

THE PRIMITIVE MACHINE OF KRON

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ABSTRACT

Tensor analysis of electrical networks and its application to the analysis of electrical machines as developed by G. Kron is discussed. From the relations developed by considering the characteristics of a simple machine, referred to as the Primitive Machine of Kron, the method of analysis is applied to determine the performance of a shaded-pole motor.

In the early part of this century Ricci, Levi-Civita and others¹ evolved a mathematical discipline which is admirably suited to the study of problems in Non-Euclidean spaces and it was this that has been later developed into what is now known as Tensor Analysis. Einstein² was one of the earliest to apply this new approach to the solution of physical problems. As tensor analysis became better known among physicists and mathematicians it was employed with greater and greater enthusiasm in the study of classical mechanics, electromagnetic theory and electro-dynamics.

It was Gabriel Kron³ who first realised that many engineering problems in general, and electrical engineering problems in particular, were best solved by the Tensor method as the variables involved were inherently multidimensional and non-Euclidean in character. He has developed a highly organised method of attack to analyse and synthesise electrical networks in a routine manner. The classical vector analysis as developed by Maxwell⁴ and others is a severely restricted type of organisation since it represents only physical entities in three dimensions and Euclidean spaces. Tensor analysis on the other hand is an extension and generalisation of vector analysis from three to 'N' dimensions and from Euclidean to non-Euclidean spaces. It has been shown by Kron that it is possible to set up, in the language of tensor analysis, equations that truly represent the performance of a large variety of networks or rotating machines or transmission systems.

Once the tensor equations have been established it is possible to find the equations of performance of any particular network or machine or transmission systems by a routine substitution of particular constants. The versatility of tensor methods enables the engineer, in the study of a large variety of rotating machines, to select one whose structure is comparatively simple and to study the properties and equations of this simple machine

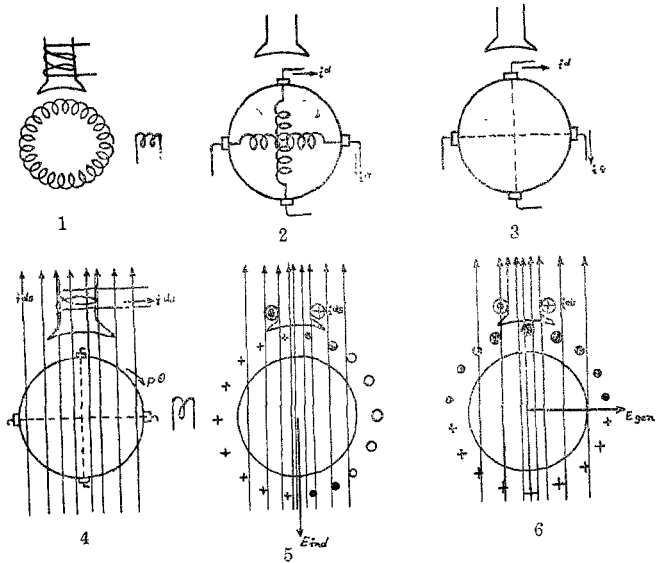
only. If the engineer learns, with the aid of tensor analysis, the general method of analysis and the physical phenomenon taking place in this particular machine he learns at the same time the method of analysis of a large number of 'mathematically' similar machines without having to learn a new trick for each new machine as is necessary with the classical methods of attack. In other words the same tensor equations, developed for the simple machine, are valid for several different types of physical phenomenon.

Electrical machine analysis consists essentially of the derivation of equations of performance for the purpose of predicting accurately the characteristics for all the various types of machines. Comprehensive analysis of an ideal machine is difficult to achieve and a segregation of the problems involved would greatly simplify the work. The electrical elements of the machine under study are considered as constituting two or more linear circuits in relative motion having lumped constants and it is a well-known fact that this assumption is amply justified by experiments.

In the case of a three phase machine supplied with balanced three phase power it is possible to analyse the performance of the machine in terms of one phase only. If, however, there is any balance in the windings or the current or any lack of symmetry in the physical structure of the magnetic circuit the machine cannot be analysed in terms of one phase alone. To overcome this difficulty two methods have been developed; namely, the method of symmetrical components and the two-axis method. The former method, originally developed by Fortescue, is widely used whenever the unbalance is in the nature of the supply while the latter method is invariably used whenever the dissymmetry is in the physical structure of the magnetic circuit.

