f}$, a uniformly continuous function on S^w. The extended function $\hat{f}$ satisfies $\tilde{\phi}(\hat{f})(s) = \hat{f}_0 \phi(s) = \hat{f}(\tilde{s}) = \tilde{f}(\tilde{s}) = f(s)$. That is, $\tilde{\phi}(\hat{f}) = f$. This proves that $\tilde{\phi}[C(S^w)] = W(S)$.

Definition 2.6. (1) Let X be a topological space, A a norm closed subspace of $C(X)$. A mean on A is any linear functional ϕ on A such that (i) $\phi \neq 0$ and $\phi(1) = 1$, if $1 \in A$; (ii) $f \in A$, $f \geqslant 0$ implies $\phi(f) \geqslant 0$. (2) Let S be a topological semigroup, A, a norm closed right (left) translation invariant vector subspace of $C(S)$. A right (left) invariant mean on A is a mean M which satisfies $M(f_t) = M(f)$ $(M({}_tf) = M(f))$ $t \in S$, $f \in A$, where f_t and ${}_tf$ denote the right and left translates of f, respectively.

If A is both left and right translation invariant and M is both a left and a right translation invariant mean, we call it a two-sided invariant mean.

For definition of kernel of a semigroup, minimal ideals in a semigroup and other related concepts we refer to Burckel[2].

3. Decomposition theorem for W(S)

Let S be a topological semigroup. Let $(\tilde{S}, \tilde{\mathscr{U}})$, S^w, ϕ and $\tilde{\phi}$ be as in Section 2. For $f \in W(S)$, let $\tilde{f}$ and $\hat{f}$ be as in the proof of theorem 2.5.

Definition 3.1. For a topological semigroup S let $\overline{O_R(f)}$ denote the weak closure of the right orbit $O_R(f)$ of f in $C(S)$. Let $W(S)_0 = \{f \in W(S): O \in \overline{O_R(f)}\}$ and $C(S^w)_0 = \tilde{\phi}^{-1}[W(S)_0]$ in $C(S^w)$.

In this section we shall prove that if G is a group and a toplogical semigroup then every function in $W(G)$ can be represented uniquely as a sum of a function in $A(G)$ and a function in $W(G)$ such that the weak closure of its orbit contains zero. To obtain this result we first need to prove a few lemmas.

Lemma 3.2. For a topological semigroup S and $f \in W(S)$, $\overline{O_R(f)} = \tilde{\phi}(O_R(\hat{f}))$, where $\hat{f}$ is the continuous extension of f to S^w.

Proof. $O_R(f) = \{f_s : s \in S\} = \tilde{\phi}\{\hat{f}_{\tilde{s}} : \tilde{s} \in \tilde{S}\} \subset \tilde{\phi}\{\hat{f}_{\hat{s}} : \hat{s} \in S^w\} = \tilde{\phi}(O_R(\hat{f}))$. Therefore,

$$\overline{O_R(f)} \subset \overline{\tilde{\phi}(O_R(\hat{f}))}. \tag{1}$$

Since $\tilde{\phi}$ is a linear isometry, by theorem V.3.15 of Dunford and Schwartz[4] it is a homeomorphism with respect to the weak topologies on $C(S^w)$ and $W(S)$. Therefore, $\tilde{\phi}$ is a closed map. Now $\tilde{\phi}(O_R(\hat{f}))$ is a weakly closed set containing $\tilde{\phi}(O_R(\hat{f}))$. But, as $\tilde{\phi}(O_R(\hat{f}))$

is the smallest set containing $\tilde{\phi}(O_R(\hat{f}))$, we have

$$\overline{\tilde{\phi}(O_R(\hat{f}))} \subset \tilde{\phi}(\overline{O_R(\hat{f})}). \tag{2}$$

