AN APPROXIMATE THEORY OF THE CAVITY RESONATOR METHOD OF DETERMINING THE DIELECTRIC LOSS OF SOLIDS AT MICROWAVE FREQUENCIES

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ABSTRACT

The field equations in a cylindrical cavity resonator and the perturbation formula due to Bethe and Schwinger (1943) are utilised to calculate the real part ϵ' and the imaginary part ϵ'' of the generalised dielectric coefficient of a small sample of solid dielectric rod. The loaded Q of the cavity is also calculated with the help of the field equations and the Poynting vector.

INTRODUCTION

At microwave frequencies, cavity resonators replace the conventional RLC tuned circuit elements of lower frequencies. They have much higher Q values than the latter. These elements are becoming increasingly important in microwave engineering because of their perfect shielding, their low loss factor and their adaptability to various electronic systems. A cavity resonator, regardless of its shape, has several eigen frequencies, corresponding to the different modes in which it can oscillate. These resonant frequencies are theoretically infinite in number and more closely spaced as the frequency increases. The total number of resonances is a function of the volume of the cavity. In order that a cavity may operate on a desired mode, free from the interfering effects of the secondary modes, it is necessary that the cavity should be designed to have minimum volume and maximum Q. A cavity designed to operate in the $TE_{0'n}$– mode gives the smallest volume for a desired value of Q.

The object of the present paper is to calculate ϵ', ϵ'' and from this the loss tangent $\tan \delta = \epsilon''/\epsilon'$ of a solid dielectric. It is also the purpose of the paper to calculate the Q of a cylindrical cavity excited in the TE_{01n}– mode when the dielectric rod is introduced into the cavity at the place of maximum electric field. Some investigations have been done by Barrow and Mieher (1940), Sproull and Linder (1944), McLean (1946), Hansen and Post (1948),

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Allan and Curling (1948), Birnbaum and Franeau (1949). The present investigations have been undertaken to evolve a simpler method of determining dielectric loss of solids at microwave frequencies. The method can also be employed for liquids and gases with suitable experimental arrangement.

BETHE AND SCHWINGER'S FORMULÆ

If an ideal cavity is designated by 1 and the same cavity by 2 when slightly deformed by the introduction of a small dielectric rod of dielectric constant $\epsilon = \epsilon' - j\epsilon''$ into the cavity then ϵ' is given in terms of the frequency shift Δf and the frequency f_2 of the loaded cavity by the following expression:

$$\frac{\Delta f}{f_2} = (\epsilon' - 1)\frac{\int_v \mathbf{E}_1 . \mathbf{E}_2 \, dv}{2\int_V \mathrm{E}_1{}^2 d\mathrm{V}} \approx (\epsilon' - 1)\frac{\int_v \mathrm{E}_1{}^2 dv}{2\int_V \mathrm{E}_1{}^2 d\mathrm{V}} \tag{1}$$

where

E_1 = amplitude of the electric field in the ideal cavity.

E_2 = amplitude of the electric field in the cavity with the dielectric rod. v and V represent the volume of the dielectric inserted into the cavity and the volume of the ideal cavity respectively.

The imaginary part ϵ'' is given in terms of the unloaded Q (Q_1) and the loaded Q (Q_2) of the cavity by the following expression:

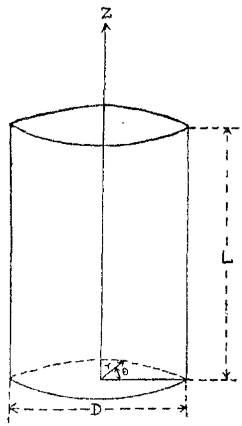
$$\frac{1}{\mathrm{Q}_2} - \frac{1}{\mathrm{Q}_1} = \epsilon'' \frac{\int_v \mathbf{E}_1 \cdot \mathbf{E}_2 \, dv}{\int_V \mathrm{E}_1{}^2 \, d\mathrm{V}}. \tag{2}$$

The relative error in making the assumption that $\mathbf{E}_1 = \mathbf{E}_2$ can be shown to be of the order of $(1/\mathrm{Q}_2) + (\Delta f/f_2)$ and this is very small.

FIELD COMPONENTS IN A CYLINDRICAL CAVITY RESONATOR

Let it be assumed that the cavity is of right circular cylindrical shape; and that the boundary conditions that **E** is normal to all boundary surfaces and **H** is tangential are satisfied. Then the normal mode fields in a completely lossless cavity excited in TE_{lmn} mode are given by the following relations (Kinzer, 1943).

$$\left.\begin{aligned}
E_r &= -l\frac{J_l(K_1r)}{K_1r}\sin l\theta \sin K_3z\\
E_\theta &= -J_l'(K_1r)\cos l\theta \sin K_3z\\
E_z &= 0\\
H_r &= \frac{K_3}{K}J_l'(K_1r)\cos l\theta \cos K_3z\\
H_\theta &= -l\frac{K_3}{K}\frac{J_l(K_1r)}{K_1r}\sin l\theta \cos K_3z\\
H_z &= \frac{K_1}{K}J_l(K_1r)\cos l\theta \sin K_3z
\end{aligned}\right\} \quad (3)$$



where,

$$n > 0,\ m > 0$$

$$K_1 = 2x_{lm}/D = x_{lm}/R$$

$$K_3 = n\pi/L$$

$$K^2 = K_1{}^2 + K_3{}^2, \text{ also } K = 2\pi/\lambda.$$

l represents the number of full period variations of the radial component of the electric field E_r along the angular or θ-co-ordinate.

m represents the number of half period variations of the angular component E_θ of electric field along the radial or r-co-ordinate.

n represents the number of half period variations of the radial component E_r of electric field along the axial or z co-ordinate.