For convenience and easy extension of tensorial methods to complicated problems Kron makes a detailed analysis of what is known as the Primitive Machine. The equations of performance of the primitive machine are first developed from the fundamental laws of electrodynamics. Then by setting up a connection tensor between the primitive machine and the machine under study, the equations of the latter can be established in a routine manner without starting its analysis all over again from the fundamentals. It is, therefore, quite evident that the study of all rotating machines will consist of three important steps: (a) the establishment of equations of the primitive machine; (b) the establishment of the connection tensor showing how the given machine differs from the primitive machine (this is usually done by mere inspection); and lastly, (c) the routine determination of the performance characteristics of the machine.

The primitive machine consists of a cylindrical stator and rotor each provided with concentric layers of windings. The stator is visualised as consisting of two independent windings one on the salient pole and the other at right angles to it between the main poles. The rotor layer of distributed winding is considered as equivalent to two hypothetical coils at right angles to each other as indicated in Figs. 1, 2 and 3. While the conductors constituting these coils are constantly in rotation the resultant coils between the brushes are stationary. The voltages induced and generated in the various windings when a current i^{ds} flows in the direct axis winding of the stator are detailed below and shown in Figs. 4, 5 and 6. The induced transformer



FIGS. 1-6

voltage arises through the alternating character of the flux induced by the current i^{ds} and is, therefore, independent of rotation. The rotational voltage arises through the physical motion of the rotor conductors in the field produced by the current i^{ds} . Thus the effect of introducing the two axis variables is to divide the set of voltages associated with the machine into two compo-

nents which follow from the fundamental laws of electrodynamics. The separation is to some extent artificial as the rotational voltages are those that would appear if the conductors moved in a field of constant strength and the transformer voltages are those that would appear if the conductors remain stationary in an alternating magnetic field. In an actual machine we have the resultant of both these effects as they cannot have independent physical existence.

If the rotor is assumed to be stationary and the current varying, the induced voltages in the direct axis windings of the stator and the rotor are $L_{ds} p i^{ds}$ and $M_{di} p i^{ds}$, respectively. (p is the differential time operator).

If the current is assumed to be constant and rotor rotating with a constant velocity of $p\theta$, the generated voltage that exists along the quadrature axis is given by $(M_{d'q} p \theta) i^{ds}$ where $M_{d'q}$ represents the proportionality factor between the generated voltage and $i^{ds} p \theta$.

All the voltages induced and generated and also the resistance drop can be written down in the form of a Matrix.

$$\begin{array}{c}
 i^{ds} \\
 \begin{array}{|c|}
 \hline
 E_{ds} \\
 \hline
 E_{dr} \\
 \hline
 E_{qr} \\
 \hline
 E_{qs} \\
 \hline
 \end{array}
 \begin{array}{|c|}
 \hline
 ds + L_{ds} p \\
 \hline
 M_{di} p \\
 \hline
 - M_{d'q} p \theta \\
 \hline
 0 \\
 \hline
 \end{array}
 \end{array}$$

In a similar fashion if positive currents are assumed to flow in each of the four hypothetical coils of the primitive machine and the transformer and generated voltages are tabulated in the form of matrix, the resultant matrix for the primitive machine becomes:—

$$\begin{array}{c}
 \begin{array}{|c|}
 \hline
 d_s \\
 \hline
 d_r \\
 \hline
 q_r \\
 \hline
 q_s \\
 \hline
 \end{array}
 \begin{array}{|c|c|c|c|}
 \hline
 d_s & d_r & q_r & q_s \\
 \hline
 \begin{array}{|c|}
 \hline
 r_{ds} + L_{ds} p \\
 \hline
 M_{di} p \\
 \hline
 - M_{d'q} p \theta \\
 \hline
 0 \\
 \hline
 \end{array}
 &
 \begin{array}{|c|}
 \hline
 M_{di} p \\
 \hline
 r_r + L_{dr} p \\
 \hline
 - L'_{dr} p \theta \\
 \hline
 0 \\
 \hline
 \end{array}
 &
 \begin{array}{|c|}
 \hline
 0 \\
 \hline
 L'_{qr} p \theta \\
 \hline
 r_r + L_{qr} p \\
 \hline
 M_{dq} p \\
 \hline
 \end{array}
 &
 \begin{array}{|c|}
 \hline
 0 \\
 \hline
 M'_{dq} p \theta \\
 \hline
 M_{dq} p \\
 \hline
 r_{qs} + L_{qs} p \\
 \hline
 \end{array}
 \end{array}
 \end{array}$$

In a squirrel cage winding $L_{dr} = L_{qr} = L_r$ and $M_{di} = M_{dq} = M$.

