MICROWAVE CAVITY RESONATORS AS CIRCUIT ELEMENTS

By S. K. Chatterjee,* M.Sc.

Abstract

The differential equation for the equivalent circuit of a double loop coupled micro-wave cavity resonator has been determined by the application of Lagrange's equation. The losses on the walls and the Q of the cavity operating in the TE_{11n} mode have been evaluated with the help of the field equations.

Introduction

A microwave cavity resonator represents a complicated oscillating system having an infinite number of natural frequencies arranged in the sequence of increasing magnitude. Microwave cavities have found wide applications as circuit elements. In practice a microwave cavity is usually coupled to an external system by means of a probe, loop or iris. The behaviour of a coupled cavity can be studied by the conventional method of circuit analysis or with the help of the field equations. The object of the present paper is to approach the problem of a double loop coupled cavity resonator from the energy point of view and to form the differential equation of the coupled system with the help of Lagrange's equation. In view of the practical importance of the method, it is worthwhile to consider first the application of Lagrange's equation to the analysis of an electrical network before discussing cavity resonators.

Lagrange's Equation and Analysis of Electrical Network

If for a holonomic system the generalised co-ordinates are represented by $q_1, q_2 \ldots q_n$ and the corresponding velocities by $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$ then the Lagrangian equation of motion for the dynamical equilibrium of the system is given by the following expression:

$$p\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = F_k \qquad k = 1, 2 \ldots n \tag{1}$$

where the operator $p = d/dt$ and F_k represents the dissipative forces and any external applied forces present in the system. The symbol L represents the Lagrangian and is a function of q and $\dot{q}$.

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The Lagrangian equation can be applied to find the performance of an electrical network if suitable co-ordinates corresponding to $q_1, q_2 \ldots q_n$ and $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$ can be found in the electrical system. The kinetic energy (T) and the potential energy (V) of the system, and hence the Lagrangian $L = T - V$ can then be found. The value of L substituted in (1) will give the differential equations of the network. The solution of the differential equations describes the characteristics of the network. The charges $Q_1, Q_2 \ldots Q_n$ and the currents $\dot{Q}_1, \dot{Q}_2 \ldots \dot{Q}_n$ in an electrical network can be considered (Wells, 1938; Olson, 1944) as equivalent to $q_1, q_2 \ldots q_n$ and $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$ respectively.

Let us consider a linear electrical network composed of n independent meshes whose elements are lumped and dissipationless. The total instantaneous magnetic energy (T) and electric energy (V) of the system are given by the following relations in terms of the mesh currents $[\dot{Q}]$ and mesh charges (Q) respectively of the network.

$$T = \frac{[\tilde{\dot{Q}}]\,[L]\,[\dot{Q}]}{2} \tag{2}$$

$$V = \frac{[\tilde{Q}]\,[S]\,[Q]}{2}, \tag{3}$$

where

$$[Q] = \begin{bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ Q_n \end{bmatrix}$$

and the total and mutual elastance coefficients

$$(S) = \begin{bmatrix} S_{11} & S_{12} \ldots S_{1n} \\ S_{21} & S_{22} \ldots S_{2n} \\ \cdot & \\ \cdot & \\ \cdot & \\ S_{n1} & S_{n2} \ldots S_{nn} \end{bmatrix}$$

As $S_{nm} = S_{mn}$ the matrix (S) is symmetric. The tilde in (2) and (3) indicate that the matrix is transposed.

If the electromotive forces (e) applied to the n meshes are given by the following column matrix

$$[e] = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{bmatrix} \tag{4}$$

then the Lagrangian equation

$$p\left(\frac{\partial L}{\partial \dot{Q}_k}\right) + \frac{\partial L}{\partial Q_k} = e_k \qquad k = 1, 2 \ldots . n \tag{5}$$

can be evaluated from the equations 2, 3 and 4. The equivalent circuit of the network is then given by the differential equation (5). If there is a dissipative force

$$F_k = -\sum_{k=1}^{n} R_k \dot{Q}_k$$

present in the system then the differential equation describing the equivalent circuit of the network is given by the following equation

$$p\left(\frac{\partial L}{\partial \dot{Q}_k}\right) - \frac{\partial L}{\partial Q_k} - F_k = e_k \tag{6}$$

