A GRAPHICAL TREATMENT OF THE SKIN EFFECT.

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The fact that an alternating current is not uniformly distributed over the cross-section of a conductor has for a long time been well known to all electrical engineers, and the changes in the resistance and self-inductance of a cylindrical conductor arising from this cause hare formed the subject of several analytical investigations. The contributions of Clerk Maxwell, Oliver Heaviside, lord Rayleigh, and Lord Kelvin to the solution of this problem are well known to those interested in the subject, and the formulæ and tabulated numerical results given by these investigators have been frequently referred to by writers on alternating currents. The most recent and thorough analytical treatment of this problem we owe to Dr. Alexander Russell.* An elementary explanation of the general nature of the skin effect has also been given; but no attempt seems hitherto to have been made to deal with the subject in a quantitative manner by a purely elementary method. It is the object of this paper to develop such a method. As will be seen, the method has the great advantage of presenting to the mind in a very vivid manner the purely physical aspect of the problem.

In what follows, we shall confine our attention to cylindrical conductors so arranged that everything is symmetrical about the axis of the conductor, the lines of magnetic induction being represented by circles having their centres on the axis of the conductor. Imagine a long, straight cylindrical conductor surrounded by a coaxial infinitely thin shell of negligible resistance lying infinitely close to the surface of the conductor but insulated from it, and let an alternating current he sent through the conductor, the surrounding shell forming the return path for the current. We may term the quotient of the potential difference by the current in such a circuit the "internal impedance" of the conductor. This may be regarded as made up of

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the "effective resistance" of the conductor (=power dissipated in conductor divided by square of current) and its "effective reactance," these three quantities forming the well-known impedance triangle. Since the reactance of the conductor in the case under consideration arises wholly from magnetic lines within its substance, we may conveniently term the effective reactance in this case the "internal reactance" of the conductor

Suppose next that the surrounding shell forming the return path for the current is made to expand, so that there is a considerable space formed between the outer surface of the cylindrical conductor and the shell. This space becomes filled with magnetic lines and increases the reactance. That portion of the reactance which arises from such lines may be conveniently termed the "external reactance." Since everything remains symmetrical about the axis, it is evident that the only effect of the external reactance is to add to the reactance component of the impressed potential difference, the current distribution within the conductor not being in any way affected. A similar result will hold good if for the coaxial shell we substitute any other form of return conductor, provided always that this return conductor is sufficiently far from the cylindrical conductor to prevent any appreciable disturbance of the symmetry of the internal magnetic field of the conductor. We are thus led to the following result, on which the method about to be explained is based :—

So long as the symmetry of the internal magnetic field of a cylindrical conductor remains unaffected, changes in the magnetic flux external to the conductor have no effect on the current distribution.

Hence if we suppose our cylindrical conductor divided into a central cylindrical core and a number of coaxial shells surrounding it, the removal of any number of shells from the outside of the conductor will not in any way affect the law of current distribution in the remaining or internal shells, since so far as these shells are concerned, the flux contributed by the shells which have been removed is an external flux. Similarly, the addition of any shells to the outside of the conductor so as to increase its diameter will leave the law of current distribution in the originally existing shells unaltered.

Making use of this principle, we first approximately determine the current distribution in the central core, and then in the successive shells which we imagine to be built up around the core, until the desired diameter of conductor is reached.

We begin by assuming a certain current density at the centre of the conductor, and suppose that the current density in each successive shell increases at a uniform rate from its inner to its outer surface.

In determining the current densities in the consecutive shells we shall suppose the current density to be split up into two components, which are in quadrature whith each other, one of these—the x-component—being in phase with the current density at the centre, and the other—the y-component—in quadrature with it.

The details of the method will be best understood by considering a numerical example.

