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On the growth of a class of plurisubharmonic functions in a unit disc

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Abstract

The concepts of growth for plurisubharmonic functions were studied by Ronkin and of generalised growth by Juneja and Sinha. The growth of plurisubharominc functions in a finite domain is presented in this paper

Keywords: Plurisubharmonic, growth parameters.

1. Introduction

A non-constant entire function of several complex variables is denoted by,

$$f(\tilde{z}) = \sum_{\|\bar{k}\|=0}^{\infty} C_{k_1 k_2,\ldots,k_n} z_1^{k_1} z_2^{k_2},\ldots, z_n^{k_n}. \tag{1.1}$$

Here we shall denote the complex n-tuple of vectors $(\sim)$, and the real n-tuple $(-)$. That is to say $\tilde{Z}\in\mathbb{C}^n$ if $\tilde{Z}=(z_1,z_2,\ldots,z_n)$ and $\bar{t}\in\mathbb{R}^n$ if $\bar{t}=(t_1,t_2,\ldots,t_n)$. We shall also denote by $\|\bar{A}\| = (A_1 + A_2 + \cdots + A_n)$. Let $D = \{(z_1, z_2,\ldots,z_n): |z_i - z_i^{(0)}| < R_i, i = 1,2,\ldots,n\}$ be a polydisc in $\mathbb{C}^n$ with centre at $\tilde{Z}^{(0)} = (z_1^{(0)}, z_2^{(0)},\ldots,z_n^{(0)})$ and polyradius equal to $\bar{R} = (R_1, R_2,\ldots,R_n)$, where each R_i is a fixed real number for $i = 1,2,3,\ldots,n$. Growth parameters for plurisubharmonic functions were studied in great detail by Ronkin[1], which were later generalised by Juneja and Sinha[2]. For definition and properties of plurisubharmonic functions see Lelong and Gruman[3].

Ronkin considered the class U of plurisubharmonic functions as introduced by him. It is known[1] that the functions $\varphi(\bar{R})$ in U satisfy the following inequality,

$$\phi(t_1^{\lambda}s_1^{\mu}, t_2^{\lambda}s_2^{\mu},\ldots,t_n^{\lambda}s_n^{\mu}) \leqslant \lambda\phi(t_1,\ldots,t_n) + \mu\phi(s_1,\ldots,s_n)\} \tag{1.2}$$

for any $\bar{t}$ and $\bar{s}\in\mathbb{R}_+^n$ and $\lambda+\mu=1$. Here $\mathbb{R}_+^n$ denotes the positive hyperoctant of $\mathbb{R}^n$. For other relevant definitions and concepts see Ronkin[1]. Šeremeta[5] generalised the definitions of order and type with the help of two classes of functions *viz.*, L^0 and Δ. Juneja and Sinha[2] further generalised class U by using Šeremeta's class of slowly varying functions. The definitions and examples of these classes can be found in Juneja and Sinha[2].

Juneja and Kapoor[4] studied the growth of analytic functions in a polydisc domain. They found the coefficient characterisation of order, studied partial orders, hypersurface of associated orders and its geometry. In this article the author has made an attempt to study the growth of plurisubharmonic functions in a general setting. Thus the results of Juneja and Kapoor[4] in this regard can be obtained as special cases.

2. Main results

First of all we define the class $E(\beta)$, $\beta \in L^0$.

