J. Indian Inst. Sci., Nov.–Dec. 1989, **69**, 429–439.

Characterisations of compact operators on the space of almost periodic functions

M. V. DESHPANDE AND M. N. PAREKH
Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400076, India

Received on February 1, 1989; Revised on September 25, 1989.

Abstract

Let X be a Banach space and $AP(\mathbb{R}, X)$ the space of continuous almost periodic functions on $\mathbb{R}$ into X with supremum norm. We obtain characterisations of compact operators on $AP(\mathbb{R}, X)$ (Theorem 3.1) This theorem is then used to prove the following: Let $AP(\mathbb{R})$ be the space $AP(\mathbb{R}, \mathbb{C})$. For a compact operator K on $AP(\mathbb{R})$, the aggregate of Fourier exponents of functions in the range of K is countable even though the range of K is uncountable. We also obtain sufficient conditions on K so that the Fourier series of all the functions in the range of K converges at the same point.

Key words: Compact operators, almost periodic functions.

1. Introduction

It is well known that a bounded linear operator on $AP(\mathbb{R})$, the space of continuous complex-valued almost periodic functions on $\mathbb{R}$, is compact if and only if it can be approximated by a trigonometric polynomial on $AP(\mathbb{R})$[1]. This appears to be the only known characterisation of compact operators on $AP(\mathbb{R})$. This result vitally depends on the fact that $AP(\mathbb{R})$ is a Banach algebra with pointwise product and supremum norm. In fact, the proof uses the Gelfand theory. If $AP(\mathbb{R}, X)$ denote the set of continuous almost periodic functions on $\mathbb{R}$ into a Banach space X, then $AP(\mathbb{R}, X)$ is a Banach space with supremum norm. Since $AP(\mathbb{R}, X)$ is not a Banach algebra, in general, the techniques of Schaeffer[1] are not applicable to the operators on $AP(\mathbb{R}, X)$. However, we show that, it is possible to obtain a similar characterisation of compact operators on $AP(\mathbb{R}, X)$, by using elementary properties of almost periodic functions. We prove that an operator K on $AP(\mathbb{R}, X)$ is compact if and only if it is approximated by an operator-valued trigonometric polynomial on $AP(\mathbb{R}, X)$. In Theorem 3.1 some more characterisations are obtained.

For Banach spaces X and Y, let $BL(X, Y)$ denote the space of bounded linear operators on X into Y with uniform operator topology. We denote by $KL(X, Y)$, the subspace of $BL(X, Y)$ consisting of compact operators. When $X = Y$ we write $BL(X)$ and $KL(X)$ for $BL(X, Y)$ and $KL(X, Y)$, respectively. Let $A = AP(\mathbb{R}, X)$. For K in $BL(A)$ and $t \in \mathbb{R}$, define

$K_t: A \to A$ by $K_t f = (Kf)_t$, where $f \in A$ and $(Kf)_t$ is the translate of Kf by t. It is proved that (Theorem 3.1) an operator K on A is compact if and only if the map $F: \mathbb{R} \to KL(A, X)$ defined by $F(t)(f) = Kf(t)$, $t \in \mathbb{R}$, $f \in A$, is continuous almost periodic. We further show that this is equivalent to the fact that the function $\theta^K: t \to K_t$ from $\mathbb{R}$ into $KL(A, X)$ is continuous almost periodic. In Corollary 3.2, it is proved that a compact operator K on $AP(\mathbb{R}, X)$ can be represented by an operator-valued almost periodic function up to an isometric isomorphism.

When $X = \mathbb{C}$, we write $AP(\mathbb{R})$ for the space $AP(\mathbb{R}, X)$. Let B be the unit ball in $AP(\mathbb{R})$. If K is a compact operator on $AP(\mathbb{R})$, we obtain in Section 4, sufficient conditions on the map $\theta^K: t \to K_t$ from $\mathbb{R}$ into $KL(AP(\mathbb{R}))$, so that the Fourier series of Kf, for all f in $AP(\mathbb{R})$, converge at the same point. Also, for a compact operator K on $AP(\mathbb{R})$, we use the results of Section 3 to prove the following: The union of sets of Fourier exponents of functions in the range of K is countable, even though the range of K contains uncountably many functions. In fact, we show that this set is contained in the set of Fourier exponents of θ^K.

2. Preliminaries

The following definitions are from Corduneanu[2] and Burckel[3]:

Definition 2.1. Let X be a Banach space. A continuous function $f: \mathbb{R} \to X$ is called almost periodic, if for any number $\varepsilon > 0$, one can find a number $l(\varepsilon) > 0$ such that any interval of the real line of length $l(\varepsilon)$ contains at least one point of abscissa τ with the property that $\| f(t+\tau) - f(t) \| < \varepsilon$ for all $t \in \mathbb{R}$.

