REVIEWS

Minimal Triangulations of Manifolds

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Abstract | Finding vertex-minimal triangulations of closed manifolds is a very difficult problem. Except for spheres and two series of manifolds, vertex-minimal triangulations are known for only few manifolds of dimension more than 2 (see the table given at the end of Section 5). In this article, we present a brief survey on the works done in last 30 years on the following: (i) Finding the minimal number of vertices required to triangulate a given pl manifold. (ii) Given positive integers n and d, construction of n-vertex triangulations of different d-dimensional pl manifolds. (iii) Classifications of all the triangulations of a given pl manifold with same number of vertices.

In Section 1, we have given all the definitions which are required for the remaining part of this article. A reader can start from Section 2 and come back to Section 1 as and when required. In Section 2, we have presented a very brief history of triangulations of manifolds. In Section 3, we have presented examples of several vertex-minimal triangulations. In Section 4, we have presented some interesting results on triangulations of manifolds. In particular, we have stated the Lower Bound Theorem and the Upper Bound Theorem. In Section 5, we have stated several results on minimal triangulations without proofs. Proofs are available in the references mentioned there. We have also presented some open problems/conjectures in Sections 3 and 5.

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1. Preliminaries

Affine Subspaces of $\mathbb{R}^n$ and Linear Maps

The space $\{(x_1,\dots,x_n) : x_i \text{ is real for } 1 \le i \le n\}$ is denoted by $\mathbb{R}^n$. For us, $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n \ge 0\}$, $I = [-1,1] \subseteq \mathbb{R}$ and $\mathbb{N} = \{0,1,2,\dots\} \subseteq \{0,\pm1,\pm2,\dots\} = \mathbb{Z}$.

An *affine* subspace $V \subseteq \mathbb{R}^m$ (of dimension n) is a translated vector subspace (of dimension n). So, $V \subseteq \mathbb{R}^m$ is an affine subspace if $a_1,\dots,a_r \in V$ and $\lambda_1,\dots,\lambda_r \in \mathbb{R}$ with $\sum_{i=1}^r \lambda_i = 1$ implies $\sum_{i=1}^r \lambda_i a_i \in V$.

A map $f : V \to \mathbb{R}^d$, from an affine subspace V of $\mathbb{R}^m$, is called (affine) *linear* if $f(\sum_{i=1}^r \lambda_i a_i) = \sum_{i=1}^r \lambda_i f(a_i)$.

Clearly, if $V \subseteq \mathbb{R}^m$ is an $(m-1)$-dimensional affine subspace then $\mathbb{R}^m \setminus V$ has two connected components, say H_1 and H_2. The subsets $V \cup H_1$ and $V \cup H_2$ are called the (closed) *half-spaces* determined by V.

If the smallest affine subspace in $\mathbb{R}^m$ containing n points $v_1,\dots,v_n$ is $(n-1)$ dimensional (equivalently, $v_2 - v_1,\dots,v_n - v_1$ are linearly independent), then we say that the points $v_1,\dots,v_n$ in $\mathbb{R}^m$ are *affinely independent*.

Joins and Cones

If A, B are subsets of $\mathbb{R}^n$, then their *join* AB is the subset $\{\lambda a + \mu b : a \in A, b \in B, \lambda, \mu \in [0,1] \text{ and } \lambda + \mu = 1\}$. So, AB consists of all points on line segments (arcs) with endpoints in each of A and B. If $A = \emptyset$ then we define $AB = B$. If $A = \{a\}$ then AB is also denoted by aB. A join aB is called a *cone* (with vertex a and base B) if $a \notin B$ and $b_1, b_2 \in B$, $b_1 \neq b_2$ then $ab_1 \cap ab_2 = \{a\}$.