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The above field components representing the normal mode fields are expressed in terms of trigonometric and Bessel functions and the resultant frequencies include the roots of Bessel functions which are given by the following relation.

$$\lambda = 2/[(x_{lm}/\pi R)^2 + (n/L)^2]^{\frac{1}{2}} \tag{4}$$

where λ represents the free space wavelength of the resonant frequencies and

$$x_{lm} = m^{th} \text{ root of } J_l'(x) = 0$$

In the case of TE_{01n} mode, $l = 0$, $m = 1$ and hence $K_1 = x_{01}/R$ and $K_3 = n\pi/L$. So for TE_{01n} mode equation (3) reduces to

$$\left.\begin{aligned} E_r &= E_z = H_\theta = 0 \\ E_\theta &= -J_0'(K_1 r)\sin K_3 z \\ H_r &= \frac{K_3}{K} J_0'(K_1 r)\cos K_3 z \\ H_z &= \frac{K_1}{K} J_0(K_1 r)\sin K_3 z \end{aligned}\right\} \tag{5}$$

CALCULATION OF ϵ'

The value of ϵ' can be calculated from equations (1) and (5) as follows:

$$\int_v E_1^2 dv = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{d} r\,[J_0'(K_1 r)]^2 \sin^2 K_3 z \; d\theta\, dr\, dz$$

$$= \frac{\pi^3 a^2 n^2 d^3}{3L^2}\left[J_0^2\left(\frac{x_{01}a}{R}\right) + J_1^2\left(\frac{x_{01}a}{R}\right) - \frac{2RJ_0\left(\frac{x_{01}a}{R}\right)J_1\left(\frac{x_{01}a}{R}\right)}{x_{01}a}\right] \tag{6}$$

neglecting powers of sine higher than the third

a = radius of the dielectric rod

d = depth of insertion of the dielectric rod into the cavity.

Similarly,

$$\int_V E_1^2 dV = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{z=0}^{L} r\,[J_0'(K_1 r)]^2 \sin^2 K_3 z \; d\theta\, dr\, dz = \frac{\pi R^2 L}{2} J_0^2(x_{01}) \tag{7}$$

since $\sin 2K_3L = 0$, n being an integer and $-J_1(x_{01}) = J_0'(x_{01}) = 0$.

From equations 1, 6 and 7, ϵ' is given by the following expression,

$$\frac{\Delta f}{f_2}=(\epsilon'-1)\frac{\pi^2 n^2 d^2 v}{3L^2 V}\left[\frac{J_0^2(x_{01}a/R)+J_1^2(x_{01}a/R)-\dfrac{2RJ_0(x_{01}a/R)\,J_1(x_{01}a/R)}{x_{01}a}}{J_0^2(x_{01})}\right] \tag{8}$$

The frequency shift Δf and the frequency f_2 of the loaded cavity can be measured accurately with the available modern technique. The value of n can be calculated from the cavity wavelength and the length of the cavity.

THE LOADED Q OF THE CAVITY

The loaded Q (Q_2) of the cavity can be defined as follows:

$$Q_2=2\pi\left[\frac{\text{Maximum energy stored in magnetic } W_m \text{ or electric } W_e \text{ field}}{\text{Energy loss in one period}}\right]$$

$$=\frac{\omega\,(W_m \text{ or } W_e)}{P}$$

where P = Total power lost on the cylindrical walls (P_r), end plates (P_z) and dielectric rod (P_r')

Or $$P=P_r+P_z+P_r'$$

Let it be assumed that the values of the field components (eq. 5) deduced on the assumption of perfect conductivity ($\sigma=\infty$) of boundary surfaces of the cavity remains the same even when some power is absorbed by the cavity walls and end plates due to imperfect conductivity. For microwave cavities, made of silver-plated brass $\sigma \gg \omega\epsilon$, and so this assumption about the field structure does not involve any appreciable error. Let us also assume that the field structure given by equation (5) remains unaltered when the dielectric rod is introduced into the cavity. This assumption is valid only if a and d are very small compared to the dimensions of the cavity. So this method of determining ϵ'' is applicable only to the case of a small sample of dielectric so that its presence into the cavity may not distort the original field inside the cavity.

Energy W_e stored in electric field per unit volume is $\epsilon E^2/2$. In the case of a cavity filled with air ϵ can be taken to be unity.