Definition 2.2. A set S is called a topological semigroup if S is a semigroup with identity e and if S has a Hausdorff topology such that the multiplication on S is separately continuous. That is, for each t in S, the maps $s \to st$ and $s \to ts$ are continuous functions.

Let S be a topological semigroup, X, a Banach space and $C(S, X)$ the Banach space of bounded continuous functions from S to X with supremum norm. For f in $C(S, X)$ and s in S, let f_s, the right translate of f by s, be defined by $f_s(t) = f(ts)$, $t \in S$. Let the right orbit of f, $O_R(f) = \{f_s : s \in S\}$.

Definition 2.3. A function f in $C(S, X)$ is called almost periodic if $O_R f$ is relatively compact in $C(S, X)$.

We shall denote by $AP(S, X)$, the set of all almost periodic functions on S to X. When $X = \mathbb{C}$, we write $AP(S)$ for $AP(S, \mathbb{C})$. It is easy to see that $AP(S, X)$ is a Banach space with the norm defined by $\| f \| = \sup_{t \in S} \| f(t) \|$. The space $AP(S, X)$ has been studied by Goldberg and Irwin[4]. When $S = \mathbb{R}$, the equivalence of definition 2.1 and definition 2.3 is proved in by Corduneanu[2] (Theorem 6.6).

Definition 2.4. A function $T: \mathbb{R} \to X$ defined by

$$T(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}, \quad t \in \mathbb{R},$$

where, for $1 \leqslant k \leqslant n$, λ_k are real numbers and c_k are in X, is called a trigonometric polynomial with values in X.

Definition 2.5. (Approximation property). A function $f:\mathbb{R}\to X$ is called a function with the approximation property, if for any $\varepsilon > 0$, one can determine a trigonometric polynomial T_ε with values in X, such that $\|f(t) - T_\varepsilon(t)\| < \varepsilon, t\in\mathbb{R}$.

Remark 1. It is proved[5] [Theorem 2.1] that $f:\mathbb{R}\to X$ is almost periodic if and only if it has the approximation property.

Definition 2.6. A family $\mathscr{F}$ in $AP(\mathbb{R}, X)$ is said to be equialmost periodic, if to any $\varepsilon > 0$, there corresponds a number $l(\varepsilon) > 0$, such that any interval of length $l(\varepsilon)$ contains at least one number τ for which $\|f(t+\tau) - f(t)\| < \varepsilon$ for all $f\in\mathscr{F}$ and for all $t\in\mathbb{R}$.

The following theorem is from Corduneanu[2] [Theorem 6.9].

Theorem 2.7. A finite family of functions in $AP(\mathbb{R}, X)$ is equialmost periodic.

For general theory of almost periodic functions on $\mathbb{R}$ with values in a Banach space X, we refer to Corduneanu[2] and Levitan & Zhikov[7]. It may be recalled that, for Banach space X, Y, an operator K in $BL(X, Y)$ is defined to be compact if the set $\{Kx:\|x\| \leqslant 1\}$ is relatively compact in Y.

3. Characterisations of compact linear operators on AP($\mathbb{R}, X$)

Throughout this section, X is a Banach space and A stands for the space $AP(\mathbb{R}, X)$. We shall obtain here some characterisations of compact linear operators on A in terms of operator-valued trigonometric polynomials and translates of operators defined in introduction. The proof of the following uses only the elementary properties of almost periodic functions in A.

Theorem 3.1. Let $K\in BL(A)$. Then the following are equivalent:

(i) K is compact

(ii) The map $F:\mathbb{R}\to KL(A, X)$ defined by $F(t)(f) = Kf(t)$, $f\in A$, $t\in\mathbb{R}$, is continuous almost periodic.

(iii) For each $\varepsilon > 0$, there exists a trigonometric polynomial T_ε in $KL(A)$ such that $\|K - T_\varepsilon\| < \varepsilon$.

(iv) The map $\theta^K:\mathbb{R}\to KL(A)$ defined by $\theta^K(t) = K_t$, $t\in\mathbb{R}$ is continuous almost periodic.