Also, as S^w is a compact Hausdorff topological semigroup (theorem 1.7 of Burckel[2]), $O_R(\hat{f})$ is weakly compact in $C(S^w)$ and hence it is weakly closed. Therefore, from (1) and (2) we have $\overline{O_R(f)} \subset \tilde{\phi}(O_R(\hat{f}))$. To prove the reverse inclusion, let $\hat{f}_{\hat{s}} \in O_R(\hat{f})$, where $\hat{s} \in S^w$. Then, there exists a net $\{s_\alpha\}$ in S such that $\tilde{s}_\alpha$ converges to $\hat{s}$. Since S^w is a topological semigroup and $\hat{f}$ is continuous, $\tilde{s}_\alpha \to \hat{s}$ implies $\hat{f}(\tilde{t}\tilde{s}_\alpha) \to \hat{f}(\tilde{t}\hat{s})$. That is $\hat{f}(\tilde{t}\hat{s}) = \lim_\alpha \hat{f}(\tilde{t}\tilde{s}_\alpha) = \lim_\alpha \tilde{f}(\tilde{t}\tilde{s}_\alpha) = \lim_\alpha f(ts_\alpha)$. Hence $\hat{f}_{\hat{s}}(\phi(t)) = \lim_\alpha f_{s_\alpha}(t)$. This shows that $\tilde{\phi}(\hat{f}_{\hat{s}}) = \hat{f}_{\hat{s}} 0 \phi$ is a point-wise limit of f_{s_α}. Since f is weak almost periodic, $\overline{O_R(f)}$ is weak compact in $C(S)$. Now, as the weak topology of $C(S)$ is stronger than the point-wise topology and the latter is Hausdorff, the two topologies coincide on $\overline{O_R(f)}$. Hence $\tilde{\phi}(\hat{f}_{\hat{s}}) \in \overline{O_R(f)}$. Thus $\tilde{\phi}(O_R(\hat{f})) \subset \overline{O_R(f)}$.

Lemma 3.3. $W(S)_0 = \{f \in W(S): 0 \in O_R(\hat{f})\}$.

Proof. Since $\tilde{\phi}$ is an isometric isomorphism, the proof follows immediately from lemma 3.2.

Definition 3.4. Let K be the kernel of S^w and $E(K)$ the set of idempotents in K. Foe $\hat{e}$ in $E(K)$, define $\ker \hat{e} = \{f \in W(S): \hat{f}_{\hat{e}} = 0\}$.

Lemma 3.5. $W(S)_0 = U\{\ker \hat{e}: \hat{e} \in E(K)\} = \{f \in W(S): \hat{f}_{\hat{e}} = 0 \text{ for some } \hat{e} \in E(K)\}$.

Proof. Suppose that $f \in W(S)$ is such that $\hat{f}_{\hat{e}} = 0$ for some $\hat{e} \in E(K)$. Then $0 = \hat{f}_{\hat{e}} \in 0_R(\hat{f})$. Therefore, by lemma 3.3 $f \in W(S)_0$. Conversely, if $f \in W(S)_0$, then, again from lemma 3.3, $\hat{f}_{\hat{s}} = 0$ for some $\hat{s} \in S^w$. Hence,

$$\hat{f}_{\hat{s}}(S^w) = \hat{f}(S^w \hat{s}) = 0. \tag{1}$$

Now, $S^w \hat{s}$ is a left ideal in S^w and hence there exists a minimal left ideal I contained in $S^w \hat{s}$ (theorem 2.1 of Burckel[2]). But by theorem 2.2 of Burckel[2], $I = S^w \hat{e}$ for some $\hat{e} \in E(K)$. Therefore, from (1), $\hat{f}(I) = \hat{f}(S^w \hat{e}) = \hat{f}_{\hat{e}}(S^w) = 0$. That is, $\hat{f}_{\hat{e}} = 0$. This proves the lemma.

Corollary 3.6. If $W(S)$ has a right invariant mean and $\hat{e}_0$ is the identity of the kernel K of S^w, then $W(S)_0 = \ker \hat{e}_0$.

Proof. Since $W(S)$ has a right invariant mean, from lemma 2.6 of Burckel[2] and theorem 2.5 of Section 2, it follows that $C(S^w)$ has a right invariant mean. Then by lemma 2.4 of Burckel[2], S^w has a unique minimal left ideal which coincides with kernel K of S^w. Now, for any $\hat{e} \in E(K)$, since $S^w \hat{e}$ is a minimal left ideal in S^w, we have $K = S^w \hat{e} = S^w \hat{e}_0$. Hence, from the above lemma, we have $f \in W(S)_0$ if and only if $0 = \hat{f}_{\hat{e}}(S^w) = \hat{f}(S^w \hat{e}) = \hat{f}(S^w \hat{e}_0) = \hat{f}_{\hat{e}_0}(S^w)$. Equivalently $f \in \ker \hat{e}_0$.

Corollary 3.7. $W(S)_0 = \{\tilde{\phi}(\hat{f}): \hat{f}(I) = 0 \text{ for some } I \in \mathscr{L}(S^w)\}$ where $\mathscr{L}(S^w)$ denotes the set of minimal left ideals in S^w.