Let the conductor be 1,000 metres long, and have a resistivity at the working temperature of $1{\cdot}8\times10^{-6}$ ohm/cm. cube. We shall assume the standard frequency of 50, and shall suppose the conductor to be built up by gradually adding coaxial shells around a central core of 1 mm, radius, the radial depth of each shell being also 1 mrn.

Let us assume a current density of 10 amperes/sq. cm. at the centre of the core. This gives for the resistance drop per centimetre length $1{\cdot}8\times10^{-6}\times10=1{\cdot}8\times10^{-5}$ volt, and for the total drop along the 1,000 metres 1'8 volt. We provisionally assume that the current density over the central core of 1 mm. radius is uniform, so that the total current is $\pi\times0.1^2\times10=0.3142$. The value of H at the surface of the core is thus $2\times0{\cdot}03142/0{\cdot}1=0{\cdot}6283$, and since II increases at a uniform rate from the centre—where its value is zero—to the surface, the mean value of H is 0 3142, and the flux linked with an infinitely thin filament coincident with the axis of the core is $0{\cdot}3142\times10^4$ lines. The reactance potential difference corresponding to this flux is given by *pX* flux, where $p=2\pi$ x frequency$=314{\cdot}2$. Thus the reactance potential difference for the axial filament is—

$$314{\cdot}2\times0{\cdot}3142\times10^4\times10^{-8}=0{\cdot}00988 \text{ volt,}$$

and this potential difference is in quadrature with the current density at the axis. Now, at the surface of the core, where there is no internal reactance, the reactance component of the potential difference will give rise to a current density in quadrature with that at the axis. This current density is obtained by dividing the potential gradient of the reactance potential difference by the resistivity. Thus y-component of current density at surface of core—

$$=0{\cdot}00988\times10^{-5}/1{\cdot}8\times10^{-6}=0{\cdot}0549.$$

The x-component being 10, we see that the resultant current density at the surface is practically identical with that at the axis, but that it is in advance of it as regards phase by the angle $\tan^{-1}\frac{0.0549}{10}=0^\circ\ 19'$. Assuming the y-component of the current density to increase at a uniform rate from a zero value at the axis to the value 0·0549 at the surface of the core, we find for the total y-component of the current in the core the value 0·00115 ampere. This gives for the y-component of H at the surface $\frac{2\times0{\cdot}000115}{0{\cdot}1}=0{\cdot}0023$.

The above results may be tabulated as follows:—

x-component of current density at axis $=d_x'=10$ amperes/sq. cm.
y-component of current density at axis $=d_y'=0$.
x-component of current density at surface $=d_x=10$.
y-component of current density at surface $=d_y=0{\cdot}0549$.
x-component of magnetic force at surface $=H_x=0{\cdot}6283$.
y-component of magnetic force at surface $=H_y=0{\cdot}0023$.
Total x-component of current in core $=i_x=0{\cdot}3142$.
Total y-component of current in core $=i_y=0{\cdot}00115$.
Phase displacement of surface current density $=\theta=0^\circ\ 19'$.

We next place around our core a coaxial shell of 1 mm. radial thickness, and proceed to determine the current densities d_x and d_y at its outer surface, the current densities at the inner surface being identical with those at the outer surface of the core. Since the current in the shell will give rise to additional magnetic lines, it is evident that if the current in the core is to remain unaltered a higher impressed potential difference must now be provided, owing to the increased reactance of the axial filament. The current distribution in the shell may be determined as follows.

We provisionally assume that the rate of change of d_x remains unaltered—*i. e.*, equal to zero, so that we still have $d_x=$ 10 at the outer surface of the shell. The total x-component of the current is now $i_x=1{\cdot}257$.* Hence at the outer surface of the shell we have—

$$H_x=2\times 0{\cdot}1257/0{\cdot}2=1257.$$

Since H_x as before increases uniformly from zero on the axis to 1-257 at the outer surface of the shell, its mean value is 0 6285, and the x-component F_x of the total flux linked with the axial filament is $0{\cdot}6285\times 0{\cdot}2\times 10^5=1{\cdot}257\times 10^4$. The reactance potential difference V_y is therefore $314{\cdot}2\times 1{\cdot}257\times 10^4\times 10^{-8}=0{\cdot}0395$. This gives rise to a current density $d_y=\dfrac{0{\cdot}0395\times 10^{-5}}{1{\cdot}8\times 10^{-6}}=0{\cdot}219$ at the outer surface of the shell.