Proof. (i)$\Rightarrow$(ii). Assume that K is compact. Let B be the closed unit ball in A and $\varepsilon > 0$. Since KB is relatively compact in A, we can choose a finite set $\{f^{(1)}, \ldots, f^{(n)}\}$ in B so that $\{Kf^{(i)}: 1 \leqslant i \leqslant n\}$ is an $\varepsilon/3$-net for KB. Given $f\in B$, choose $i\in\{1, \ldots, n\}$ so that $\|Kf - Kf^{(i)}\| < \varepsilon/3$. Since the family $\{Kf^{(i)}: 1 \leqslant i \leqslant n\}$ is uniformly equicontinuous, there exists $\delta > 0$ such that, whenever $|s - t| < \delta$, $\|Kf^{(i)}(s) - Kf^{(i)}(t)\| < \varepsilon/3$ for all $i = 1, \ldots, n$. Hence, whenever

$|s-t|<\delta$, we have

$$\begin{aligned}\|F(s)(f)-F(t)(f)\| &= \|Kf(s)-Kf(t)\|\\ &\leqslant \|Kf(s)-Kf^{(i)}(s)\| + \|Kf^{(i)}(s)-Kf^{(i)}(t)\|\\ &\quad + \|Kf^{(i)}(t)-Kf(t)\|\\ &\leqslant \|Kf-Kf^{(i)}\| + \|Kf^{(i)}(s)-Kf^{(i)}(t)\| + \|Kf^{(i)}-Kf\|\\ &\leqslant \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.\end{aligned}$$

Therefore, $\|F(s)-F(t)\| = \sup_{\|f\|\leqslant 1} \|F(s)(f)-F(t)(f)\| \leqslant \varepsilon$, whenever $|s-t|<\delta$. This proves the uniform continuity of F. Now to prove the almost periodicity of F, note that as above

$$\|F(s+\tau)(f)-F(s)(f)\| \leqslant \|Kf-Kf^{(i)}\| + \|Kf^{(i)}(s+\tau)-Kf^{(i)}(s)\| + \|Kf^{(i)}-Kf\|.$$

It is then clear that the equialmost periodicity of $\{Kf^{(i)}:1\leqslant i\leqslant n\}$ implies F is almost periodic.

(ii)$\Rightarrow$(iii). If F is continuous almost periodic, then F has the approximation property (Section 2, *Remark* 1). Therefore, for each $\varepsilon>0$, there exists a trigonometric polynomial P_ε in $AP(\mathbb{R}.KL(A,X))$ such that $\|F-P_\varepsilon\|<\varepsilon$, where $P_\varepsilon(t)=\sum_{k=1}^{n} a_k e^{-i\lambda_k t}$, $a_k\in KL(A,X)$, $\lambda_k\in\mathbb{R}$, $1\leqslant k\leqslant n$. Now, define $T_\varepsilon:A\to A$ by $T_\varepsilon f(t)=P_\varepsilon(t)(f)$. We first prove that T_ε is compact. Let B be the unit ball in A. Since P_ε is almost periodic[2] [Theorem 6.5], the set $\{P_\varepsilon(t):t\in\mathbb{R}\}$ is relatively compact in $KL(A,X)$. Let $\{P_\varepsilon(t^{(i)}):1\leqslant i\leqslant m\}$ be a finite $\varepsilon/3$-net for $\{P_\varepsilon(t):t\in\mathbb{R}\}$. Also since a_ks are compact operators, the terms $a_k e^{t\lambda_k t}$ are compact. As $P_\varepsilon(t)$ is a finite sum of such terms, it is compact. In particular, $P_\varepsilon(t^{(i)})$ is compact for each $i=1,\ldots,m$. Now if $H:A\to X^m$ is defined by $Hf=(P_\varepsilon(t^1)(f),\ldots,P_\varepsilon(t^m)(f))$, then it is easy to see that HB is relatively compact in X^m. Let $Hf^{(1)},\ldots,Hf^{(p)}$ be a finite $\varepsilon/3$ – net for HB. Then for any f in B, there exists $f^{(j)}$ such that $\|Hf-Hf^{(j)}\|_{X^m}<\varepsilon/3$. But

$$\|Hf-Hf^{(j)}\|_{X^m}=\sum_{i=1}^{m}\|P_\varepsilon(t^{(i)})(f)-P_\varepsilon(t^{(i)})(f^{(j)})\|.$$

Hence

$$\|P_\varepsilon(t^{(i)})(f)-P_\varepsilon(t^{(i)})(f^{(j)})\|<\varepsilon/3 \text{ for all } i=1,\ldots,n \tag{I}$$

Let $t\in\mathbb{R}$ and $f\in B$. Choose $i\in\{1,\ldots,n\}$ such that $\|P_\varepsilon(t)-P_\varepsilon(t^{(i)})\|<\varepsilon/3$ and then $j\in\{1,\ldots,n\}$ so that (I) holds. Then

$$\begin{aligned}\|T_\varepsilon f(t)-T_\varepsilon f^{(j)}(t)\| &= \|P_\varepsilon(t)(f)-P_\varepsilon(t)(f^{(j)})\|\\ &\leqslant \|P_\varepsilon(t)(f)-P_\varepsilon(t^{(i)})(f)\|\\ &\quad + \|P_\varepsilon(t^{(i)})(f)-P_\varepsilon(t^{(i)})(f^{(j)})\|\\ &\quad + \|P_\varepsilon(t^{(i)})(f^{(j)})-P_\varepsilon(t)(f^{(j)})\|\\ &\leqslant \|P_\varepsilon(t)-P_\varepsilon(t^{(i)})\| + \|P_\varepsilon(t^{(i)})(f)-P_\varepsilon(t^{(i)})(f^{(j)})\|\\ &\quad + \|P_\varepsilon(t^{(i)})-P_\varepsilon(t)\|\\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.\end{aligned}$$