The final equation of the motor is given by $e = Zi$.

where

$$e = \begin{array}{|c|c|c|c|} \hline d_s & d_r & q_r & q_s \\ \hline e_{d_s} & e_{d_r} & e_{q_r} & e_{q_s} \\ \hline \end{array}$$

and

$$i = \begin{array}{|c|c|c|c|} \hline d_s & d_r & q_r & q_s \\ \hline i_{d_s} & i_{d_r} & i_{q_r} & i_{q_s} \\ \hline \end{array}$$

the multiplication indicated is a matrix multiplication.

The impedance matrix developed above can be split up into three components: the first one consisting of the coefficients of the differential time operator p , denoted by L , as given in Table I: the second one consisting of

TABLE I

	d_s	d_r	q_r	q_s
d_s	r_{d_r}	0	0	0
d_r	0	r_r	0	0
q_r	0	0	r_r	0
q_s	0	0	0	r_{q_s}

	d_s	d_r	q_r	q_s
d_s	L_{d_s}	M_d	0	0
d_r	M_d	L_{d_r}	0	0
q_r	0	0	L_{q_r}	M_q
q_s	0	0	M_q	L_{q_s}

	d_s	d_r	q_r	q_s
d_s	0	0	0	0
d_r	0	0	L'_{q_r}	M'_q
q_r	$-M'_d$	$-L'_{d_r}$	0	0
q_s	0	0	0	0

the coefficients of $p\theta$ denoted by G also given in Table I; and the last one consisting of the resistances of the various windings denoted by R 's in Table I.

The performance characteristics of the machine can be determined by solving the four simultaneous equations:

$$e_{ds} = (r_{ds} + L_{ds}) i^{ds} + M_{d1} p i^{1r}$$

$$e_{dr} = M_{d1} p i^{ds} + (r_r + L_{dr} p) i^{dr} + L'_{q1} p \theta i^{qr} + M'_{q1} p \theta i^{qs}$$

$$e_{qr} = -M'_{q1} p \theta i^{ds} - L'_{q1} p \theta i^{dr} + (r_r + L_{qr} p) i^{qr} + M_{q1} p i^{qs}$$

$$e_{qs} = M_{q1} p i^{qr} + (r_{qs} + L_{qs} p) i^{qs}$$

The inductance tensor contains the self and mutual inductances of the four hypothetical windings of the primitive machine. The resistance tensor contains the resistances of the four windings; while the torque tensor contains the mutual inductances existing entirely due to rotation.

$$Z = R + L(p) + G(p\theta)$$

The flux density tensor representing the resultant flux density cut by each coil is given by $B = G \cdot I$.

It therefore follows that the electromagnetic torque is given by

$$T = I \cdot B = I \cdot G \cdot I$$

In alternating current steady state calculation the steady component of the torque $T = I^* \cdot B$; where I^* is the complex conjugate of I .

If the connection Tensor C shows the manner in which the new variables for the machine are related to those of the primitive machine, Kron has shown that it is possible to put down the equations of performance of the new machine as $e' = Z' \cdot I'$.

Where

$$Z' = C_t \cdot Z \cdot C; \quad e' = C_t e; \quad I' = Z'^{-1} \cdot e'$$

C_t is the matrix transpose of C .

The torque tensor may be established quickly by considering those components of Z that contain $p\theta$. It can be independently derived by the transformation $C_t \cdot G \cdot C$, where the multiplications indicated are matrix multiplications.

As an example, the analysis of performances of a shaded pole motor is worked out by the method of Tensor.

The impedance tensor of a primitive machine having three stator windings and two rotor windings is given by Z , in Table II.