The dissipative force F_k may be derived in terms of the Rayleigh's dissipation function F* (Goldstein, 1950) and is given by

$$F_k = -\frac{\partial F}{\partial \dot{Q}_k}$$

So the differential equation for the network is given as follows

$$p\left(\frac{\partial L}{\partial \dot{Q}_k}\right) - \frac{\partial L}{\partial Q_k} + \frac{\partial F}{\partial \dot{Q}_k} = e_k, \tag{7}$$

where R_k's represent the resistances in the different meshes of the network. The two scalar functions L and F have to be evaluated in order to describe the equivalent circuit of an electrical network.

* In a system where frictional forces are present, it frequently happens that the frictional forces are proportional to the velocity of the particles. In this case, F is defined as

$$F = \tfrac{1}{2} \sum_i (k_x \dot{q}_{ix}^2 + k_y \dot{q}_{iy}^2 + k_z \dot{q}_{iz}^2)$$

where the summation is taken over the particles of the system. The dissipation function can also be interpreted as one-half the rate of energy dissipation due to friction.

As an illustration, let us find the differential equation of the following network (Fig. 1) which finds wide applications in radio engineering.

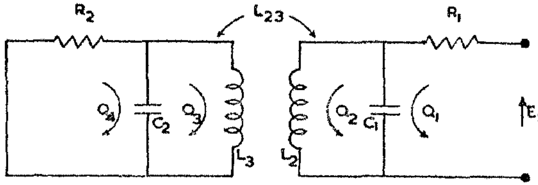

FIG. 1

In the above network $Q_1, Q_2 \ldots . Q_4$ are charges and $\dot{Q}_1 \ldots . \dot{Q}_4$ are currents. The kinetic and the potential energies and the dissipative forces for the above network can be written as follows:—

$$T = \tfrac{1}{2} L_3\dot{Q}_3^2 + \tfrac{1}{2} L_2\dot{Q}_2^2 + L_{23}\dot{Q}_3\dot{Q}_2$$

$$V = -E_1Q_1 + \frac{(Q_1 - Q_2)^2}{2c_1} + \frac{(Q_4 - Q_3)^2}{2c_2}$$

$$F = -R_1\dot{Q}_1 - R_2\dot{Q}_4 \tag{8}$$

In the above equations $-E_1Q_1$ indicates energy extracted from the source. Hence the Lagrangian is given by the following expression:

$$L = \tfrac{1}{2} L_3\dot{Q}_3^2 + \tfrac{1}{2} L_2\dot{Q}_2^2 + L_{23}\dot{Q}_3\dot{Q}_2 + E_1Q_1 - \frac{(Q_1 - Q_2)^2}{2c_1} - \frac{(Q_4 - Q_3)^2}{2c_2} \tag{9}$$

So the differential equations of the network can be written from 7, 8 and 9 as follows:

$$E_1 = \frac{\iota_1}{pc_1} - \frac{\iota_2}{pc_1} + R_1\iota_1$$

$$0 = -\frac{\iota_1}{pc_1} + \iota_2\left(L_2p + \frac{1}{pc_1}\right) + L_{23}p\iota_3$$

$$0 = p\iota_2L_{23} + \iota_3\left(pL_3 + \frac{1}{pc_2}\right) - \frac{\iota_4}{pc_2}$$

$$0 = -\frac{\iota_3}{pc_2} + \iota_4\left(\frac{1}{pc_2} + R_2\right) \tag{10}$$

where $\iota_1 = \dot{Q}_1$, $\iota_2 = \dot{Q}_2$, $\iota_3 = \dot{Q}_3$, $\iota_4 = \dot{Q}_4$ and $p = d/dt$.

The equations (10) can be solved for steady or transient currents for any applied voltage E. The coefficients of the currents can be directly written by the inspection of the network in Fig. 1. (Stigant, 1947) as follows:

$$\begin{bmatrix} E_1 \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} R_1 + \dfrac{1}{pc_1} & -\dfrac{1}{pc_1} & & \\ -\dfrac{1}{pc_1} & L_2p + \dfrac{1}{pc_1} & L_{23}p & \\ & L_{23}p & L_3p + \dfrac{1}{pc_2} & -\dfrac{1}{pc_2} \\ & & -\dfrac{1}{pc_2} & \dfrac{1}{pc_2} + R_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

This agrees with the results that can be derived from (10).