Therefore,

$$\| T_\varepsilon f - T_\varepsilon f^{(j)} \| = \sup_{t \in \mathbb{R}} \| T_\varepsilon f(t) - T_\varepsilon f^{(j)}(t) \| \leqslant \varepsilon.$$

This shows that $\{ T_\varepsilon f^{(j)} : 1 \leqslant j \leqslant n \}$ is a finite ε-net for $T_{\varepsilon B}$. Hence T_ε is compact. Finally, since

$$\| K - T_\varepsilon \| = \sup_{\|f\| \leqslant 1} \| Kf - T_\varepsilon f \| = \sup_{\|f\| \leqslant 1} \sup_{t \in \mathbb{R}} \| Kf(t) - T_\varepsilon f(t) \|$$

$$= \sup_{t \in \mathbb{R}} \sup_{\|f\| \leqslant 1} \| F(t)(f) - P_\varepsilon(t)(f) \| = \sup_{t \in \mathbb{R}} \| F(t) - P_\varepsilon(t) \| = \| F - P_\varepsilon \|,$$

assertion (iii) follows.

(iii)⇒(i) If (iii) holds then K is compact, since it is a uniform limit of compact operators T_ε.

(ii)⇒(iv). Suppose that F is continuous almost periodic and let $\varepsilon > 0$. Since F is uniformly continuous on $\mathbb{R}^2$ [Theorem 6.2], there exists $\delta > 0$ such that, whenever $|s - t| < \delta$, $\| F(x + s) - F(x + t) \| < \varepsilon$ for all $x \in \mathbb{R}$. But as

$$\begin{aligned} \sup_{x \in \mathbb{R}} \| F(x + s) - F(x + t) \| &= \sup_{x \in \mathbb{R}} \sup_{\|f\| \leqslant 1} \| F(x + s)(f) - F(x + t)(f) \| \\ &= \sup_{\|f\| \leqslant 1} \sup_{x \in \mathbb{R}} \| Kf(x + s) - Kf(x + t) \| \\ &= \sup_{\|f\| \leqslant 1} \| (Kf)_s - (Kf)_t \| \\ &= \sup_{\|f\| \leqslant 1} \| K_s(f) - K_t(f) \| \\ &= \| K_s - K_t \| \\ &= \| \theta^K(s) - \theta^K(t) \|, \end{aligned}$$

it follows that θ^K is continuous. Also, from the above equalities we have, for any t,

$$\begin{aligned} \| \theta^K(t + \tau) - \theta^K(t) \| &= \sup_{x \in \mathbb{R}} \| F(x + t + \tau) - F(x + t) \| \\ &= \sup_{x' = x + t \in \mathbb{R}} \| F(x' + \tau) - F(x') \|. \end{aligned}$$

Hence, the almost periodicity of θ^K follows from that of F. (iv)⇒(ii). From the equalities in (ii)⇒(iv), it is clear that the continuity and almost periodicity of F follow from that of θ^K.

Remark 1. The proof of the above theorem vitally depends on the fact that the functions in $AP(\mathbb{R}, X)$ have approximation property. If G is a locally compact abelian topological group then the algebra of trigonometric polynomials in G, that is, the algebra of finite linear combinations of the continuous characters on G, is norm dense in $AP(G)$ [Larsen[7], Theorem 10.7.4]. More generally, if S is a topological semigroup and algebraically an abelian group then by Burckel[3], Corollary 5.6, the space of finite linear combinations of semicharacters of S is norm dense in $AP(S)$. Hence in the above theorem, when $X = \mathbb{C}$, we can replace $\mathbb{R}$ by either a locally compact abelian group or a locally compact topological semigroup which is algebraically an abelian group. The exponential functions are then replaced by continuous characters or semicharacters.

Corollary 3.2. $KL(A)$, the Banach space of compact operators is isometrically isomorphic to a subspace of $AP(\mathbb{R}, KL(A))$.