We thus have rf_y=0'0549 at the inner, and 0·219 at the outer surface of the shell. Prom this we find for the y-component of the current in the shell the value 0·01376*. Adding this to the y-component previously found for the core—*viz.*, 0·00115, we get for the total y-component the value 0·01491. Hence $H_y=2\times 0{\cdot}001491/0{\cdot}2=0{\cdot}01491$ at the outer surface of the shell. If, now, we plot the values $H_y=0{\cdot}0023$ and $H_y=0{\cdot}01491$ as ordinates against the values 0·1 and 0'2 (distances from axis) as abscissæ, and draw a curve through the origin and the two points so determined, we find for the area of the curve the value 8×10^{-4}. This represents the y-component of the flux per centimetre length of our conductor, so that the total y-component of the flux linked with the axial filament is 80. The corresponding induced E.M.F. is—

$$314\times 80\times 10^{-8}=0{\cdot}000251,$$

which is negligible in comparsion with the resistance drop (1·8) along the axial filament. It is to be noted that this E. M. F. is co-phasal with the axial current density, so that its effect (if appreciahle) would be to reduce the x-component of the impressed potential difference, and hence to reduce d_x at the outer surface of the shell.

*If the current density have the values d_1 and d_2 at the inner and outer surfaces respectively of a shell whose inner and outer radii are r_1 and r_2 respectively, and if the density increases at a uniform rate from the inner to the outer surface, then it may be shown (by a simple integration) that the total current in the shell is—

area of shell x

As an alternative method, the current may be determined by a process of graphical integration.

The phase displacement of the current density at the outer surface of the shell relatively to the axial current density is—

$$\tan^{-1}\frac{0\cdot219}{10}=1^\circ\ 15'.$$

We now place another shell of 1 mm. radial thickness around our conductor, thereby increasing its radius to 3 mm. We again provisionally assume d_x to remain unaltered, and, proceeding as before, we find the following values :—

i_x (total x-component of current in conductor)$=2\cdot827$.
H_x $=1\cdot885$ at surface.
F_x (total x-component of flux) $=2\cdot827\times10^4$.
V_y (reactance potential difference)$=0\cdot0888$.
d_y $=0\cdot4935$ at surface.
y-component of current in shell$=0\cdot0574$.
i_y (total y-component of current in conductor)$=0\cdot0723$.
H_y $=0\cdot0482$ at surface.

On plotting the value $H_y=0\cdot0482$ in the diagram connecting H_y with distance from axis, finding the area of the curve as before, and multiplying by 10^5 (length of conductor), we find for the total y-component F_y of the flux linked with the axial filament the value $F_y=366$. The corresponding induced E. M. F. is $E_x=0\cdot00115$. Since, as already mentioned, this E. M. F. is in phase with the axial current density, the component V_x of the impressed potential difference in phase with the axial current density is $V_x=1\cdot8-0\cdot00115=1\cdot7988$. Hence the assumed value $d_x=10$ at the surface of the conductor requires a small correction, the true value being $10\times\frac{1\cdot7988}{1\cdot8}=9\cdot993$.

The phase displacement of the surface current density relatively to that along the axis is $\tan^{-1}$

We next place an additional shell of 1 mm. radial thickness around the conductor, thus making its radius equal to 4 mm. We provisionally assume that the rate of change of d_x with distance from axis changes by the same amount as before—*i.e.*, by $0\cdot007$ per millimetre. The new rate of change will thus be $0\cdot014$ per millimetre, giving for the value of d_x at the new surface $9\cdot993-0\cdot014=9\cdot979$. The work is then continued as before, the values of the various other quantities being determined, and a correction being applied if necessary to the provisionally assumed value of d_x at the surface.