Definition 2.1: Let $E(\beta)$ be the class of functions $\varphi(\bar{R})$ satisfying the following:

(i) $\varphi(\bar{R})$ in $E(\beta)$ is upper semi-continuous on D.
(ii) $\varphi(\bar{R})$ is monotone non-decreasing in each of the variables $R_1, R_2, \ldots, R_n$.
(iii) $\varphi(\bar{R})$ is pluriconvex in $-\beta(\log(1-R_1)), -\beta(\log(1-R_2)), \ldots, -\beta(\log(1-R_n))$, meaning thereby for every $\bar{t} = (t_1, t_2, \ldots, t_n)$ and $\bar{s} = (s_1, s_2, \ldots, s_n)$ in I^n_+ and for all λ, μ such that $\lambda + \mu = 1$

$$\begin{aligned}&\varphi[1 - \exp\{\beta^{-1}(\lambda\beta(\log(1-t_1)) + \mu\beta(\log(1-s_1))\}, \ldots,\\ &\quad 1 - \exp\{\beta^{-1}(\lambda\beta(\log(1-t_n)) + \mu\beta(\log(1-s_n))\}]\\ &\quad \leqslant \lambda\varphi(t_1, \ldots, t_n) + \mu\varphi(s_1, \ldots, s_n).\end{aligned} \tag{2.1}$$

Here $I^n = \{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n : 0 \leqslant r_i \leqslant 1,\ i = 1, 2, \ldots, n\}$ and I^n_+ is the positive part of I^n. It is worthwhile to see that by substituting $\beta(x) = x$, the identity function in (2.1) class $E(\beta)$ reduces to the class of functions $\varphi(\bar{R})$ defined by Juneja and Kapoor[4].

Definition 2.2: Let

$$M_D(t, \varphi) = \max_{\bar{R} \in tD} \varphi(\bar{R}), \quad 0 < t < 1$$

be the maximum modulus of the function $\varphi(\bar{R})$, and D the unit polydisc. The generalised order of $\varphi(\bar{R})$ is defined as,

$$\rho(\varphi) = \limsup_{t \to 1} \frac{\alpha(M_D(t, \varphi))}{-\beta(\log(1-t))} \tag{2.2}$$

where $\alpha \in \Delta$ and $\beta \in L^0$.

Example: Consider the function,

$$\varphi(R_1, R_2) = \alpha^{-1}[-\beta(\log(1-R_1)) - \beta(\log(1-R_2))].$$

Then it can be easily seen from (2.2) that $\varphi(R_1, R_2)$ has generalised order 2.

We now introduce the system of generalised associated orders and the hypersurface of generalised associated orders.

Definition 2.3: Let $\alpha\in\Delta$ and $\beta\in L^0$, and let $\phi(\bar{R})\in E(\beta)$ be a function of finite generalised order $\rho(\varphi)$ as defined above. Let

$$B_\rho = B_\rho(\varphi) = \left\{(a_1, a_2, \ldots, a_n)\in\mathbb{R}^n_+ : \text{as } \|\bar{R}\| \to 1,\right.$$
$$\left.\varphi(\bar{R}) < \sum_{i=1}^{n} \alpha^{-1}[-a_i\beta(\log(1-R_i))]\right\}. \tag{2.3}$$

From the properties of the functions α and β, it is very easy to see the following:

(a) The set $B_\rho(\varphi)$ is octant like.
(b) The boundary points of the set $B_\rho(\varphi)$ form a certain hypersurface $S_\rho = S_\rho(\varphi)$, which divides the hyperoctant $\mathbb{R}^n_+$ into two parts, one in which the inequality (2.3) is true and the other in which it is false. Thus we shall call it the hypersurface of generalised associated orders of $\varphi(\bar{R})$ in $E(\beta)$, and any system of numbers $\rho_1, \rho_2, \ldots, \rho_n$ such that $(\rho_1, \rho_2, \ldots, \rho_n)\in S_\rho(\varphi)$ will be called a system of generalised associated orders of the function $\varphi(\bar{R})$ in $E(\beta)$.

Remark: Taking $\alpha(x) = \log x$, $\beta(x) = x$ and $\varphi(\bar{R}) = \log M(\bar{R}, f)$ where

$$M(\bar{R}, f) = \max_{\substack{|z_i| = r_i \\ i = 1, 2, \ldots, n}} |f(\tilde{Z})|. \tag{2.4}$$

We get back equation (5.3.1) of Juneja and Kapoor [4].