Proof. From the above theorem, if K is compact then the map θ^K defined by $\theta^K(t) = K_t$ is continuous almost periodic. This defines a map $\psi: K \to \theta^K$ from $KL(A)$ into $AP(\mathbb{R}, KL(A))$. It is easy to see that ψ is a linear homomorphism. Also as $\|(Kf)_s\| = \|Kf\|$, for any $s \in \mathbb{R}$ we have

$$\begin{aligned}\|\psi(K)\| = \|\theta^K\| &= \sup_{s \in \mathbb{R}} \|\theta^K(s)\| = \sup_{s \in \mathbb{R}} \|K_s\| \\ &= \sup_{s \in \mathbb{R}} \sup_{\|f\| \leqslant 1} \|K_s f\| = \sup_{s \in \mathbb{R}} \sup_{\|f\| < 1} \|(Kf)_s\| \\ &= \sup_{\|f\| \leqslant 1} \sup_{s \in \mathbb{R}} \|Kf\| = \|K\|,\end{aligned}$$

which shows that ψ is an isometry. This completes the proof.

Remark 2. When $X = \mathbb{C}$, Theorem 3.1 can be generalised to obtain characterisations of collectively compact sets of operators on $A = AP(\mathbb{R})$. It may be recalled that for Banach spaces X, Y a set $\mathscr{K} \subset BL(X, Y)$ is collectively compact if $\{Kx: \|x\| \leqslant 1, K \in \mathscr{K}\}$ is relatively compact in Y[8]. Let A^* denote the topological dual of $AP(\mathbb{R})$. For each $K \in BL(A)$ define $F^K: \mathbb{R} \to A^*$ by $t \to F^K(t)$, where $F^K(t)(f) = Kf(t)$, $f \in A$, and $\theta^K: \mathbb{R} \to BL(A)$ by $\theta^K(t) = K_t$, $t \in \mathbb{R}$. Theorem 3.1 then can be generalised as follows:

Theorem 3.3. For a set of operators $\mathscr{K}$ in $BL(A)$, the following are equivalent:

(i) $\mathscr{K}$ is collectively compact.

(ii) The family $\{F^K: K \in \mathscr{K}\}$ is uniformly equicontinuous and equialmost periodic.

(iii) The family $\{\theta^K: K \in \mathscr{K}\}$ is uniformly equicontinuous and equialmost periodic.

4. Fourier series

Let K be a compact operator on $AP(\mathbb{R})$ and θ^K, as in Theorem 3.1. In this section, we apply the results of Section 3 to obtain sufficient conditions on the map θ^K so that the Fourier series of Kf, for all f in A, converge at the same point. We also investigate the relation between the Fourier exponents of functions in the range of K with those of θ^K. We show that even though the range of K is uncountable the union of sets of Fourier exponents of functions in the range of K is countable and in fact this set is contained in the set of Fourier exponents of θ^K.

Throughout this section, A denotes the space $AP(\mathbb{R})$. To obtain the desired results, we first defined vector-valued functions of bounded variation in a way suggested by scalar functions of bounded variation.

Definition 4.1. Let f be a function defined on the interval $[a, b]$ in $\mathbb{R}$ with values in Banach space X. If P is the partition of $[a, b]$ given by $a = t_0 < t_1 \cdots < t_{n-1} < t_n = b$, put $V(P, f) =$

$\sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \|$. Then the function f is said to be of bounded variation on $[a,b]$ if and only if $\sup_P V(P,f) < \infty$.

The following theorem is from Corduneanu[2] [Theorem 1.21].

Theorem 4.2. Assume that the almost periodic function $f(x)$ is such that $\lambda_{n+1} - \lambda_n \geqslant \alpha > 0$, $n = 1, 2, \ldots$, so that the unique limit point of its Fourier exponents is the point at infinity. If x_0 is a point in the neighbourhood of which $f(x)$ has bounded variation, then the Fourier series at x_0 converges to $f(x_0)$.

Theorem 4.3. Let $K \in KL(A)$ and B the closed unit ball in A. If θ^K is as in Theorem 3.1 and satisfies

(i) θ^K has bounded variation in the neighbourhood of a point (say) t_0 in $\mathbb{R}$.

(ii) The Fourier exponents of θ^K are such that $\lambda_{n+1} - \lambda_n \geqslant \alpha > 0$, $n = 1, 2, \ldots$ (That is, the unique limit point of the Fourier exponents of θ^K is the point at infinity).

Then, (a) The Fourier exponents of Kf, for all $f \in A$, belong to the set of Fourier exponents of θ^K.

(b) For all $f \in A$, the Fourier series of Kf converges at the point $2t_0$.