Polytopes and Simplices

A subset $C \subseteq \mathbb{R}^m$ is called *convex* if for each pair of points $a, b \in C$ the arc $ab \subseteq C$. For a set A (possibly empty) in $\mathbb{R}^m$, the smallest convex set containing A is called the *convex hull of* A and is denoted by $\langle A \rangle$. A *polytope* is a convex hull of a finite set. A polytope C is said to be an n *dimensional* polytope (or n-*polytope*) if the smallest affine subspace containing C is n dimensional. By convention, the empty set is a polytope of dimension -1.

A point v in a polytope C is called a *vertex* if $v \in$ arc $ab \subset C$ implies v is a or b.

Clearly, an n-polytope has at least $n+1$ vertices. If an n-polytope has exactly $n+1$ vertices then it is also called an n-*simplex*. So, $\langle \{v_0, v_1, \ldots, v_n\} \rangle$ is an n-simplex if and only if $v_0, v_1, \ldots, v_n$ are affinely independent. An n-simplex with vertices $v_0, v_1, \ldots, v_n$ is denoted by $v_0 v_1 \cdots v_n$.

If $A = v_0 \cdots v_k$ is a k-simplex then $\hat{A} := \sum_{i=0}^{k} \frac{1}{k+1} v_i$ is called the *barycentre* of A.

Faces of a Polytope

Let C be an n-polytope in $\mathbb{R}^m$. If V is an $(m-1)$-dimensional affine subspace such that C is in one of the half-space determined by V then $C \cap V$ is called a *face* of C and is denoted by $C \cap V < C$. Clearly, a face of a polytope is a polytope. If $\emptyset \neq D < C$ and $D \neq C$ then D is called a *proper face* of C. The union of all the proper faces of an n-polytope C ($n \geq 1$) is called the *frontier* of C and is denoted by $\dot{C}$. The subset $\mathring{C} := C \setminus \dot{C}$ is called the *interior* of C. For a 0-polytope (i.e., for a vertex) v we define $\mathring{v} = v$ and $\dot{v} = \emptyset$. So, for a polytope C, $C = \mathring{C} \sqcup \dot{C}$.

Simplicial and Stacked Polytopes

A polytope is called *simplicial* if its proper faces are simplices.

A simplicial d-polytope P is called *stacked* if there is a sequence $P_1, \ldots, P_k$ of simplicial d-polytopes such that P_1 is a simplex, $P_k = P$ and P_{j+1} can be constructed from P_j by attaching a d-simplex along a $(d-1)$-face of P_j for $1 \leq j \leq k-1$.

Polyhedra and Subpolyhedra

A subset $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if each point a in P has a cone neighbourhood (in P) $N = aL$, where L is compact; N and L are called the *star* and the *link* of a in P respectively. We write $N = N_a(P)$ and $L = L_a(P)$. If P and Q are polyhedra and $Q \subseteq P$ then Q is called a *subpolyhedron* of P.

Piecewise-Linear (PL) Maps

A map $f : P \to Q$, where P and Q are polyhedra, is called *piecewise-linear* (in short *pl*) if each $a \in P$ has a star $N = aL$ such that $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$, for all $x \in L$ and $\lambda, \mu \geq 0$, $\lambda + \mu = 1$. Moreover, if f is a homeomorphism then f is called a *pl homeomorphism*. A pl map $f : P \to Q$ is called a *pl embedding* if f is injective and $f(P)$ is a subpolyhedron of Q.

[Check that $f : P \to Q$ a pl homeomorphism implies $f^{-1} : Q \to P$ is pl.]

PL Manifolds

A polyhedron M is called an n-dimensional *pl manifold* (or a *pl n-manifold*) if each $x \in M$ has a neighbourhood in M which is pl homeomorphic to an open set in $\mathbb{R}^n_+$. The set ∂M consisting of points corresponding to $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n_+$ is called the *boundary* of M. If $\partial M = \emptyset$ then M is called a *pl manifold without boundary*. A compact pl manifold without boundary is also called a *closed pl manifold*.