So, for TE_{0in}- mode, from equation (5)

$$W_e=\frac{1}{2}\int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{z=0}^{L} rE_\theta^2\,dr\,d\theta\,dz$$

$$=\frac{\pi R^2 L}{4}J_0^2(x_{01})$$

When $\sigma = \infty$ the tangential component of the electric field $\mathbf{E}_{tan}$ at the boundary surfaces of the cavity is zero but $\mathbf{H}_{tan} \neq 0$. When σ is finite, in order that power may be absorbed by the metal surface there must be a component of both $\mathbf{E}_{tan}$ and $\mathbf{H}_{tan}$ present at the boundary surfaces so that both are at right angles to each other and the direction of $\mathbf{E}_{tan}$ is such that the Poynting vector is directed into the metal. The average power absorbed per unit area by the metal surfaces is given by the following relation:

$$P' = \tfrac{1}{2} \text{ Real } (\mathbf{E}_{tan} \times \mathbf{H}_{tan}{}^*)$$

When $\sigma = \infty$, both $\mathbf{E}_{tan}$ and $\mathbf{H}_{tan}$ are related to each other according to the following relation:

$$\bar{E}_{tan} = \eta\, \mathbf{H}_{tan}$$

which can be applied to the present case $\sigma >> \omega\epsilon$ without introducing appreciable error.

Where,

η = Intrinsic impedance of the metal $= (\pi f \mu/\sigma)^{\frac{1}{2}}$

σ = Conductivity of the metal

μ = Permeability of the metal

f = Frequency of excitation of the cavity

So $\quad P' = \tfrac{1}{2} \text{ Real } |\eta \mathbf{H}_{tan}{}^2|$

The average power $\mathbf{P}_r$ lost in the cylindrical wall per unit length is given by the following expression

$$P_r = \tfrac{1}{2}\eta \int_{z=0}^{L} \int_{\theta=0}^{2\pi} H_z{}^2 \, R \, d\theta \, dz,$$

where R $d\theta\, dz$ is the element of surface of the cylindrical wall.

From the above expression for P_r and equation (5)

$$P_r = \tfrac{1}{2}\eta \int_0^{L} \int_0^{2\pi} \frac{K_1{}^2}{K^2} R J_0{}^2 (K_1 r) \sin^2 K_3 z \, d\theta \, dz$$

$$= 1 \cdot 836 \frac{\lambda^2 l}{\pi R} \left(\frac{\pi f \mu}{\sigma}\right)^{\frac{1}{2}} J_0{}^2 (x_{01})$$

The average power P_r' lost into the dielectric per unit length is given by

$$P_r' = \tfrac{1}{2}\eta \int_{z=0}^{d} \int_{\theta=0}^{2\pi} H_z{}^2 \, a \, d\theta \, dz$$

$$= 1 \cdot 224 \, \frac{\pi a n^2 \lambda^2 d^3}{R^2 L^2} \left(\frac{\mu_d}{\epsilon_l}\right)^{\frac{1}{2}} J_0{}^2 \left(\frac{x_{01} a}{R}\right)$$

neglecting powers of sine higher than the third. μ_d and ϵ_d refer to the dielectric.

The average power lost P_z for both end plates is given by

$$P_z = 2\left[\tfrac{1}{2}\eta \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r H_r^2 \, d\theta \, dr\right]$$

$$= \frac{\pi n^2 \lambda^2 R^2}{4L^2}\left(\frac{\pi f \mu}{\sigma}\right)^{\frac{1}{2}} J_0^2(x_{01})$$

as $\cos K_3 z$ at either of the end plates where $z = 0$ or $z = L$ is unity and as $J_1(x_{01}) = -J'_0(x_{01}) = 0$.

The loaded Q (Q_2) of the cavity is given by the following expression

$$Q_2 = \left[\omega V J_0^2(x_{01})\right] \Big/ \left[7\cdot 344\,\frac{\lambda^2 L}{\pi R}\left(\frac{\pi f \mu}{\sigma}\right)^{\frac{1}{2}} J_0^2(x_{01}) + 4\cdot 896\,\frac{\pi a n^2 \lambda^2 d^3}{R^2 L^2}\left(\frac{\mu_d}{\epsilon_d}\right)^{\frac{1}{2}} J_0^2\left(\frac{x_{01} a}{R}\right) + \frac{\pi n^2 \lambda^2 R^2}{L^2}\left(\frac{\pi f \mu}{\sigma}\right)^{\frac{1}{2}} J_0^2(x_{01})\right], \tag{9}$$

The Unloaded Value of Q (Q_1)

The unloaded Q of the cavity operating in the $TE_{0'n}$ mode is given (Wilson, etc., 1946) by the following expression:

$$Q_1 = \frac{3\cdot 831 \lambda f^{\frac{1}{2}}}{(10^3 \rho)^{1/3}} \frac{[1 + 0\cdot 168\,(D/L)^2 n^2]^{3/2}}{[1 + 0\cdot 168\,(D/L)^3 n^2]} \tag{10}$$

where ρ is the resistivity of the material of the cavity in ohms/cm.

The values of μ_d and ϵ_d can be measured by any of the well-known technique outlined by Westphal (1950). From equations 2, 6, 7, 9 and 10, ϵ'' can be determined. The ratio ϵ''/ϵ' gives $\tan \delta$.

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