We now introduce the generalised order of the function $\varphi(\bar{R})$ in $E(\beta)$, with respect to one of the variables R_i (keeping the other $i \neq j$ fixed).

Definition 2.4: Let $\alpha\in\Delta$ and $\beta\in L^0$, then the generalised order of $\varphi(\bar{R})$ in $E(\beta)$ with respect to the variable R_i (keeping the other variables $i \neq j$ fixed) is defined as,

$$\rho_i^*(\varphi) = \limsup_{R_i \to 1} \frac{\alpha(\varphi^+(\bar{R}))}{-\beta(\log(1-R_i))} \tag{2.5}$$

Definitions (2.2) and (2.3) team up to prove

Theorem 2.1: The hypersurface of generalised associated order of the function $\varphi(\bar{R})$ in $E(\beta)$ determines its order $\rho(\varphi)$.

From the definition of the generalised order with respect to one of the variables we first prove,

Theorem 2.2: The generalised order ρ_i with respect to one of the variables keeping the others $(i \neq 1)$ fixed is independent of the values assigned to the fixed variables.

Theorem 2.3: Let $\rho(\varphi)$ be the generalised order of $\varphi(\bar{R})$ and $\rho_i^*(\varphi)$ the generalised order of $\varphi(\bar{R})$ with respect to one of the variables i, $1 \leqslant i \leqslant n$, then for any $\varphi(\bar{R})$ in $E(\beta)$ we have

$$\rho(\varphi) \leqslant \sum_{i=1}^{n} \rho_i^*(\varphi). \tag{2.6}$$

Also we have a relation between $\rho_i^*(\varphi)$ and the system of associated orders.

Theorem 2.4: Let $\alpha\in\Delta$ and $\beta\in L^0$, then if $\rho(\varphi)<\infty$ is the generalised order of $\varphi(\bar{R})$ in $E(\beta)$ and $\rho_i^*(\varphi)$ the generalised order with respect to one of the variables, then

$$\rho_i^*(\varphi)=\inf_{(\rho_1,\rho_2,\ldots,\rho_n)\in S_\rho(\varphi)}\rho_i(\varphi). \tag{2.7}$$

Finally a geometrical characterisation of the hypersurface of generalised associated orders is given in the shape of the following theorem.

Theorem 2.5: Let $\varphi(\bar{R})$ be in $E(\beta)$, for $\alpha\in\Delta$, $\beta\in L^0$. Also let B_ρ^0 be the domain consisting of the interior points of the set $B_\rho(\varphi)$. Then the domain B_ρ^{-1} that is the image of B_ρ^0 under the transformation $a_i'=1/a_i$, $i=1,2,\ldots,n$ is a complete convex domain.

3. Proofs

Proof of Theorem 2.1: Let $(t,t,\ldots,t)\in S_\rho(\varphi)$. Then for $\|\bar{R}\|\to 1$ and any $\varepsilon>0$, we have

$$\varphi(\bar{R})<\sum_{i=1}^{n}\alpha^{-1}[-(t+\varepsilon)\beta(\log(1-R_i))]. \tag{3.1}$$

Therefore, for $\bar{R}^*=(R,R,\ldots,R)$ such that R is sufficiently close to 1,

$$\varphi(\bar{R}^*)<n\alpha^{-1}[-(t+\varepsilon)\beta(\log(1-R))]. \tag{3.2}$$

But, on the other hand, there exists a sequence $\{\bar{R}_n^*\}$ such that for $\|\bar{R}_n^*\|\to 1$ as $n\to\infty$

$$\varphi(\bar{R}_n^*)>\alpha^{-1}[-(t-\varepsilon)\beta(\log(1-R_n))] \tag{3.3}$$