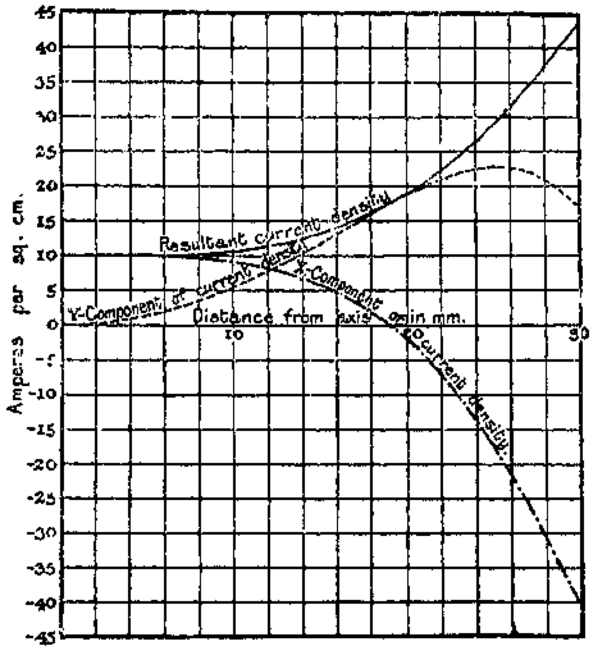

FIG. 1.—Variation of Current Density with Distance from Axis.

By proceeding in this manner we can build up our conductor to any desired thickness. Table I. contains the results of the calculation up to a radius of 3 cm.

Some of these results are exhibited graphically in Figs. 1 and 2. Referring to Fig. 1, we see that the x-component of the current density steadily decreases, passes through a zero value, and then increases in the negative direction. The y-component starts from a zero value, increases very slowly at first, then more rapidly, passes through a maximum value and decreases. Had we extended Table I, to larger distances from the axis, we should have found that each component of the current density is represented by a wave which passes through a succession of maxima, zero, and minima values. The total or resultant current density steadily increases, and so does its phase angle of advance relatively to the current density at the axis. The connection between the value of the current density and its phase angle of advance is clearly exhibited in the polar diagram of Fig. 2, in which the radius vector represents the current density, and the vectorial angle the phase angle of advance of the current density relatively to that at the axis. The numbers along the polar curve (which is the locus of the current-density vector) represent distances from the axis.

Knowing the current-density distribution, we can easily determine from it the total power dissipated in a conductor of given radius. For this purpose we first plot a curve, as in Fig. 3, whose abscissa represents the area of a conductor of given

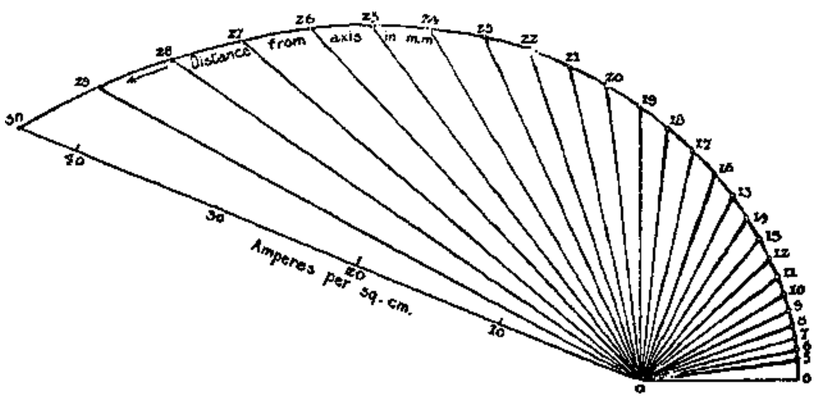

FIG. 2.—Polar Diagram of current Density and Phase Angle.

radius, while its ordinate gives us the rate of dissipation of energy per square centimetre of conductor cross-section at the surface of the conductor. In order to obtain points along this curve, we take the consecutive radii given in Table I., and find the areas of the corresponding circles; this gives us a number of abscissæ fur our curve.