TABLE II

	d_{s2}	d_{s1}	q_{s1}	d_r	q_r
d_{s2}	$r_{ds2} + jx_{ds2}$	jx_{ms}	0	jx_{md}	0
d_{s1}	jx_{ms}	$r_{s1} + jx_{ds1}$	0	jx_{md}	0
$Z = q_{s1}$	0	0	$r_{s1} + jx_{qs1}$	0	jx_{mq}
d_r	jx_{md}	jx_{md}	$x_{mq}V$	$r_2 + jx_{dr}$	$x_{qr}V$
q_r	$-x_{md}V$	$-x_{md}V$	jx_{mq}	$-x_{dr}V$	$r_2 + jx_{qr}$

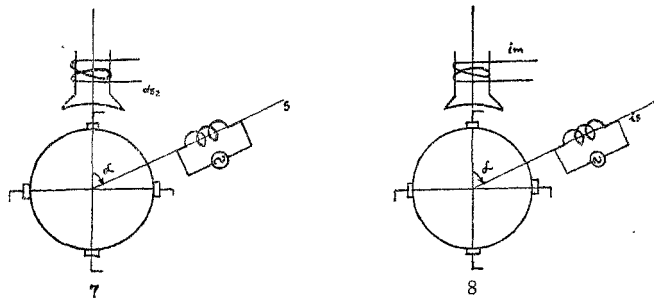
The coefficients of V , the ratio of actual motor speed and synchronous speed, constitute the torque tensor G , in Table III.

TABLE III

	d_{s2}	d_{s1}	q_s	d_r	q_r
d_r	0	0	x_{mq}	0	x_{qr}
q_r	$-x_{md}$	$-x_{md}$	0	$-x_{dr}$	0

Figs. 7 and 8 show the schematic diagram of the shaded pole motor. The shading coil is shifted by angle α from the direct axis (main winding axis) and it has 'a' number of turns, where

$$a = \frac{\text{No. of turns of shading coil}}{\text{No. of turns of main winding}}$$



FIGS. 7 & 8

By mere inspection the connection tensor C is obtained and is given in Table IV.

TABLE IV

	m	s	d_r	q_r
d_{s2}	1	0	0	0
d_{s1}	0	$a \cos \alpha$	0	0
$C = q_{s1}$	0	$a \sin \alpha$	0	0
d_r	0	0	1	0
q_r	0	0	0	1

The impedance of the machine, then, is $Z' = C_r Z C$, given in Table V. G , the torque tensor is also obtained by inspection from Z' , and is given in Table VI.

TABLE V

	m	s	d_r	q_r
m	$r_{dm} + jX_{dm}$	$jX_{ms}a \cos \alpha$	jX_{md}	0
s	$jX_{ms}a \cos \alpha$	$a^2 r_{s1} + jX_{ds1}a^2 \cos^2 \alpha + jX_{qs1}a^2 \sin^2 \alpha$	$jX_{ms}a \cos \alpha$	$jX_{mq}a \sin \alpha$
d_r	jX_{md}	$jX_{ms}a \cos \alpha + X_{mq}\sqrt{a} \sin \alpha$	$r_r + jX_{dr}$	$X_{qr}\sqrt{a}$
q_r	$-X_{md}\sqrt{a}$	$jX_{mq}a \sin \alpha - X_{md}\sqrt{a} \cos \alpha$	$X_{dr}\sqrt{a}$	$r_r + jX_{qr}$

G , by inspection, is

TABLE VI

	m	s	d_r	q_r
d_r	0	$X_{mq}a \sin \alpha$	0	X_{qr}
q_r	$-X_{md}$	$-X_{md}a \cos \alpha$	$-X_{dr}$	0

$$e' = \begin{array}{c|c|c|c} m & s & d_r & q_r \\ \hline e_m & e_s & 0 & 0 \end{array} \quad i \quad \begin{array}{c|c|c|c} m & s & d_r & q_r \\ \hline i_m & i_s & i^{ds} & i^{qr} \end{array}$$

With the aid of the equation established for Z' it is not quite possible to establish the equivalent circuits which lend themselves to easy extension to show the effect of space harmonics which are of considerable strength in shaded pole motors.

$$e' = Z'i' ; \text{ and } T = i_1^* G_1$$

represent the basic equations of the shaded pole motor in the crossfield theory. There is, however, a serious drawback in that the direct and quadrature axes are mixed up in an inconvenient form and hence the equations are not well adapted for the introduction of symmetrical components nor for the development of a simple and conventional equivalent circuit. At this stage Kron hits on the idea of introducing the axes of magnetising force (Fig. 8).