Proof. By theorem 2.2 of Burckel[2], we have $\mathscr{L}(S^w)=\{S^w\hat{e}:\hat{e}\in E(K)\}$. Also for each $f\in W(S)$ there exists $\hat{f}\in C(S^w)$ such that $\tilde{\phi}(\hat{f})=f$ (Theorem 2.5). Hence the corollary follows from lemma 3.5.

Corollary 3.8. If $W(S)$ has a right invariant mean, then $W(S)_0=\tilde{\phi}\{\hat{f}\in C(S^w):\hat{f}(K)=0\}$ where K is the kernel of S^w.

Proof. From theorems 2.6 and 2.4 of Burckel[2], if follows that S^w has a unique minimal left ideal which must coincide with the kernel of S^w. Hence, corollary 3.7 proves the assertion.

Corollary 3.9. If $W(S)$ has a right invariant mean then $W(S)_0$ is a right and left translation invariant closed ideal in $W(S)$.

Proof. From corollary 3.8, it can be easily seen that $W(S)_0$ is an ideal in $W(S)$. We shall show that $W(S)_0$ is right translation invariant. The proof of left translation invariance of $W(S)_0$ is similar.

Let $f\in W(S)_0$ and $s\in S$. Then, by corollary 3.8, $\hat{f}(K)=0$, where $\tilde{\phi}(\hat{f})=f$. Now for any $t\in S$, $f_s(t)=f(ts)=\tilde{\phi}(\hat{f})(ts)=\hat{f}_0\phi(ts)=\hat{f}(\phi(t)\phi(s))=\hat{f}_{\phi(s)}(\phi(t))=\tilde{\phi}(\hat{f}_{\phi(s)})(t)$. That is $f_s=\tilde{\phi}(\hat{f}_{\phi(s)})$. Since K is two-sided ideal, $K\phi(s)=K\bar{s}\subset K$. But as $\hat{f}$ vanishes on the whole of K, we have $\hat{f}_{\phi(s)}(K)=\hat{f}(K\phi(s))=0$. Hence from corollary 3.8 again it follows that $f_s\in W(S)_0$. This completes the proof.

Corollary 3.10. If $W(S)$ has two-sided invariant mean M, then $W(S)_0=\{f\in W(S):M(|f|)=0\}$.

Proof. From theorem 2.7 of Burckel[2], $M(|f|)=\int_K\tilde{\phi}^{-1}(|f|)\,d\mu$, where μ is normalised Haar measure on the compact group K. Since $\tilde{\phi}^{-1}(|f|)=|\tilde{\phi}^{-1}(f)|$, $M(|f|)=\int_K|\tilde{\phi}^{-1}(f)|\,d\mu$. Hence from corollary 3.8, $f=\tilde{\phi}(\hat{f})\in W(S)_0\Leftrightarrow\hat{f}(K)=0\Leftrightarrow\tilde{\phi}^{-1}(f)(K)=0\Leftrightarrow M(|f|)=O$.

Definition 3.11. Let S be a topological semigroup for which $W(S)$ has two-sided invariant mean. Let $\hat{e}$ denote the identity of kernel K of S^w. Define $C(S^w)_p=\{\hat{f}_{\hat{e}}:\hat{f}\in C(S^w)\}$ and $W(S)_p=\tilde{\phi}(C(S^w)_p)$.

Throughout the remaining part of this section the identity of the kernel of S^w is denoted by $\hat{e}$.

Proposition 3.12. Let S be a topological semigroup for which $W(S)$ has two-sided invariant mean. Then $W(S)_p=\{f\in W(S):\hat{f}_{\hat{e}}=\hat{f}\}$.

Proof. Let $f\in W(S)$ be such that $\hat{f}_{\hat{e}}=\hat{f}$. Since $\hat{f}_{\hat{e}}\in C(S^w)_p$, $f=\tilde{\phi}(\hat{f})=\tilde{\phi}(\hat{f}_{\hat{e}})\in\tilde{\phi}(C(S^w)_p)=W(S)_p$. Conversely, if $f\in W(S)_p$, $f=\tilde{\phi}(\hat{g}_{\hat{e}})$ for some $\hat{g}\in C(S^w)_p$. Then $\hat{f}=\tilde{\phi}^{-1}(f)=\hat{g}_{\hat{e}}$ and since $\hat{e}$ is the identity of the kernel, $\hat{f}_{\hat{e}}=\hat{g}_{\hat{e}\hat{e}}=\hat{g}_{\hat{e}}=\hat{f}$. This proves the proposition.