where

$$\bar{R}_n^*=(R_n,R_n,\ldots,R_n).$$

(3.3) implies that for $\bar{R}=(R_1,R_2,\ldots,R_n)$ with $\|\bar{R}\|$ sufficiently close to 1

$$\begin{aligned}\varphi(\bar{R})&\leqslant\varphi(\bar{R}^{**})\\ &\leqslant\max\{\alpha^{-1}[-(t-\varepsilon)\beta(\log(1-R_1))],\ldots,\alpha^{-1}[-(t-\varepsilon)\beta(\log(1-R_n))]\}\\ &<\sum_{i=1}^{n}\alpha^{-1}[-(t-\varepsilon)\beta(\log(1-R_n))]\end{aligned} \tag{3.4}$$

where

$$\bar{R}^{**}=(R,R,\ldots,R).$$

Inequality (3.4) contradicts the fact that $(t,t,\ldots,t)\in S_\rho(\varphi)$. But $\varphi(R,R,\ldots,R)=M_D(t,\varphi)$, where D is the unit polydisc. Therefore $t=\rho(\varphi)$. Thus geometrically the generalised order $\rho(\varphi)$ can be obtained as the intersection of the hypersurface $S_\rho(\varphi)$ with the ray

$$\{\bar{a}\colon\bar{a}\in\mathbb{R}_+^n,\ a_i=t,\ i=1,2,\ldots,n,\ 0\leqslant t\leqslant\infty\}.$$

Proof of Theorem 2.2: Without the loss of generality, we can assume $i=n$. Set

$$\rho_n^*(R_1^*,\ldots,R_{n-1}^*)=\limsup_{R_n^*\to\infty}\frac{\alpha(\varphi^+(\bar{R}^*))}{-\beta(\log(1-R_n^*))} \tag{3.5}$$

ρ_n^* is monotone non-decreasing from the properties of α and β. From (3.5) we get,

$$\varphi^+(\bar{R}) < \alpha^{-1}[-(\rho_n^* + \varepsilon)\beta(\log(1 - R_n^*)) + C_\varepsilon(R_1^*, \ldots, R_{n-1}^*)] \tag{3.6}$$

where $C_\varepsilon(R_1^*, \ldots, R_{n-1}^*)$ is some function less than infinity. Estimating $\varphi(\bar{R})$ by the inequality (2.1) with

$$t_i = R_i^*$$

$$s_i = 1 - \exp\left[\frac{\beta^{-1}[\beta(\log(1 - R_i)) - \lambda\beta(\log(1 - R_i^*))]}{\mu}\right] \quad \text{for } 1 \leqslant i \leqslant n-1.$$

and

$$t_n = 1 - \exp\left[\beta^{-1}\left\{\frac{1}{\lambda}\beta(\log(1 - R_n))\right\}\right], \quad s_n = 0$$

and using (3.6) we obtain,

$$\varphi(R_1, R_2, \ldots, R_n) \leqslant \lambda\alpha^{-1}\left[-\left(\frac{\rho_n^* + \varepsilon}{\lambda}\right)\beta(\log(1 - R_n)) + C_\varepsilon(R_1^*, \ldots, R_{n-1}^*)\right] + \mu\varphi[s_1, s_2, \ldots, 0]$$

applying α on both the sides and using the definition of the class Δ we get

$$\limsup_{R_n \to 1} \frac{\alpha(\varphi^+(R_1, \ldots, R_n))}{-\beta(\log(1 - R_n))} \leqslant \frac{\rho_n^* + \varepsilon}{\lambda}$$

or

$$\rho_n^*(R_1, R_2, \ldots, R_{n-1}) \leqslant \frac{\rho_n^* + \varepsilon}{\lambda}.$$

Since ε and λ are arbitrary, we have,

$$\rho_n^*(R_1, R_2, \ldots, R_{n-1}) \leqslant \rho_n^*(R_1^*, \ldots, R_{n-1}^*). \tag{3.7}$$

We can similarly prove that,

$$\rho_n^*(R_1^*, R_2^*, \ldots, R_{n-1}^*) \leqslant \rho_n^*(R_1, R_2, \ldots, R_{n-1}). \tag{3.8}$$

Combining (3.7) and (3.8) we prove the assertion.