Some Fundamental Relations in a Microwave Cavity Resonator

The properties of microwave cavities of simple geometry, and oscillating in normal modes have been studied by Borgnis (1939), Hansen and Richtmyer (1939), Condon (1942) and Slater (1942, 1946) using Maxwell's equations of the electromagnetic field which for a source-free cavity are given in m.k.s. units as follows:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \\ \nabla \times \mathbf{H} &= -\dot{\mathbf{D}} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{D} &= 0 \end{aligned}$$

where, $B = \mu H$, $D = \epsilon E$, $\mu = \mu_0 k_m$ and $\epsilon = \epsilon_0 k_\varepsilon \cdot k_m$ and k_ε are respectively the magnetic and the electric specific inductive capacities of vacuum and μ_0 and ϵ_0 are the magnetic and the electric inductive capacities of vacuum. The normal mode fields for a microwave cavity can be expressed in terms of a pair of vector functions $\mathbf{E}_a$ and $\mathbf{H}_a$ associated with the ath mode. The mode vectors satisfy the following equations:

$$\begin{aligned} \nabla^2 \mathbf{E}_a + K_a^2 \mathbf{E}_a &= 0 \\ \nabla^2 \mathbf{H}_a + K_a^2 \mathbf{H}_a &= 0 \\ \nabla \cdot \mathbf{E}_a &= 0 \\ \nabla \cdot \mathbf{H}_a &= 0 \end{aligned}$$

The wave equations have an infinite sets of solutions, each set being characterised by the particular wave number K_a given by $K_a^2 = \omega_a^2 \mu\epsilon$ and subject to the following boundary conditions

$$n \times \mathbf{E}_a = 0$$

$$n \cdot \mathbf{H}_a = 0$$

where n represents the outward normal unit vector. It can be shown that the mode vectors form orthogonal sets and are normalised in such a way that

$$\int_v \mathrm{E}_a \cdot \mathrm{E}_b dv = \begin{cases} 0 & a \neq b \\ v & a = b \end{cases}$$

and

$$\int_v \mathrm{H}_a \cdot \mathrm{H}_b dv = \begin{cases} 0 & a \neq b \\ v & a = b \end{cases}$$

EQUIVALENT CIRCUIT OF A DOUBLE LOOP COUPLED CAVITY

The problem of a single loop coupled cavity has been studied by Hansen (1938), Condon (1941), Slater (1946), Crout (1944, 1948) and Banos (1944). In the present paper the case of a double loop coupled cavity is treated with the help of Lagrange's equation and the Maxwellian field equations.

The energy stored in the magnetic field of the cavity is equivalent to the kinetic energy (T) of the mechanical system and is given as follows:

$$\mathrm{T} = \tfrac{1}{2}\mu \int_v \mathrm{H}^2 dv \tag{11}$$

The energy stored in the electric field of the cavity is equivalent to the potential energy (V) of the mechanical system and is given by

$$\mathrm{V} = \frac{\epsilon}{2} \int_v \mathrm{E}^2 dv \tag{12}$$

where **H** and **E** represent the field inside the cavity and can be expressed in terms of solenoidal and irrotational fields as follows:—

$$\mathbf{E} = \sum_a e_a \mathbf{E}_a + \sum_a f_a \mathbf{E}_a$$

$$\mathbf{H} = \sum_a m_a \mathbf{H}_a + \sum_a l_a \mathbf{H}_a$$

The vector functions $\mathbf{E}_a$ and $\mathbf{H}_a$ satisfy the conditions $\nabla \cdot \mathbf{E}_a = 0$ and $\nabla \cdot \mathbf{H}_a = 0$. Inside the empty cavity $\nabla \cdot \mathbf{E}_a = 0$; and so there is only the solenoidal part of E. And as $\mathbf{H}$ has no irrotational part

$$\mathbf{E} = \sum_a e_a \mathbf{E}_a \text{ and } \mathbf{H} = \sum_a m_a \mathbf{H}_a.$$