$$i^{ds} = i^m + i^s a \cos \alpha$$

$$i^{qs} = i^s a \sin \alpha$$

Solving the above two equations for i^m and i^s ,

$$i^m = i^{ds} - i^{qs} \cot \alpha$$

$$i^s = \frac{i^{qs}}{a \sin \alpha}$$

Put into matrix form

$$C_2 = \begin{matrix} & d_s & q_s & d_r & q_r \\ \begin{matrix} m \\ s \\ d_r \\ q_r \end{matrix} & \begin{bmatrix} 1 & -\cot \alpha & 0 & 0 \\ 0 & \frac{1}{a \sin \alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

By applying this transformation to the impedance matrix given in Table V the following expression for the new value of the impedance matrix is obtained.

$$Z'' = C_{2t} \cdot Z' C_2$$

where C_{2t} is the matrix transpose of C_2

TABLE VII

	d_s	q_s	d_r	q_r
d_s	$r_{ds2} + jx_{ds2}$	$(r_{ds2} + jx_{ds2}) \cot \alpha$	jx_{md}	0
q_s	$(r_{ds2} + jx_{ds2}) \cot \alpha$	$(r_{qs1} + jx_{qs1}) + [(r_{ds2} + jx_{ds2}) + (r_{dr2} + jx_{dr2})] \cot^2 \alpha$	0	jx_{mq}
d_r	jx_{md}	$x_{mq}V$	$r_r + jx_{dr}$	$x_{qr}V$
q_r	$-x_{md}V$	jx_{mq}	$-x_{dr}V$	$r_r + jx_{qr}$

	d_s	q_s	d_r	q_r
e_1''	d_r	0	x_{mq}	0
	q_r	$-x_{md}$	0	$-x_{dr}$
	d_s	q_s	d_r	q_r
e_2''	e_m	$\frac{e_s}{\alpha \sin \alpha} - e_m \cot \alpha$	0	0

By this simple transformation the equations have been reduced to the form of those of an unsymmetrical split phase induction machine having windings along the direct and quadrature axes. There is, however, a remarkable difference between the usual split phase machine and an unsymmetrical machine of this type in that mutual impedance exists between the two stator windings at right angles in space. In an actual shaded pole motor the mutual inductance consists of one component which arises on account of the angle between the main winding and auxiliary winding being different from 90° electrical and another component due to the magnetic bridge which provides a leakage path of low reluctance for the flux.

In the case of a machine having a smooth air gap and a cage winding on the rotor the impedance matrix given in Table V can be written in a simple form in terms of the self-impedances of the various windings and the mutual impedances that exist between them. This is shown in Tables VIII and IX. Z'' can be rewritten in the following way:

TABLE VIII

	m	s	d_r	q_r
m	Z_m	jx_{ms}	jx_{mr}	0
s	jx_{ms}	Z_s	$ajx_m \cos \alpha$	$ajx_m \sin \alpha$
d_r	jx_m	$ajx_m \cos \alpha + ax_m \sqrt{V} \sin \alpha$	Z_r	$x_2 \sqrt{V}$
q_r	$-x_m \sqrt{V}$	$ajx_m \sin \alpha - ax_m \sqrt{V} \cos \alpha$	$-x_2 \sqrt{V}$	Z_r

where

$$x_{qr} = x_{dr} = x_2$$

$$x_{mq} = x_{ms} = x_m$$

and

x_m = mutual reactance between main and rotor winding.

x_{ms} = Total mutual reactance between main and shading coil.

R_2, x_2 = Resistance and reactance, respectively, of the rotor, referred to the main winding.

$$Z_2 = R_2 + jx_2.$$

R_s, x_s = Resistance and reactance, respectively, of shading coil.

$$Z_s = R_s + jx_s.$$

TABLE IX

By inspection:—

$$= \begin{bmatrix} E_m & E_s & 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \end{bmatrix}; E_1 = \begin{bmatrix} E_m & E_s \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_m & I_s & I_{dr} & I_{qr} \end{bmatrix} = \begin{bmatrix} I_1 & I_2 \end{bmatrix}; I_1 = \begin{bmatrix} I_m & I_s \end{bmatrix}; I_2 = \begin{bmatrix} I_{dr} & I_{qr} \end{bmatrix}$$