Proof of Theorem 2.3: From the inequality (2.1) we can easily see that $\varphi(\bar{R})$ satisfies the following generalised inequality,

$$\begin{aligned}\varphi[1 - \exp\{\beta^{-1}(\lambda_1\beta(\log(1 - t_{11})) + \lambda_2\beta(\log(1 - t_{12})) + \cdots + \lambda_n\beta(\log(1 - t_{1n}))\}, \ldots,\\ 1 - \exp\{\beta^{-1}(\lambda_1\beta(\log(1 - t_{n1})) + \cdots + \lambda_n\beta(\log(1 - t_{nn})\}] \\ \leqslant \sum_{i=1}^{n} \lambda_i\varphi(t_{i1}, \ldots, t_{in}).\end{aligned} \tag{3.9}$$

Estimating the function $\varphi(R, \ldots, R)$ by the above inequality (3.9) with the choice,

$$t_{ii} = 1 - \exp\left[\beta^{-1}\left\{\frac{1}{\lambda_i}\beta(\log(1 - R))\right\}\right]$$

and

$$t_{ij}=0 \quad \text{for } i\neq j$$

and by definition (2.4) we get

$$\varphi(R,R,\ldots,R)\leqslant \sum_{i=1}^{n}\lambda_i\alpha^{-1}\left[-\left(\frac{\rho_i^*+\varepsilon}{\lambda_i}\right)\beta(\log(1-R))\right]$$

$$<\sum_{i=1}^{n}\alpha^{-1}\left[-\left(\frac{\rho_i^*+\varepsilon}{\lambda_i}\right)\beta(\log(1-R))\right].$$

Setting,

$$\lambda_i=\frac{\rho_i^*+\varepsilon}{\sum_{i=1}^{n}\rho_i^*+n\varepsilon}$$

$$\varphi(\bar{R})<n\alpha^{-1}\left[-\left(\sum_{i=1}^{n}\rho_i^*+n\varepsilon\right)\beta(\log(1-R))\right]$$

or

$$\frac{\alpha\left(\frac{1}{n}\varphi(\bar{R})\right)}{-\beta(\log(1-R))}\leqslant\sum_{i=1}^{n}\rho_i^*+n\varepsilon.$$

Taking limit and using the fact that $\alpha\in\Delta$, we get,

$$\rho(\varphi)\leqslant\sum_{i=1}^{n}\rho_i^*$$

Remark: By taking $\alpha(x)=\log x$ and $\beta(x)=x$ we can get theorem 5.4.2 of Juneja and Kapoor[4] from our theorem.

Proof of Theorem 2.4: Without the loss of generality we assume that $i=n$. Now from definitions 2.2 and 2.4 we obtain,

$$\varphi^+(\bar{R})<\alpha^{-1}[-(\rho+\varepsilon)\beta(\log(1-R))] \tag{3.10}$$

and

$$\varphi^+(0,0,\ldots,0,R_n^*)<\alpha^{-1}[-(\rho_n^*+\varepsilon)\beta(\log(1-R_n^*))]. \tag{3.11}$$