[Well-defineness of ∂M follows from the following: *Let U and V be open in $\mathbb{R}^n_+$ and $f : U \to V$ be a pl homeomorphism. If $x \in \mathbb{R}^{n-1} \times \{0\} \cap U$ then $f(x) \in \mathbb{R}^{n-1} \times \{0\}$.*]

Clearly, if M and N are pl manifolds of dimensions m and n respectively then $M \times N$ is a pl $(m+n)$-manifold and $\partial(M \times N) = (M \times \partial N) \cup (\partial M \times N)$.

Let $T = ([-2, 2] \times [-2, 2] \setminus (-1, 1) \times (-1, 1)) \times [-1, 1] \subseteq \mathbb{R}^3$. Clearly, $\mathring{T}$ (the interior of T) $= ((-2, 2) \times (-2, 2) \setminus [-1, 1] \times [-1, 1]) \times (-1, 1)$. Then T and $T \setminus \mathring{T}$ are pl manifolds and $\partial T = T \setminus \mathring{T}$. Observe that ∂T is homeomorphic to the torus $S^1 \times S^1$.

PL Balls and PL Spheres

A polyhedron M is called a *pl n-ball* if it is pl homeomorphic to I^n. A polyhedron M is called a *pl n-sphere* if it is pl homeomorphic to ∂I^{n+1}. So, a pl $(n+1)$-ball is a pl $(n+1)$-manifold having a pl n-sphere as boundary. If C is an n-polytope then C is a pl n-ball with boundary $\dot{C}$.

Simplicial Complex

A finite collection K of simplices in some $\mathbb{R}^n$ is called a *simplicial complex* if (i) $\alpha \in K$, $\beta < \alpha$ imply $\beta \in K$ and (ii) $\sigma, \gamma \in K$ imply $\sigma \cap \gamma < \sigma, \gamma$.

For $i = 0, 1$, the i-simplices in a simplicial complex K are also called the *vertices* and *edges* of K, respectively. The set of vertices is called the *vertex set* of K and is denoted by $V(K)$. For a simplicial complex K, the maximum of k such that K has a k-simplex is called the *dimension* of K.

A simplex σ in a simplicial complex K is called *maximal* if $\sigma < \gamma \in K$ implies $\gamma = \sigma$. Clearly, a simplicial complex is uniquely determined by its maximal simplices.

A simplicial complex is called *pure* if all the maximal simplices are of same dimension. A maximal simplex in a pure simplicial complex is also called a *facet*.

A simplicial complex of dimension ≤ 1 is called a *graph*.

Geometric Carrier
If K is a simplicial complex then $|K| := \bigcup_{\sigma\in K}\sigma$ is a compact polyhedron and is called the *geometric carrier* of K or the *underlying polyhedron* corresponding to K.

Subcomplex
If K and L are simplicial complexes and $L \subseteq K$ then L is called a *subcomplex* of K. We consider $\emptyset$ to be a subcomplex of every simplicial complex.

For a simplicial complex K, if $U \subseteq V(K)$ then $K[U]$ denotes *induced subcomplex* of K on the vertex-set U (i.e., $K[U] = \{\sigma \in K :$ vertices of σ are in $U\}$).

Simplicial Maps
Let K and L be two simplicial complexes. A map $f : |K| \to |L|$ is called *simplicial* if $f|_\sigma$ is linear and $f(\sigma)$ is a simplex of L for each $\sigma \in K$.

Abstract Simplicial Maps
Let K and L be two simplicial complexes. A map $\varphi : V(K) \to V(L)$ is called an *abstract simplicial map* if $\langle A\rangle$ is a simplex in K implies $\langle\varphi(A)\rangle$ is a simplex in L, for every $A \subseteq V(K)$.