Multiplying the above expressions by E_a and H_a respectively and integrating over the volume v of the cavity the following expressions for the coefficients e_a and m_a are obtained due to the Fourier nature of the expansions and the properties of orthogonality.

$$e_a = \frac{1}{v} \int_v \mathbf{E} \cdot \mathbf{E}_a \, dv$$

$$m_a = \frac{1}{v} \int_v \mathbf{H} \cdot \mathbf{H}_a dv$$

As the mode vector functions $\mathbf{E}_a$ and $\mathbf{H}_a$ have zero divergence, they can be expressed as the curl of another vector. Let it be assumed that the following relations hold good

$$\nabla \times \mathbf{E}_a = \mathrm{K}_a \mathbf{H}_a$$

$$\nabla \times \mathbf{H}_a = \mathrm{K}_a \mathbf{E}_a$$

The vector fields E and H can be expanded in the following form (Crout, 1944)

$$\mathbf{E} = \frac{1}{\epsilon} \Sigma \, \mathrm{K}_a^2 \mathrm{Q}_a \mathrm{E}_a$$

$$\mathbf{H} = \Sigma \, \mathrm{K}_a \dot{\mathrm{Q}}_a \mathrm{H}_a \tag{13}$$

where Q_a and $\dot{\mathrm{Q}}_a$ are the normal coordinates of the system. So the energy expressions can be written as follows from 11, 12 and 13:

$$\mathrm{T} = \tfrac{1}{2}\mu \int_v \sum_a \mathrm{K}_a^2 \dot{\mathrm{Q}}_a^2 \, \mathrm{H}_a^2 dv$$

$$= \tfrac{1}{2}\mu \sum_a \mathrm{K}_a^2 \dot{\mathrm{Q}}_a^2 v$$

and

$$\mathrm{V} = \frac{1}{2\epsilon} \int_v \sum_a \mathrm{K}_a^4 \mathrm{Q}_a^2 \mathrm{E}_a^2 dv$$

$$= \frac{1}{2\epsilon} \sum_q \mathrm{K}_a^4 \mathrm{Q}_q^2 v$$

The differentials of the Lagrangian are then given as follows:

$$p\frac{\partial L}{\partial \dot{Q}_a} = \mu \sum_a K_a^2 \ddot{Q}_a v$$

$$\frac{\partial L}{\partial Q_a} = -\frac{1}{\epsilon}\sum_a K_a^4 Q_a^2 v$$

The differential equation of a dissipationless cavity is then given by

$$\mu \sum_a K_a^2 \ddot{Q}_a v + \frac{1}{\epsilon}\sum_a K_a^4 Q_a v = 0$$

or

$$L_a \ddot{Q}_a + Q_a/c_a = 0 \tag{14}$$

where

$$L_a = \mu \sum_a K_a^2 v \qquad \text{and} \qquad c_a = \epsilon/\sum_a K_a^4 v$$

The dimensions of L_a and c_a are $[ML^2Q^{-2}]$ and $[M^{-1}L^{-2}T^2Q^2]$ respectively and so L_a and c_a may be defined as the equivalent inductance and capacitance respectively of the cavity. So equation (14) represents the equivalent circuit of a dissipationless microwave cavity as a parallel resonant circuit composed of L_a and c_a.

Let the cavity be excited in the ath mode by two identical loops and carrying equal currents I_a. In deriving the differential equation for the equivalent circuit let us make the assumptions (a) that the dimensions of both the loops and the holes are so small compared to the size of the resonator and the wavelength that the normal mode fields inside the unperturbed cavity remain unaltered by the introduction of the loops through the holes made through the walls of the cavity; and (b) that there is coupling only between each loop and the cavity but no coupling between the loops themselves.

It is clear that there is no mutual coupling between the different modes of the cavity in consequence of the orthogonality relations

$$\int_v E_a \cdot E_b dv = 0 \qquad a \neq b$$

of the vector functions.

The voltage induced by each loop is $M_a \dot{I}_a$, where the mutual inductance M_a between the loop and the cavity is given by (Condon, 1941)

$$M_a = \int_s H_a dS \tag{15}$$

The integration is performed over the surface of the loop and H_a represents the vector sum of the components of the magnetic field for the ath oscillation mode. So the mutual inductance between the loop and the cavity will differ for different modes present in the cavity.