To find the corresponding ordinates we multiply the square of the current density d in Table I. by 0·18 $(=1{\cdot}8\times10^{-6}\times10^{5}$, the product of the resistivity into the length of the conductor). We next proceed to find the area of the curve so obtained, this area representing, for any given value of the abscissa, the total watts dissipated in a conductor whose cross-section is given by the abscissa. On dividing the total watts dissipated by the square of the total current in the conductor $(=i_x^2+i_y^2)$, we find its resistance to alternating currents of frequency 50, and the quotient of this resistance by the resistance to continuous currents (calculated in the ordinary way from the resistivity, length and cross-section of the conductor) gives us the ratio of the two resistances. This ratio is plotted against the diameter of the conductor in Fig. 4. It remains at a value not far removed from unity until the diameter of the conductor exceeds 5 mm., then begins to increase, very slowly at first, then more rapidly, and after a diameter of 20 mm. has been reached, it nearly follows a straight-line law.

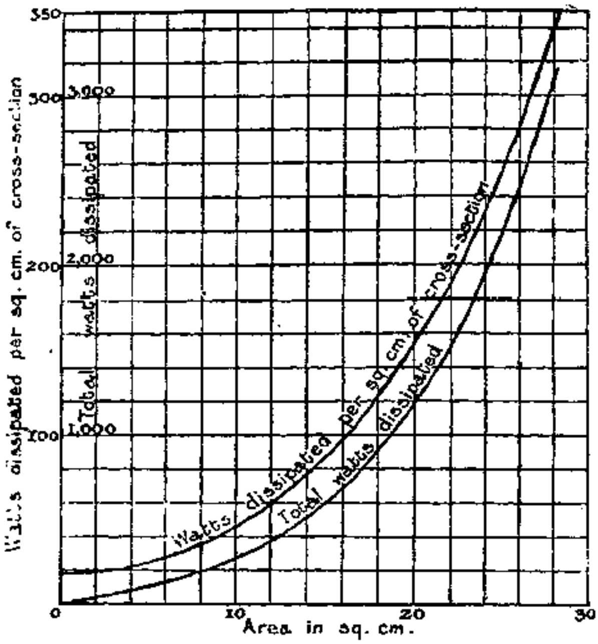

FIG. 3.—Determination of power lost in conductor.

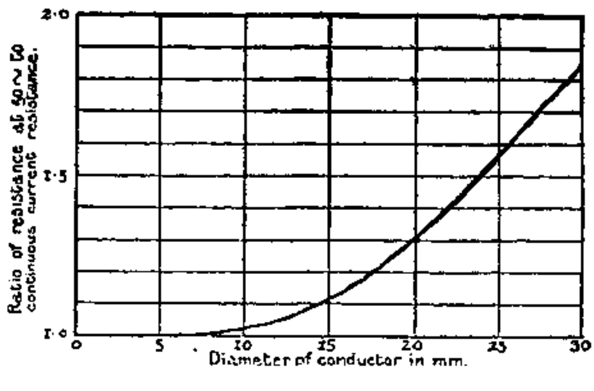

FIG. 4.—Ratio of resistances corresponding to various diameters of conductor.