Proof. Since θ^K is of bounded variation in the neighbourhood, say $[a,b]$, of t_0,

$$\sup_P \sum_{i=1}^{n} \| \theta^K(t_i) - \theta^K(t_{i-1}) \| < \infty,$$

where supremum is taken over all partition P of $[a,b]$. But

$$\begin{aligned} \| \theta^K(t_i) - \theta^K(t_{i-1}) \| &= \| K_{t_i} - K_{t_{i-1}}) \| = \sup_{f \in B} \| K_{t_i} f - K_{t_{i-1}} f \| \\ &= \sup_{f \in B} \sup_{t \in \mathbb{R}} | Kf(t + t_i) - Kf(t + t_{i-1}) | \\ &\geqslant | Kf(t_0 + t_i) - Kf(t_0 + t_{i-1}) |, \end{aligned}$$

for all $f \in B$. Now, if $0 \neq f \in A$ is arbitrary then $(f/\|f\|) \in B$ and we have

$$\| \theta^K(t_i) - \theta^K(t_{i-1}) \| \geqslant \frac{1}{\|f\|} \| Kf(t_0 + t_i) - Kf(t_0 + t_{i-1}) \|.$$

Therefore, if $u_i = t_0 + t_i, \quad 1 \leqslant i \leqslant n,$

$$\sup_{P'} \sum_{i=1}^{n} | Kf(u_i) - Kf(u_{i-1}) | \leqslant \left(\sup_P \sum_{i=1}^{n} \| \theta^K(t_i) - \theta^K(t_{i-1}) \| \right) \| f \| < \infty.$$

This shows that for all f in A, Kf is of bounded variation in $[a + t_0, b + t_0]$, a neighbourhood of $2t_0$. Now let $\sum_{k=1}^{\infty} A_k e^{i\lambda_k(\cdot)}$ be the Fourier series associated with θ^K, where

$$A_k = a(\lambda, \theta^K) = \lim_{n \to \infty} \frac{1}{n} \int_0^n \theta^K(t) e^{-i\lambda_k t} \, dt.$$

Then

$$\begin{aligned}
\| a(\lambda, \theta^K) \| &= \sup_{\|f\| \leq 1} \| a(\lambda, \theta^K)(f) \| \\
&= \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} |a(\lambda, \theta^K)(f)(x)| \\
&= \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \frac{1}{n} \int_0^n \theta^K(t)(f) e^{-i\lambda_k t} \, dt \right| \\
&= \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \frac{1}{n} \int_0^n K_t f(x) e^{-i\lambda_k t} \, dt \right| \\
&= \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} \left| \lim_{n \to \infty} \frac{1}{n} \int_0^n Kf(x+t) e^{-i\lambda_k t} \, dt \right| \\
&\geq \left| \lim_{n \to \infty} \frac{1}{n} \int_0^n Kf(x+t) e^{-i\lambda_k t} \, dt \right| \quad \text{for all } f \in B,\ x \in \mathbb{R}.
\end{aligned}$$

Changing t to $t - x$, we have

$$\begin{aligned}
\text{RHS} &= \left| \lim_{n \to \infty} \frac{1}{n} \int_x^{x+n} Kf(t) e^{-i\lambda_k (t-x)} \, dt \right| \\
&= \left| \lim_{n \to \infty} \frac{1}{n} \int_0^n Kf(t) e^{-i\lambda_k t} \, dt \right| \quad \text{by Corduneanu}^2\text{; Theorem 1.12 and} \\
&\quad \text{since } |e^{-i\lambda x}| = 1.
\end{aligned}$$

Hence, RHS $= |a(\lambda, Kf)|$, where $a(\lambda, Kf)$ is the Fourier coefficient of Kf. Thus $\| a(\lambda, \theta^K) \| \geq |a(\lambda, Kf)|$ for all $f \in B$. Now, if $0 \neq f \in A$ be any element then $(f / \|f\|) \in B$ and we have $\| a(\lambda, \theta^K) \| \geq (1/\|f\|)\, |a(\lambda, Kf)|$. Therefore, for any $f \in A$, if $a(\lambda, Kf) \neq 0$ then $a(\lambda, \theta^K) \neq 0$. This shows that, for all $f \in A$ if λ is a Fourier exponent of Kf, then it is a Fourier exponent of θ^K. In other words, the union of sets of Fourier exponents of functions in the range of K is contained in the set of Fourier exponents of θ^K. This proves (a). Therefore, if θ^K satisfies (ii) so does Kf, for all $f \in A$. The assertion (b) then follows from Theorem 4.2.

We now give an example of a compact operator K on A and estimate the set of Fourier exponents of θ^K.