A portion of the power coupled into the cavity from the loops will be lost if the boundary walls of the cavity have finite conductivity. The loss of power in the resonator walls per unit length is given by

$$\frac{R}{2}\oint_s H_0^2 dS, \tag{16}$$

where the integration is extended around the periphery of the cross-sections of the resonator and H_0 is the amplitude of $\mathbf{H}$ at the surface of the cavity for the particular mode. R is the surface resistance of the resonator walls and is given by $R = \sqrt{\mu\omega/2\sigma}$, where σ is the conductivity of the walls in mhos per metre, μ is the permeability and ω is the angular frequency of the mode. The loss of power in each loop due to its resistance R_1 is $R_1 I_a^2$. So the total power lost in the two loop coupled cavity is given by

$$F_k \dot{Q}_a = 2R_1 I_a^2 + \frac{R}{2}\oint_s H_0^2\, dS.$$

So, the differential equation for the double loop coupled cavity is given by the following expression

$$\mu \sum_a K_a^2 \ddot{Q}_a v + \frac{1}{\epsilon}\sum_a K_a^4 \dot{Q}_a v + 2R_1 I_a + \frac{R}{2I_a}\oint_s H_0^2 dS = M_{1a}\dot{I}_a + M_{2a}\dot{I}_a,$$

where M_{1a} and M_{2a} refer to the two loops. The above equation can also be written in the following form to represent the equivalent circuit shown in Fig. 2.

$$L_a \dot{I}_a + \frac{\int \iota_a dt}{c_a} + 2R_1 I_a + \frac{R}{2I_a}\oint_s H_0^2\, dS = M_{1a}\dot{I}_a + M_{2a}\dot{I}_a.$$

where ι_a represents the instantaneous value of I_a.

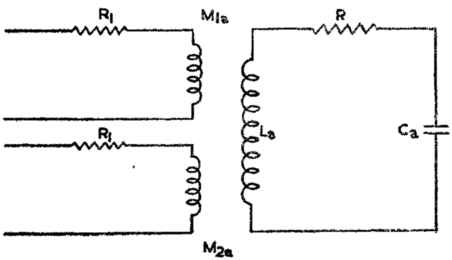

FIG. 2

POWER LOST ON THE WALLS OF THE CAVITY (TE_{11n} MODE)

Let it be assumed that the two loops are excited in proper phase and are so displaced spatially inside a cylindrical cavity that the cavity is excited in the TE_{11n} mode. The H-components of the TE_{11n} mode for the cylindrical cavity can be deduced from the field components of the TE_{lmn} mode (Kinzer, 1943) as follows:

$$\left.\begin{aligned} H_r &= \frac{K_3}{K} J_1'(K_1 r)\cos\theta\cos K_3 Z \\ H_\theta &= -\frac{K_3}{K}\frac{J_1(K_1 r)}{K_1 r}\sin\theta\cos K_3 Z \\ H_z &= \frac{K_1}{K} J_1(K_1 r)\cos\theta\sin K_3 Z \end{aligned}\right\} \quad \begin{matrix} n > 0 \\ m > 0 \end{matrix} \tag{17}$$

where, $K_1 = x_{11}/d$ $\qquad d =$ radius of the cavity.

$K_3 = n\pi/L$ $\qquad L =$ length of the cavity

$K^2 = K_1^2 + K_3^2$ $\qquad K = 2\pi/\lambda$

$x_{11} =$ 1st root of $J_1'(x) = 0$

l, m, n have their usual meanings.