We shall now obtain the main results of this section.

Theorem 3.13. Let S be a topological semigroup for which $W(S)$ has two-sided invariant mean. Then $W(S) = W(S)_0 \oplus W(S)_p$.

Proof. For each f in $W(S)$, we write $f = f - \tilde{\phi}(\hat{f}_{\hat{e}}) + \tilde{\phi}(\hat{f}_{\hat{e}})$. Since $\hat{f}_{\hat{e}} \in C(S^w)_p$, we have $\tilde{\phi}(\hat{f}_{\hat{e}}) \in \tilde{\phi}(C(S^w)_p) = W(S)_p$. We shall show that the function $h = f - \tilde{\phi}(\hat{f}_{\hat{e}}) \in W(S)_0$. By corollary 3.6, it is enough to show that $h \in \ker \hat{e}$. Let $\tilde{\phi}(\hat{f}_{\hat{e}}) = g$. Then $\hat{g} = \tilde{\phi}^{-1}(g) = \hat{f}_{\hat{e}}$ and $\hat{g}_{\hat{e}} = \hat{f}_{\hat{e}\hat{e}} = \hat{f}_{\hat{e}}$. Therefore, $\hat{h}_{\hat{e}} = [\hat{f} - \widehat{\tilde{\phi}(\hat{f}_{\hat{e}})}]_{\hat{e}} = \hat{f}_{\hat{e}} - \hat{g}_{\hat{e}} = 0$, which proves that $h \in \ker \hat{e}$. Now if $g \in W(S)_o \bigcap W(S)_p$, then $\hat{g}_{\hat{e}} = 0$ and also by proposition 3.12 $\hat{g}_{\hat{e}} = \hat{g}$, which implies $\hat{g} = 0$ and hence $g = 0$. This proves that the above decomposition of $W(S)$ is unique.

We note that if G is a group and a topological semigroup, $W(G)$ has a two-sided invariant mean (theorem 1.26, Burckel[2]). Hence every function in $W(G)$ has a unique representation as in theorem 3.13. In fact, the following theorem shows that a stronger result is possible to obtain in this case.

Theorem 3.14. If G is a group and a topological semigroup then $W(G)_p = A(G)$ and $W(G) = W(G)_0 \oplus A(G)$.

Proof. Define a map $\rho: G^w \to G^w \hat{e} = K$ by $\rho(\hat{s}) = \hat{s}\hat{e}$, where $\hat{e}$ is the identity of the kernel of G^w. Then ρ is a homomorphism. For, $\rho(\hat{s}\hat{t}) = \hat{s}\hat{t}\hat{e} = \hat{s}(\hat{e}\hat{t}\hat{e}) = (\hat{s}\hat{e})(\hat{t}\hat{e}) = \rho(\hat{s})\rho(\hat{t})$. Let $\tilde{\rho}: C(K) \to C(G^w)$ be the induced map. Then, for $\hat{f} \in C(K)$ and $\hat{s} \in G^w$, $\tilde{\rho}(\hat{f})(\hat{s}) = \hat{f}_0 \rho(\hat{s}) = \hat{f}(\hat{s}\hat{e}) = \hat{f}_{\hat{e}}(\hat{s})$. That is, $\tilde{\rho}(\hat{f}) = \hat{f}_{\hat{e}}$. Now define $\hat{g}: G^w \to \mathbb{C}$ by $\hat{g}(\hat{s}) = \hat{f}(\hat{s}\hat{e})$. Then $\hat{g}_{\hat{e}}(\hat{s}) = \hat{f}(\hat{s}\hat{e}\hat{e}) = \hat{f}(\hat{s}\hat{e}) = \hat{f}_{\hat{e}}(\hat{s})$. Thus, $\hat{g}_{\hat{e}} = \hat{f}_{\hat{e}}$. But since $\hat{g} \in C(G^w)$, $\hat{f}_{\hat{e}} = \hat{g}_{\hat{e}} \in C(G^w)_p$. Hence $\tilde{\rho}(C(K)) \subset C(G^w)_p$. Using similar arguments one can easily see that $C(G^w)_p \subset \tilde{\rho}(C(K))$. Thus, $\tilde{\rho}(C(K)) = C(G^w)_p$. Since K is a compact group and ρ is a homomorphism, by theorems 1.7 and 1.8 of Burckel[2] we have $A(K) = C(K)$ and $C(G^w)_p = \tilde{\rho}(C(K)) = \tilde{\rho}(A(K)) \subset A(G^w)$. Again by theorem 1.8 of Burckel[2], $W(G)_p = \tilde{\phi}(C(G^w)_p) \subset \tilde{\phi}(A(G^w)) \subset A(G)$. On the other hand, given $f \in A(G)$, let $f = f_1 + f_2$ where $f_1 \in W(G)_0$ and $f_2 \in W(G)_p$. Then $f_1 = f - f_2 \in A(G)$, since we just showed that $W(G)_p \subset A(G)$. Also as $A(G)$ is closed under the operation of taking absolute value we have $|f_1| \in A(G)$. By corollary 1.26 of Burckel[2], $W(G)$ has unique two-sided invariant mean which annihilates no non-negative function in $A(G)$ except zero. As $f_1 \in W(G)_0$, from corollary 3.9, $M(|f_1|) = 0$. But $|f_1| \in A(G)$ and hence $|f_1| = 0$. Therefore, $f = f_2 \in W(G)_p$ and we have the other inclusion $A(G) \subset W(G)_p$. Thus $W(G)_p = A(G)$ and from theorem 3.13, $W(G) = W(G)_0 \oplus A(G)$.