Example. For $f \in A$ define $K: A \to A$ by

$$Kg(s) = \lim_{n \to \infty} \frac{1}{n} \int_0^n f(s-t) g(t) \, dt.$$

Since f is almost periodic, from Theorem 3.1 it can be easily proved that K is compact. Let θ^K be as in Theorem 3.1. We shall show that the set of Fourier exponents of θ^K is precisely the set of Fourier exponents of f. Let

$$a(\lambda, \theta^K) = \lim_{m \to \infty} \frac{1}{m} \int_0^m \theta^K(s) e^{-i\lambda s} \, ds.$$

Then

$$\begin{aligned}\|a(\lambda,\theta^K)\| &= \sup_{\|g\|\leqslant 1} \|a(\lambda,\theta^K)(g)\| \\ &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} |a(\lambda,\theta^K)(g)(x)| \\ &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} \left|\lim_{m\to\infty}\frac{1}{m}\int_0^m \theta^K(s)(g)(x)\mathrm{e}^{-i\lambda s}\,\mathrm{d}s\right| \\ &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} \left|\lim_{m\to\infty}\frac{1}{m}\int_0^m (K_s g)(x)\mathrm{e}^{-i\lambda s}\,\mathrm{d}s\right| \\ &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} \left|\lim_{m\to\infty}\frac{1}{m}\int_0^m Kg(x+s)\mathrm{e}^{-i\lambda s}\,\mathrm{d}s\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{RHS} &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} \left|\lim_{m\to\infty}\frac{1}{m}\int_0^m \left[\lim_{n\to\infty}\frac{1}{n}\int_0^n f(x+s-t)g(t)\,\mathrm{d}t\right]\mathrm{e}^{-i\lambda s}\,\mathrm{d}s\right| \\ &= \sup_{\|g\|\leqslant 1}\sup_{x\in\mathbb{R}} \left|\lim_{m\to\infty}\frac{1}{m}\int_0^m \left(\lim_{n\to\infty}\phi_n(s)\,\mathrm{d}s\right)\right| \qquad \text{(I)}\end{aligned}$$

where

$$\phi_n(s)=\frac{1}{n}\int_0^n f(x+s-t)g(t)\mathrm{e}^{-i\lambda s}\,\mathrm{d}t.$$

We shall show that $\phi_n(s)$ converges uniformly in s. It is enough to prove that $\phi_n(s)$ is Cauchy uniformly in s. Since f is almost periodic, there exists $s_1,\ldots,s_k$ in $\mathbb{R}$ such that for any $s\in\mathbb{R}$, there is s_i, $1\leqslant i\leqslant k$ with

$$\|f_s-f_{s_i}\|\leqslant\frac{\varepsilon}{3\|g\|}.$$

Let $x\in\mathbb{R}$. For each $i=1,\ldots,k$ define $g_i(t)=f(x+s_i-t)g(t)$. Then g_i, $1\leqslant i\leqslant i\leqslant k$, is almost periodic function. Therefore

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n g_i(t)\,\mathrm{d}t$$

exists for each i. Hence there exists $N(i)$ such that

$$\left|\frac{1}{n}\int_0^n g_i(t)\,\mathrm{d}t-\frac{1}{n'}\int_0^{n'} g_i(t)\,\mathrm{d}t\right|<\frac{\varepsilon}{3}$$

for all n, $n'\geqslant N(i)$. Let $N=\max\{N(i){:}1\leqslant i\leqslant k\}$. Then, for n, $n'\geqslant N$ we have

$$\left|\frac{1}{n}\int_0^n f(x+s_i-t)g(t)\,\mathrm{d}t-\frac{1}{n'}\int_0^{n'} f(x+s_i-t)g(t)\,\mathrm{d}t\right|<\frac{\varepsilon}{3}$$

for all $i \leqslant k$. Now, for $n, n' \geqslant N$ and $s \in \mathbb{R}$,

$$|\phi_n(s) - \phi_{n'}(s)| = \left| \frac{1}{n} \int_0^n f(s+x-t) g(t) e^{-i\lambda s} \, dt - \frac{1}{n'} \int_0^{n'} f(s+x-t) g(t) e^{-i\lambda s} \, dt \right|$$

$$\leqslant \left| \frac{1}{n} \int_0^n [f(s+x-t) - f(s_i+x-t)] g(t) \, dt \right| + \left| \frac{1}{n} \int_0^n f(s_i+x-t) g(t) \, dt - \frac{1}{n'} \int_0^{n'} f(s_i+x-t) g(t) \, dt \right| + \left| \frac{1}{n'} \int_0^{n'} [f(s_i+x-t) - f(s+x-t)] g(t) \, dt \right|$$

$$\leqslant \| f_s - f_{s_i} \| \, \| g \| + \frac{\varepsilon}{3} + \| f_{s_i} - f_s \| \, \| g \|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus

$$|\phi_n(s) - \phi_{n'}(s)| < \varepsilon \text{ for all } n, n' \geqslant N \text{ and for all } s \in \mathbb{R}. \tag{II}$$

Hence, $\lim_{n\to\infty} \phi_n(s)$ exists uniformly for all $s \in \mathbb{R}$. But then from (I) we have

$$\| a(\lambda, \theta^K) \| = \sup_{\|g\| \leqslant 1} \sup_{x \in \mathbb{R}} \left| \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{m} \int_0^m \phi_n(s) \, ds \right| \tag{III}$$