$$= \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$$

where, referring to Table VIII,

$$Z_1 = \begin{array}{|c|c|} \hline Z_m & jx_{ms} \\ \hline jx_{ms} & Z_s \\ \hline \end{array} \quad \text{and} \quad Z_2 = \begin{array}{|c|c|} \hline jx_m & 0 \\ \hline ajx_m \cos \alpha & ajx_m \sin \alpha \\ \hline \end{array}$$

$$Z_3 = \begin{array}{|c|c|} \hline jx_m & (ajx_m \cos \alpha + aVx_m \sin \alpha) \\ \hline -x_m V & (ajx_m \sin \alpha - aVx_m \cos \alpha) \\ \hline \end{array}$$

$$Z_4 = \begin{array}{|c|c|} \hline Z_r & x_2 V \\ \hline -x_2 V & Z_r \\ \hline \end{array}$$

Thus

$$\begin{array}{|c|c|} \hline E_1 & E_2 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline I_1 \\ \hline I_2 \\ \hline \end{array}$$

whence

$$E_1 = Z_1 I_1 + Z_2 I_2 \quad (1)$$

$$E_2 = Z_3 I_1 + Z_4 I_2 \quad (2)$$

$$Z_4 I_2 = E_2 - Z_3 I_1 \quad (3)$$

$$I_2 = Z_4^{-1} (E_2 - Z_3 I_1) \quad (4)$$

$$E_1 = Z_1 I_1 + Z_2 Z_4^{-1} (E_2 - Z_3 I_1) \quad (5)$$

$$= Z_1 I_1 + Z_2 Z_4^{-1} E_2 - Z_2 Z_4^{-1} Z_3 I_1 \quad (6)$$

$$E_1 = Z_2 Z_4^{-1} E_2 + (Z_1 - Z_2 Z_4^{-1} Z_3) I_1 \quad (7)$$

$$(E_1 - Z_2 Z_4^{-1} E_2) = (Z_1 - Z_2 Z_4^{-1} Z_3) I_1 \quad (8)$$

$$E' = Z' I_1 \quad (9)$$

If we are finally interested in knowing the values of I_m and I_s , only the two unwanted axes d_r and q_r can be eliminated by the process outlined.

Since $E_2 = 0$ for a shaded pole motor in which the shaded coil is short-circuited on itself

$$E' = E_1$$

$$\therefore I_1 = (Z')^{-1} \cdot E' = (Z')^{-1} \cdot E \quad (10)$$

$$I_2 = -Z_4^{-1} \cdot Z_3 \cdot I_1 \quad (11)$$

$$Z_4^{-1} = \frac{1}{Z_r^2 + x_2^2 V^2} \cdot \begin{matrix} d_r & q_r \\ Z_r & -x_2 V \\ q_r & x_2 V \\ & Z_r \end{matrix} \quad (12)$$

$$Z_2 \cdot Z_4^{-1} = \frac{1}{Z_r^2 + x_2^2 V^2} \cdot \begin{matrix} m & q_r \\ ajx_m Z_r & -jx_m x_2 V \\ s & ajx_m Z_r \cos \alpha + ajx_m x_2 V \sin \alpha & -ajx_m x_2 V \cos \alpha + ajx_m Z_r \sin \alpha \end{matrix}$$

$$Z_2 Z_4^{-1} Z_3 = \frac{1}{Z_r^2 + x_2^2 V^2} \cdot \begin{matrix} m & s \\ ajx_m Z_r & -jx_m x_2 V \\ s & ajx_m Z_r \cos \alpha + ajx_m x_2 V \sin \alpha & -ajx_m x_2 V \cos \alpha + ajx_m Z_r \sin \alpha \end{matrix}$$

$$\begin{matrix} m & s \\ d_r & jx_m & ajx_m \cos \alpha + aVx_m \sin \alpha \\ q_r & -x_m V & ajx_m \sin \alpha - aVx_m \cos \alpha \end{matrix} \downarrow$$

$$Z_2 Z_4^{-1} Z_3 = \frac{jx_m^2}{Z_r^2 + x_2^2 V^2} \cdot \begin{matrix} m & s \\ ajZ_r + x_2 V^2 & a [\cos \alpha (jZ_r + x_2 V^2) - V \sin \alpha (jx_2 - Z_r)] \\ s & a [\cos \alpha (x_2 V^2 + jZ_r) + V \sin \alpha (jx_2 - Z_r)] & a^2 (jZ_r + x_2 V^2) \end{matrix}$$