Estimating the function $\varphi(\bar{R})$ by inequality (2.1) with

$$R=\max[R_1,R_2,\ldots,R_{n-1}]$$

$$t_i=1-\exp\left[\beta^{-1}\left\{\frac{1}{\lambda}\beta(\log(1-R))\right\}\right]$$

$$s_i=0 \quad \text{for } i=1,2,\ldots,n-1 \text{ and } t_n=0$$

$$s_n=1-\exp\left[\beta^{-1}\left\{\frac{1}{\lambda}\beta(\log(1-R_n^*))\right\}\right]$$

to get,

$$\varphi(R_1,R_2,\ldots,R_n^*)\leqslant\varphi(R,R,\ldots,R_n^*)$$

$$\leqslant \lambda\varphi\left(1-\exp\left[\beta^{-1}\left\{\frac{1}{\lambda}\beta(\log(1-R))\right\}\right],\ldots,0\right)$$
$$+\mu\varphi\left(0,0,0,\ldots,1-\exp\left[\beta^{-1}\left\{\frac{1}{\lambda}\beta(\log(1-R_n^*))\right\}\right]\right).$$

Then using (3.10) and (3.11) we obtain,

$$\varphi(R_1,R_2,\ldots,R_n^*)\leqslant\sum_{i=1}^{n}\alpha^{-1}\left[-\left(\frac{\rho+\varepsilon}{\lambda}\right)\beta(\log(1-R))\right]$$
$$+\alpha^{-1}\left[-\left(\frac{\rho_n^*+\varepsilon}{\lambda}\right)\beta(\log(1-R_n^*))\right]$$

which, however, means that,

$$\left(\frac{\rho+\varepsilon}{\lambda},\ldots,\frac{\rho_n^*+\varepsilon}{\lambda}\right)\in B_\rho(\varphi).$$

But this implies that there exists a point $(\rho_1,\rho_2,\ldots,\rho_n)$ on $S_\rho(\varphi)$ such that $\rho_n<(\rho_n^*+\varepsilon)/\lambda$. But $\varepsilon>0$ and λ are arbitrary. We, therefore, have,

$$\inf_{(\rho_1,\rho_2,\ldots,\rho_n)\in S_\rho(\varphi)}\{\rho_n\}\leqslant\rho_n^*. \tag{3.12}$$

The reverse inequality

$$\rho_n^*\leqslant\inf_{(\rho_1,\rho_2,\ldots,\rho_n)\in S_\rho(\varphi)}\{\rho_n\} \tag{3.13}$$

is obvious. Combining (3.12) and (3.13) we get our desired result.

Remark: The above result also generalises theorem 5.4.3 of Juneja and Kapoor[4].

Proof of Theorem 2.5: It is easy to see that the domain B_ρ^{-1} is complete. We now prove its convexity. Let $\bar{a}$, $\bar{b}\in\mathring{B}_\rho$. Estimating the function $\varphi(\bar{R})$ by the inequality (2.1) with

$$t_i=1-\exp\left[\beta^{-1}\left(\frac{a_i}{\lambda a_i+\mu b_i}\beta(\log(1-R_i))\right)\right]$$

and

$$s_i=1-\exp\left[\beta^{-1}\left(\frac{b_i}{\lambda a_i+\mu b_i}\beta(\log(1-R_i))\right)\right]$$

for $i=1,2,\ldots,n$ we obtain,

$$\varphi(R_1,R_2,\ldots,R_n)<\lambda\sum_{i=1}^{n}\alpha^{-1}\left[-\frac{a_ib_i}{\lambda a_i+\mu b_i}\beta(\log(1-R_i))\right]$$
$$+\mu\sum_{i=1}^{n}\alpha^{-1}\left[-\frac{a_ib_i}{\lambda a_i+\mu b_i}\beta(\log(1-R_i))\right]$$

or

$$\varphi(R_1, R_2, \ldots, R_n) < \sum_{i=1}^{n} \alpha^{-1}\left[-\frac{1}{(\lambda/b_i + \mu/a_i)}\beta(\log(1 - R_i))\right]$$

which thus show that for any λ and μ with $\lambda + \mu = 1$ the point

$$(\lambda/b_1 + \mu/a_1, \ldots, \lambda/b_n + \mu/a_n) \in B_\rho^{-1}$$

proving the convexity of the domain.

Remark: The above result generalises theorem 5.3.3 of Juneja and Kapoor[4].

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