Let

$$a_n^m = \frac{1}{m} \int_0^m \phi_n(s) \, ds.$$

Write $a^m = \lim_{n\to\infty} a_n^m$ and $a_n = \lim_{m\to\infty} a_n^m$. We shall prove that a_n^m converges to a^m uniformly in m. But again, it is enough to show that a_n^m is uniformly cauchy in m. From (II) if $n, n' \geqslant N$, we have

$$|a_n^m - a_{n'}^m| = \left| \frac{1}{m} \int_0^m \phi_n(s) \, ds - \frac{1}{m} \int_0^m \phi_{n'}(s) \, ds \right|$$

$$\leqslant \frac{1}{m} \int_0^m |\phi_n(s) - \phi_{n'}(s)| \, ds < \varepsilon.$$

This shows that $\lim_{n\to\infty} a_n^m$ exists uniformly in m. Therefore, the sequence $\{a_n\}$ converges and $\lim_{m\to\infty} a^m = \lim_{n\to\infty} a_n$. Equivalently,

$$\lim_{m\to\infty} \lim_{n\to\infty} a_n^m = \lim_{n\to\infty} \lim_{m\to\infty} a_n^m.$$

Hence, from (III),

$$\| a(\lambda, \theta^K) \| = \sup_{\|g\| \leqslant 1} \sup_{x \in \mathbb{R}} \left| \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{m} \int_0^m \phi_n(s) \, ds \right|$$

$$= \sup_{\|g\|\leq 1} \sup_{x\in\mathbb{R}} \left| \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{m} \int_0^m \left[\frac{1}{n} \int_0^n f(x+s-t)g(t)e^{-i\lambda s}\,dt \right] ds \right|$$

$$= \sup_{\|g\|\leq 1} \sup_{x\in\mathbb{R}} \left| \lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{mn} \int_0^n \left[\int_0^m f(x+s-t)e^{-i\lambda s}\,ds \right] g(t)\,dt \right|$$

If $\psi_m(t) = (1/m)\int_0^m f(x+s-t)g(t)e^{-i\lambda s}\,ds$, then it can be shown that $\psi_m(t)$ converges uniformly in t. Therefore, taking limit inside the integral and changing s to $s+t$ we have

$$\text{RHS} = \sup_{\|g\|\leq 1} \sup_{x\in\mathbb{R}} \left| \lim_{n\to\infty} \frac{1}{n} \int_0^n \left[\lim_{m\to\infty} \frac{1}{m} \int_{-t}^{m-t} f(x+s)e^{-i\lambda s}\,ds \right] g(t)e^{-i\lambda t}\,dt \right|$$

$$= \sup_{\|g\|\leq 1} \sup_{x\in\mathbb{R}} \left| \lim_{n\to\infty} \frac{1}{n} \int_0^n \left[\lim_{m\to\infty} \frac{1}{m} \int_0^m f(x+s)e^{-i\lambda s}\,ds \right] g(t)e^{-i\lambda t}\,dt \right|$$

$$= \sup_{\|g\|\leq 1} \sup_{x\in\mathbb{R}} |a(\lambda, f_x)| \left| \lim_{n\to\infty} \frac{1}{n} \int_0^n g(t)e^{-i\lambda t}\,dt \right|$$

$$= |a(\lambda, f)| \sup_{\|g\|\leq 1} |a(\lambda, g)|, \text{ since } |a(\lambda, f_x)| = |a(\lambda, f)|$$

$$= |a(\lambda, f)|, \text{ since } \sup_{\|g\|\leq 1} |a(\lambda, g)| = 1.$$

Thus, $\| a(\lambda, \theta^K) \| = |a(\lambda, f)|$, which shows that λ is a Fourier exponent of θ^K if and only if it is a Fourier exponent of f.

References

1. Schaefer, H. H. — *Banach lattices and positive operators*, Springer-Verlag, 1974.
2. Corduneanu, C. — *Almost periodic functions*, Wiley Interscience, 1968.
3. Burckel, R. B. — *Weakly almost periodic functions on semigroups*, Gordon and Breach, 1970.
4. Goldberg, S. and Irwin, P. — Weakly almost periodic vector-valued functions, *Diss. Math.*, 1979, **157**, 1–42.
5. Amerio, L. and Prouse, G. — *Almost periodic functions and functional equations*, Van Nostrand, 1971.
6. Levitan, B. M. and Zhikov, V. V. — *Almost periodic functions and differential equations*, Cambridge University Press, 1982.
7. Larsen, R. — *Banach algebras, an introduction*, Marcel-Dekker, 1973.
8. Anselone, P. M. — *Collectively compact operator approximation theory and applications to integral equations*, 1971, Prentice-Hall.