$$= \frac{jx_m^2}{Z_r^2 + x_2^2 V^2} \cdot \begin{matrix} m & s \\ R_1 & R_2 \\ s & R_3 & R_4 \end{matrix}$$

$$Z' = Z_1 - Z_2 Z_4^{-1} Z_3 = \frac{1}{P} \cdot \begin{matrix} m & s \\ PZ_m - R_1 & Pjx_{ms} - R_2 \\ s & Pjx_{ms} - R_3 & PZ_s - R_4 \end{matrix}$$

where

$$P = \frac{Z_r^2 + x_2^2 V^2}{jx_m^2}$$

The inverse of this matrix, *i.e.*, Z' can be obtained as follows and then

the values of $I = \begin{bmatrix} I_m \\ I_s \end{bmatrix}$ estimated.

$$(Z')^{-1} = \frac{P}{m} \frac{P^2 (Z_m Z_s + x_{ms}^2) - P \{ (a^2 Z_m + Z_s) (jZ_r + x_2 V^2) - 2jx_{ms} a \cos \alpha (jZ_r + x_2 V^2) \} - a^2 \sin^2 \alpha (1 - V^2) (x_2^2 V^2 + Z_r^2)}{s}$$

$$\begin{array}{c} m \\ s \end{array} \left[\begin{array}{c|c} PZ_s - a^2 (jZ_r + x_2 V^2) & - [Pjx_{ms} - a \{ \cos \alpha (jZ_r + x_2 V^2) + V \sin \alpha (jx_2 - Z_r) \}] \\ \hline - [Pjx_{ms} - a \{ \cos \alpha (jZ_r + x_2 V^2) - V \sin \alpha (jx_2 - Z_r) \}] & PZ_m - (jZ_r + x_2 V^2) \end{array} \right]$$

Hence

$$(Z')^{-1} = \frac{1}{D} \begin{array}{c} m \\ s \end{array} \left[\begin{array}{c|c} (Z_r^2 + x_2^2 V^2) Z_s - jx_{ms}^2 a^2 (jZ_r + x_2 V^2), \text{ i.e., } Y_1 & Y_2 \\ \hline V^2 (ajx_{ms}^2 x_2 \cos \alpha - jx_{ms} x_2^2) - V (jx_{ms}^2 x_2 \sin \alpha) - (jx_{ms} Z_r^2 + ax_{ms}^2 Z_r \cos \alpha), \text{ i.e., } Y_3 & Y_4 \end{array} \right]$$

Where

$$Z_r = r_2 + jx_2; \quad Z_s = r_s + jx_s; \quad \text{and } Z_m = r_1 + jx_1$$

and

$$D = (1 - V^2) [(Z_m x_2 - jx_m^2) (ja^2 x_m^2 - x_2 Z_s) - jx_{ms}^2 a \cos \alpha (ajx_{ms}^2 \cos \alpha - 2jx_2 x_{ms}) - x_{ms}^2 x_2] \\ + r_2 [(Z_m Z_s + x_{ms}^2) (r_2 + 2jx_2) - jx_{ms}^2 (jZ_s + 2ax_{ms} \cos \alpha + ja^2 Z_m)]$$

Thus

$$\begin{array}{c} m \quad s \\ \left[\begin{array}{|c|c|} \hline I_m & I_s \\ \hline \end{array} \right] = \frac{1}{D} \cdot \begin{array}{c} m \quad s \\ \left[\begin{array}{|c|c|} \hline Y_1 & Y_2 \\ \hline Y_3 & Y_4 \\ \hline \end{array} \right] \cdot \left[\begin{array}{|c|} \hline V_m \\ \hline 0 \\ \hline \end{array} \right] \end{array}$$

and

$$I_m = \frac{V_m Y_1}{D}; \quad I_s = \frac{V_m Y_3}{D}$$

Substituting for Y_1 , Y_3 and D , I_m and I_s then are obtained as:

$$\begin{aligned} I_m = V_m [& (ja^2 x_m^2 x_2 - Z_r x_2^2) (1 - V^2) + r_B (Z_r r_2 + a^2 x_m^2 + 2jZ_r x_2) \\ & + [(1 - V^2) \{ (Z_m x_B - jx_m^2) (ja^2 x_m^2 - x_2 Z_r) \\ & - jx_m^2 a \cos \alpha (ajx_m^2 \cos \alpha - 2jx_2 x_m) - x_{m,r}^2 x_2 \} \\ & + r_B \{ (Z_m Z_r + x_m^2) (r_2 + 2jx_2) - jx_m^2 (jZ_r + 2ax_{m,c} \cos \alpha + ja^2 Z_m) \}] \end{aligned}$$

and

$$\begin{aligned} I_s = V_m [& V^2 (ajx_m^2 x_2 \cos \alpha - jx_m x_2^2) - V (jx_m^2 x_2 \sin \alpha) - (jx_m Z_r^2 + ax_m^2 Z_r \cos \alpha) \\ & \div D \text{ (as above for } I_m) \end{aligned}$$