Corollary 3.15. Let G be a group and a topological semigroup. Then G^w is a group (algebraically, and then by theorem 1.28 of Burckel[2], topologically also) if and only if $W(G) = A(G)$.

Proof. Since kernel K of G^w is a group and an ideal in G^w, it is clear that G^w is a group if and only if $K = G^w$. From corollary 3.8, if $K = G^w$ then $W(G)_0 = 0$. Conversely let $W(G)_0 = 0$ and suppose that $K \neq G^w$. Then, by Urysohn lemma there exists a non-zero function $\hat{f} \in C(G^w)$ such that $\hat{f}(K) = 0$. But this means $0 \neq \tilde{\phi}(\hat{f}) \in W(G)_0$, a contradiction. Thus $K = G^w$ if and only if $W(G)_0 = 0$. Hence, from the direct sum decomposition theorem $W(G)_0 = 0$ if and only if $W(G) = A(G)$.

Remark: The following classes of vector-valued functions have been studied by Goldberg and Irwin[5]. For a topological semigroup S and X, a Banach space,

(i) $AP(S,X)=\{f\in C(S,X): O_R(f)$ is relatively norm compact in $C(S,X)\}$.

(ii) $WRC(S,X)=\{f\in C(S,X): O_R(f)$ is weakly relatively compact in $C(S,X)$ and $f(S)$ is relatively norm compact in $X\}$.

Goldberg and Irwin[5] have extended operator theoretic techniques of Deleeuw and Glicksberg[1] to obtain a compactification S^{WRC} of S through functions in $WRC(S,X)$. It is then observed[6] that S^{WRC} is isomorphic to the weak almost periodic compactification S^w obtained through functions in $W(S)$[1]. The following theorem is from Goldberg and Irwin[5].

Theorem: Let G be a group and a topological semigroup. Then $WRC(G,X)=WRC_0(G,X)\oplus AP(G,X)$ where $WRC_0(G,X)=\{f\in WRC(G,X): 0\in\overline{O_R(f)}$, the weak closure of $O_R(f)\}$.

We observe that the results similar to those of Sections 2 and 3 can be proved for the space $WRC(S,X)$ and also the techniques used in these sections can be extended to obtain the decomposition theorem stated above. The proof of this is omitted as it can be obtained by adopting suitably the proof of theorem 3.14.

References

1. DELEEUW, K. AND GLICKSBERG, I. Applications of almost periodic compactifications, *Acta Math.*, 1961, **105**, 63–97.

2. BURCKEL, R. B. *Weakly almost periodic functions on semigroups*, 1970, Gordon and Breach.

3 RUPPERT, W. *Compact semitopological semigroup· an intrinsic theory*, 1984, Springer-Verlag.

4. DUNFORD, N. AND SCHWARTZ, J. T. *Linear operators, part I, General Theory*, 1957, Interscience.

5. GOLDBERG, S. AND IRWIN, P. Weakly almost periodic vector-valued functions, *Diss. Math.*, 1979, **157**, 1–42.

6. MILNES, P. On vector-valued weakly almost periodic functions, *J. Lond. Math. Soc.* (2), 1980, **22**, 467–472.