While the effective resistance of a conductor is *increased* by the crowding of the current towards the surface, the effective internal self inductance is *decreased* by the same effect. It is easy to understand this result from general considerations. For if we imagine a cylindrical current sheet to start from near the axis of the conductor and to expand outwards like a ripple on water,

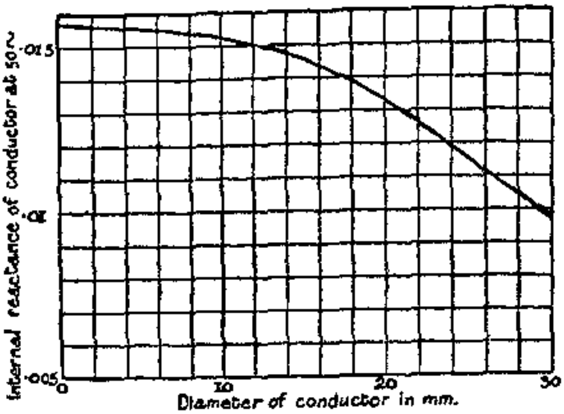

FIG 5.—Relation connecting internal reactance with diameter of conductor.

it is obvious that as the sheet expands it becomes linked with less and less flux. From the data at our disposal we can easily calculate the values of the internal reactance corrsponding to various diameters of conductors. For, using Table I., we have for the square of the internal impedance the expression $(V_x^2+V_y^2)/(i_x^2+i_y^2)$. On subtracting from this the square of the resistance to currents of frequency 50, we obtain the square of the reactance, and so the reactance itself. It is to be noted that for the smaller sizes of conductors this method does not yield accurate results, owing to the smallness of the reactance as compared with the resistance. But for the smaller sizes the internal self-inductance at 50 cycles per second will be nearly the same as that corresponding to a uniform current distribution, and this latter self-inductance is known to be (in C.G.S. units) $\frac{1}{2}$ per centimetre length of conductor, and to be independent of the size of the conductor. Hence the internal reactance, in ohms, of a cylindrical copper conductor of small diameter 1,000 metres long is $2\pi\times 50\times\frac{1}{2}\times 10^{-4}=0.01571$ at a frequency of 50. The difficulty of finding the value of the internal reactance for small diameters is thus overcome, and we are in a position to plot the curve of Fig. 5, which shows the gradual drop in the internal reactance with increasing diameter of conductor.

It is hardly necessary to point out that the method described is applicable to the case of a long, hollow, cylindrical conductor or tube, provided its internal field is represented by circles concentric with the tube. It may also be applied to solid or hollow cylindrical conductors of iron, and the effect of varying permeability is easily taken into account (we suppose the permeability to be uniform for any one shell, but to vary from shell to shell). Although the method may appear to be somewhat laborious, it has the merit of bringing within the reach of any engineer possessing a sound knowledge of the elements of alternate current theory a problem of great complexity, whose solution could hitherto only be dealt with by the aid of very advanced mathematics.

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TABLE I.