Let $\varphi : K \to L$ be an abstract simplicial map. If $x \in |K|$ then there exists a unique simplex $\sigma \in K$ such that $x \in \mathring{\sigma}$. Let $\sigma = v_0v_1\cdots v_k$ and $x = t_0v_0 + \cdots + t_kv_k$ where $t_i \in [0,1]$ for $0 \le i \le k$ and $t_0 + \cdots + t_k = 1$. Define $|\varphi|(x) = t_0\varphi(v_0) + \cdots + t_k\varphi(v_k)$. This defines a simplicial map $|\varphi| : |K| \to |L|$.

Isomorphisms
A bijection $\varphi : V(K) \to V(L)$ is called an *isomorphism* if both φ and φ^{-1} are abstract simplicial maps. Two simplicial complexes K and L are called *isomorphic* (denoted by $K \cong L$) if such an isomorphism exists. We identify two simplicial complexes if they are isomorphic. Clearly, if φ is an isomorphism then $|\varphi|$ is a pl homeomorphism.

An isomorphism from a simplicial complex K to itself is called an *automorphism* of K. All the automorphisms of K form a group under composition, which is denoted by $\mathrm{Aut}(K)$.

If $\varphi : V(K) \to V(L)$ is an isomorphism then define $\Phi : K \to L$ as $\Phi(v_0v_1\cdots v_k) = \langle\varphi(\{v_0, v_1, \ldots, v_k\})\rangle$. Clearly, Φ is a bijection and $\alpha < \beta$ if and only if $\Phi(\alpha) < \Phi(\beta)$. Conversely, any such bijection $\Phi : K \to L$ defines an isomorphism $\Phi|_{V(K)}$.

f-vector and Euler characteristic
If $f_i(K)$ denote the number of i-simplices ($0 \le i \le d$) in a d-dimensional simplicial complex K then $(f_0(K), f_1(K), \ldots, f_d(K))$ is called the *f-vector* of K and the number $\chi(K) := \sum_{i=0}^{d}(-1)^i f_i(K)$ is called the *Euler characteristic* of K. (Formally we take $f_{-1} := 1$.)

A simplicial complex K is called *k-neighbourly* if the convex hull of any set of k vertices is a $(k-1)$-simplex of K (i.e., $f_{k-1}(X) = \binom{f_0(K)}{k}$).

Face polynomial and h-vector
The *face polynomial* of a d-dimensional simplicial complex K is

$$f_K(x) := \sum_{i=-1}^{d} f_i(K)x^{d-i}.$$

The polynomial $h_K(x) := f_K(x-1)$ is called the *h-polynomial* of K. The *h-vector* of K is $(h_0(K), \ldots, h_{d+1}(K))$, where $h_K(x) = \sum_{j=0}^{d+1} h_j x^{d+1-j}$. Equivalently,

$$h_j(K) = \sum_{i=-1}^{j-1}(-1)^{j-i-1}\binom{d-i}{j-i-1}f_i(K)$$

for $0 \le j \le d+1$. Observe that $h_{d+1}(K) = (-1)^{d+1}(1-\chi(K))$ and, for $0 \le i \le d$,

$$f_{i-1}(K) = \sum_{j=0}^{i}\binom{d+1-j}{i-j}h_j(K).$$

Join of Complexes
Two simplicial complexes K and L (in $\mathbb{R}^N$) are called *independent* if $\alpha\beta$ is an $(m+n+1)$-simplex for each m-simplex α in K and each n-simplex β in L for $m, n \ge 0$. If K and L are independent then we define $K * L = K \cup L \cup \{\alpha\beta : \alpha \in K, \beta \in L\}$. The simplicial complex $K * L$ is called the (simplicial) *join* of K and L.