The power lost on the walls of the cavity can then be determined from (17) as follows:

$$\oint_s H_0^2 dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{d}(H_r^2 + H_\theta^2 + H_z^2)\, r d\theta dr$$

$$\int_0^d\int_0^{2\pi} H_r^2 r dr d\theta = \pi\frac{K_3^2}{K^2}\cos^2 K_3 Z\left[\tfrac{1}{2}d^2\{J_1^2(x_{11}) + J_0^2(x_{11})\} + K_1^{2s}\frac{d^{2s+2}}{2s+2}\{\pi_1 - \pi_2\}\right],$$

where

$$\pi_1 = \sum_{s=0}^{\infty}\frac{(-1)^s(2+2s)!}{2^{2+2s}s!\,(2+s)!\,[(1+s)!]^2}$$

and

$$\pi_2 = 2\sum_{s=0}^{\infty}\frac{(-1)^s(1+2s)!}{2^{1+2s}[s!]^2[(1+s)!]^2}$$

$$\int_0^d\int_0^{2\pi} H_\theta^2 dS = \pi\frac{K_3^2}{K^2}\cos^2 K_3 Z\frac{d^{2s+2}}{2s+2}K_1^{2s}\pi_1$$

$$\int_0^d\int_0^{2\pi} H_z^2 dS = \pi\frac{K_1^2}{K^2}\sin^2 K_3 Z\,\tfrac{1}{2}d^2\left[\{J_1'(x_{11})\}^2 + \left(1 - \frac{1}{K_1^2 d^2}\right)\{J_1(x_{11})\}^2\right]$$

So, loss per unit length of the cavity wall is given by

$$\frac{R}{2}\oint_s H_0^2 dS = \frac{R}{2}\pi\frac{K_3^2}{K^2}\cos^2 K_3 Z\left[0\cdot2194d^2 + K_1^{2s}\frac{d^{2s+2}}{2s+2}\{2\pi_1-\pi_2\}\right]$$
$$+\frac{R}{2}\pi\frac{K_1^2}{K^2}\sin^2 K_3 Z\,\tfrac{1}{2}d^2\left\{0\cdot3386\left(1-\frac{1}{K_1^2 d^2}\right)\right\}$$

as $J_1'(x_{11}) = 0$, $J_1(x_{11}) = 0\cdot5819$ and $J_0(x_{11}) = 0\cdot3167$.

Mutual Inductance Between the Loop and the Cavity

The mutual inductance $M_a = \int_s H_a dS$ for the TE_{11n} mode can be evaluated from the field components (17) in a similar way as above. Let it be assumed that the loops are placed symmetrically about the axis.

$$\int_s H_a dS = \sqrt{\left[\pi\frac{K_3^2}{K^2}\cos^2 K_3 Z + \frac{1}{2s+2}K_1^{2s}\{(r+a)^{2s+2} - (r-a)^{2s+2}\}\right.}$$
$$\left.\times\{2\pi_1-\pi_2+\pi_3\} + \pi\frac{K_1^2}{K^2}\sin^2 K_3 Z\,\frac{1}{4+2s}K_1^{2+2s}\pi_1\{(r+a)^{4+2s} - (r-a)^{4+2s}\}\right]$$

where,

$$\pi_3 = \sum_{s=0}^{\infty}\frac{(-1)^s(2s)!}{2^{2s}s!\,(s!)^2}$$

and a = radius of the loop.

When the loops are placed at either end of the cavity the expression above for mutual inductance reduces to the following:

$$\int_s H_a dS = \sqrt{\left[\pi\frac{K_3^2}{K^2}K_1^{2s}\frac{1}{2s+2}\{r+a)^{2s+2}-(r-a)^{2s+2}\}\{2\pi_1-\pi_2+\pi_3\}\right]}$$

as $\sin K_3 Z = 0$ at $Z = 0$ and $Z = L$.

It will be observed from the above relation that as the loops shrink in size, $a \to 0$, $r \to d$ and so $M_a \to 0$.

Q of the Cavity (TE_{11n} Mode)

If the medium inside the cavity has conductivity σ_1 and dielectric constant ϵ, then the total loss inside the cavity will be due to the loss in the dielectric and the loss due to the finite conductivity of the cavity wall. The Q (Q) of the cavity is then given by the following well-known relation:

$$\frac{1}{Q} = \frac{1}{Q_{\text{dielectric}}} + \frac{1}{Q_{\text{wall}}}. \tag{18}$$

It can be shown from the field equations that

$$Q_{\text{wall}} = \frac{2}{\delta}\frac{\iiint H_a{}^2 dv}{\iint H_a{}^2 dS}. \tag{19}$$

where, $\delta = \sqrt{(2/\omega\sigma\mu)}$ = skin depth.