TORQUE AND OUTPUT

The torque equation is given, in the case of a machine having a constant airgap length and smooth surface for the rotor, by

$$T = I_2^* \cdot B = I_2^* \cdot G_2 \cdot I_1,$$

where I_2^* is the complex conjugate of I_2 and B is the flux density tensor.

With the aid of the equations already given it is easily shown that in a shaded pole motor, the synchronous torque is

$$T = [-I_1^* \cdot (Z_4^{-1} \cdot Z_3)_l] \cdot G_2 \cdot I_1$$

since

$$I_2 = (-Z_4^{-1} \cdot Z_3 \cdot I_1)$$

G_3 is obtained by retaining the coefficients of the V in Z_3 and putting down zeros everywhere else.

$$G_3 = \begin{array}{c} \begin{array}{cc} m & s \end{array} \\ \begin{array}{c} d_r \\ q_r \end{array} \begin{array}{|cc|} \hline 0 & ax_m \sin \alpha \\ \hline -x_m & -ax_m \cos \alpha \\ \hline \end{array} \end{array}$$

$$Z_4^{-1} \cdot Z_3 = \frac{x_m}{(Z_r^2 + x_2^2 V^2)} \begin{array}{c} \begin{array}{cc} m & s \end{array} \\ \begin{array}{c} d_r \\ q_r \end{array} \begin{array}{|cc|} \hline jZ_r + x_2^2 V^2 & Z_r (aj \cos \alpha + aV \sin \alpha) - x_2 V (aj \sin \alpha - aV \cos \alpha) \\ \hline jx_2 V - Z_r V & x_2 V (-aj \cos \alpha + aV \sin \alpha) + Z_r (aj \sin \alpha - aV \cos \alpha) \\ \hline \end{array} \end{array}$$

$$(Z_4^{-1} \cdot Z_3)_t \cdot G_3 = \frac{x_m}{Z_r^2 + x_2^2 V^2} \begin{array}{c} \begin{array}{cc} m & s \end{array} \\ \begin{array}{c} d_r \\ q_r \end{array} \begin{array}{|cc|} \hline Vx_m (jx_2 + Z_r) & aVx_m \cos \alpha (jx_2 + Z_r) + ax_m \sin \alpha (x_2^2 V^2 - jZ_r) \\ \hline Vax_m \cos \alpha (jx_2 + Z_r) - ax_m \sin \alpha (x_2^2 V^2 - jZ_r) & a^2 Vx_m (jx_2 + Z_r) \\ \hline \end{array} \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} m & s \end{array} \\ \begin{array}{c} d_r \\ q_r \end{array} \begin{array}{|cc|} \hline B_1 & B_2 \\ \hline B_3 & B_4 \\ \hline \end{array} \end{array}$$

Torque, in Synchronous watts, is

$$\begin{array}{c}
 \begin{array}{cc} m & \longrightarrow & s \\ \hline I_m^* & I_s^* \end{array} \cdot \begin{array}{cc} m & s \\ \hline B_1 & B_2 \\ \hline B_3 & B_4 \end{array} \downarrow \cdot \begin{array}{c} I_m \\ \hline I_s \end{array}
 \end{array}$$

$$= I_m (B_1 \cdot I_m^* + B_3 I_s^*) + I_s (B_2 \cdot I_m^* + B_4 \cdot I_s^*)$$

Thus, it is shown that the impedance matrices of any machine can be quickly obtained from the primitive machine. An advantage of the method is that the initial steps for analysis is done once for all for the primitive machine and the steps of analysis for other machines can start later; sometimes only the final step being necessary. The correlation between machines is so complete that analysis of a new machine is built on that of machines already analysed. Herein lies the power in this method of machine analysis.

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