Shell.	dx	i*	H_x	F_x	V_y	d_y	i_y	H_y	F_y	E_x	V_x	9	d
Millimetres													
0—1	10·00	0·314	0·628	3,140	0·0099	0·5500	0·00115	0·0023			1·800	0° 19′	10·00
1—2	10·00	1·260	1·260	12,570	0·0395	0·2190	0·01490	0 1400	80	0·00025	1·800	1° 15′	10·00
2—3	9·99	2·830	1·865	28,270	0·0888	0·4935	0·07230	0·4820	3?6	0·00115	1·799	2° 50′	10·00
3-4	9 98	5·020	2-510	50,240	0·1578	0·8770	0·22500	0·1125	1,140	0·00360	1·796	5° 1′	10·02
4—5	9·95	7·840	3.140	78,470	0·2465	1·3700	0·54500	0·2180	2,740	0.008CO	1·791	7° 50′	10·04
5—6	9.90	11·270	3·760	112,940	0·3550	1·9700	1·12500	0·3750	5,670	0-01780	1·782	11° 16′	10·09
6—7	9·82	15·290	4·370	153,570	0·4825	2·6800	2 08000	0·5940	10,513	0·02309	1·767	15" 16′	10·17
7-8	9.C9	19·880	4·970	200,270	0·6290	3·4950	3·54000	0·8850	17.000	0·05620	1·744	19° 50′	10·30
8-9	9·47	25·000	5·560	252,900	0·7950	4·4140	5·65000	1·?570	30,470	0·09570	1·70i	25° 0′	10·45
9-10	9·21	30·550	6·110	311,230	0·9780	5·4300	8·59000	1·7190	45,350	0·14250	1·657	30° 32′	10·69
10-11	8·81	36·500	fi 640	375,000	1·1780	6·5400	12·54000	2·2800	68,160	0 21400	1·586	36" 37′	10·97
11—12	8·35	42·700	7·114	443,800	1·3940	7·7500	17·70000	2·9500	04,820	0 29600	1·50i	42° 54′	11·39
12—13	7·77	49·000	7·550	517,100	1·6250	9·0200	24·30000	3·7400	127,760	0·40100	1·399	49° 17′	11·91
13—14	6·96	55·300	7·900	594,300	1·8670	10·3700	32·50000	4·6500	174,230	0·54700	1·253	56° 8′	12·49
14—15	6·06	61.200	8·160	674,600	2·1200	11·7800	42·60000	5·6800	225,900	0.71000	1·090	62° 47′	13·25
15—16	4·97	66·600	8.330	757,100	2·3800	13·2000	54·80000	6 8500	288,500	0·90?00	0·?91	69° 23′	14·12
16—17	3·61	71·100	8·360	840,500	2·6400	14·7000	69·20000	8·1400	366,200	1·15000	0·649	76° 10′	15·11
17—18	2·06	74·200	8·240	928,500	2·9000	16·1200	86·20000	9 5700	454,800	1·43000	0·371	82° 43′	16·25
18—19	0·25	75·500	7·950	1,004,500	3.1560	17·5300	105·70000	11·1300	558,300	1·75400	0 046	89° 11′	17·53
19—20	– 1-83	74.550	7·450	1,079,000	3·3900	18·8300	128·00000	12·8000	678,000	2·13000	—0·330	95° 34′	18·92
20—21	– 4·22	70·650	6·730	1,149,900	3·6100	20·1000	153·00000	14·6000	815.000	2·56000	—0·760	101° 53′	20·51
21—22	– 6·93	63·120	5·740	1,212,300	3·8100	21·1600	181·00000	16·4400	970,200	3·05000	—1·250	108° 9′	22·27
22—23	– 9.97	51·200	4·450	1,263,200	3·9700	22·0500	211·40000	18 4000	1,144,300	3·59500	—1·795	114° 20′	24·20
23—24	—13·34	34·000	2·830	1,299,600	4·0800	22·7000	244·40000	20·4000	1,338,000	4·20000	—2·400	120° 27′	26·30
24—25	—17·08	10·600	0·850	1,318,000	4·1400	23·0000	279·50000	224000	1,551,600	4.87000	—3·070	126° 36′	28·65
25-26	—21·15	- 20·040	— 1·543	1,314,500	4·1300	22·9400	316·00000	24·3000	1,785,000	5 61000	—3·810	132° 41′	31·20
26—27	—25·57	- 58.900	- 4·370	1,270,900	3·9900	22·2000	354·00000	26·2000	2,037,800	6·40000	—4·600	139° 4′	34·14
27—28	—30·30	—107·200	- 7·660	1,210.800	3·8000	21·1300	391·00000	27·3000	2,308,600	7·25000	—5·450	145° 7′	36·94
28—29	—35·30	—166·000	— 11·450	1,115,200	3·5000	19·4600	428·00000	29·5000	2,595,800	8·16000	—6·360	151° 9′	40·32
29—30	—40·56	—236·400	- 15·740	979,400	3·0800	17·1000	461·00000	30·8000	2,807,100	9·10000	—7·340	157° 9′	44·00