The numerator in (19) can be evaluated from (17) and is given by the following expression:

$$\int_{v\,(\text{cavity})} H_a{}^2 dv = \frac{\pi L}{2}\frac{K_3{}^2}{K^2}\left[0\cdot 2194 d^2 + K_1{}^{2s}\frac{d^{2s+2}}{2s+2}\{2\pi_1 - \pi_2\}\right] + \frac{\pi L}{4}\frac{K_1{}^2}{K^2} d^2\left[0\cdot 3386\left(1 - \frac{1}{K_1{}^2 d^2}\right)\right] \tag{20}$$

as $\quad J_1'(x_{11}) = 0.$

Similarly, $\int_s H_a{}^2 dS$ can be evaluated from (17) and hence Q_{wall} can be calculated from (19). The value of $Q_{\text{dielectric}}$ is given by the following well-known relation:

$$Q_{\text{dielectric}} = \frac{\epsilon\omega_a}{\sigma_1}, \tag{21}$$

where ω_a is the angular frequency of the ath mode. So, the Q of the cavity operating in the TE_{11n} mode can be evaluated from (18) to (21) and the value of $\int_s H_a{}^2 dS$.

Input Impedance of the Cavity

The equivalent circuit (Fig. 2) can be generalised into an oscillating system having an infinite number of natural frequencies $\omega_1, \omega_2 \ldots \omega_n$. The natural frequency ω of the loops will be affected by the presence of the cavity and *vice versa*. When the loops are very small, the current distributions along the loops may be considered to be uniform. In this case the input impedance of the cavity across the terminals of each loop is given by (Schelkunoff, 1944) the following impedance function

$$Z_{\text{in}} = j\omega L + \sum_{1}^{\infty}\frac{j\omega^3 L^2{}_{mn}}{L_n(\omega_n{}^2 - \omega^2)}.$$

This is the expression for the impedance looking into the cavity through the loop and consists of a sum of resonant terms. It is also seen that $Z_{11} \to \infty$ when the frequency tends to equal the resonant frequency of any one of the modes. In the above expression L_n is the inductance of the cavity at the nth mode and can be evaluated from

$$L_n = \mu\int_{v\,(\text{cavity})} \mathbf{H}_n \cdot \mathbf{H}_n{}^* dv$$

and the **H**-components for the nth mode of oscillation. $L_{m,n}$ is the mutual inductance between the loop and the cavity and can be evaluated from the field equations (17) and relation similar to equation (15). L is the inductance of the loop.

If the cavity oscillates in the ath mode and has slight dissipation, then the above impedance function needs modification and is given by

$$Z_{in} = R + j\omega L + \frac{j\omega^3 L^2_{1a}}{L_a(\omega_a{}^2 - \omega^2 + j\delta_a\omega_a\omega)},$$

where δ_a is the reciprocal of the Q in the ath mode (Q_a) and can be found from (15) and the field equations.

If one of the loops is used for exciting the cavity and other loop for taking output from the cavity, then the system behaves as a two pair terminal network. In this case the transfer impedance Z_{12} across the cavity is given (Schelkunoff, 1944) by the following expression:

$$Z_{12} = j\omega\, L_{12}{}^0 + \sum_1^\infty \frac{j\omega^3 L_{1n} L_{2n}}{L_n(\omega_n{}^2 - \omega^2 + j\delta_n\omega_n\omega)}$$

where L^0_{12} represents the low frequency mutual inductance between the loops in the cavity. L_{1n}, L_{2n} represent the mutual inductances between the cavity and the first loop and the cavity and the second loop respectively. The values of L_{1n} and L_{2n} can be found from the field components as above. So, Z_{12} can be determined. The above expression for the impedance consists of two terms, the first term varying slowly with frequency and the other is the summation of the resonant term. The latter term shows the effect of Q, the losses in the walls on the impedance. The above relation holds good so long as the cavity operates in a non-degenerate mode. If the two resonant modes are close together, so that the two resonant peaks overlap, then the Q of the cavity and the losses will be different and the expressions for Z_{12} and Z_{in} will not hold good. In the case of degeneracy, more elaborate methods will be necessary.

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