If K and L are two simplicial complexes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then we can define their join in a bigger space. More explicitly, let $i_1 : \mathbb{R}^n \to \mathbb{R}^{n+m+1}$, $i_2 : \mathbb{R}^m \to \mathbb{R}^{n+m+1}$ be the maps given by $i_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ and $i_2(x_1, \ldots, x_m) = (0, \ldots, 0, x_1, \ldots, x_m, 1)$. Let $K_1 := \{i_1(\alpha) : \alpha \in K\}$ and $L_1 := \{i_2(\beta) : \beta \in L\}$. Then $K_1 \cong K$, $L_1 \cong L$ and K_1 and L_1 are independent simplicial complexes in $\mathbb{R}^{n+m+1}$. We define $K * L = K_1 * L_1$.

Stars, Links and Degrees
Let K be a simplicial complex and $\gamma \in K$. Let $\mathrm{st}_K(\gamma)$ be the subcomplex of K whose maximal simplices are those maximal simplices of K which contain γ as a face. This subcomplex is called the *star* of γ in K.

Let K be a simplicial complex and $\gamma \in K$. Let $\mathrm{lk}_K(\gamma) := \{\beta : \beta \cap \gamma = \emptyset, \beta\gamma \in K\}$. Then, $\mathrm{lk}_K(\gamma)$ is a subcomplex of K and is called the *link* of γ in K. The number of vertices in the link of γ in K is called the *degree* of γ and is denoted by $\deg_K(\gamma)$.

If σ is a simplex in $\mathbb{R}^n$ then $\mathrm{Cl}(\sigma) := \{\beta : \beta < \sigma\}$ and $\mathrm{Bd}(\sigma) := \{\beta : \beta < \sigma, \beta \neq \sigma\}$ are simplicial complexes. Clearly, $|\mathrm{Cl}(\sigma)| = \sigma$ and $|\mathrm{Bd}(\sigma)| = \overset{\bullet}{\sigma}$.

If α is a simplex in a simplicial complex K then $\mathrm{Cl}(\sigma)$ and $\mathrm{lk}_K(\alpha)$ are independent and $\mathrm{st}_K(\alpha)$ is the join of $\mathrm{Cl}(\alpha)$ and $\mathrm{lk}_K(\alpha)$.

Subdivisions and Combinatorially Equivalent Complexes

A simplicial complex L is called a *subdivision* of a simplicial complex K (denoted by $L \lhd K$) if each simplex in L is contained in a simplex in K and $|L| = |K|$. Two simplicial complexes K and L are called *combinatorially equivalent* (denoted by $K \approx L$) if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $K' \cong L'$. So (by Proposition 1.5), $K \approx L$ if and only if $|K|$ and $|L|$ are pl homeomorphic. Clearly, '$\approx$' is an equivalence relations.

For $\gamma \in K$ and $a \in \mathring{\gamma}$, consider the simplicial complex (on the vertex-set $V(K) \cup \{a\}$) $K' = \{\delta \in K : \gamma \not< \delta\} \cup \{a, a\alpha : \alpha < \delta$ where $\gamma < \delta \in K$ and $\sigma \neq \alpha \neq \gamma, \}$. Then K' is a subdivision of K and is called the subdivision obtained from K by *starring* at a (or *starring the vertex a in* γ). We also say that K is obtained from K' by *collapsing* the vertex a.

Stellar Subdivisions

A simplicial complex K_1 is called a *stellar* subdivision of K if K_1 is obtained from K by starring (successively) at finitely many points. Two complexes K and L are called *stellar equivalent* if they have isomorphic stellar subdivisions.

Let $A_1, \dots, A_n$ be all the simplices of a simplicial complex K of dimension ≥ 1 such that $\dim(A_1) \geq \dim(A_2) \geq \cdots \geq \dim(A_n)$. Choose $a_i \in \mathring{A}_i$ for $1 \leq i \leq n$. Let $K^{(1)}$ be the stellar subdivision of K obtained by starring at $a_1, \dots, a_n$ successively. Then $K^{(1)}$ is called a *first derived subdivision* of K. For $r \geq 2$, a *r-th derived subdivision* of K is defined inductively by $K^{(r)} = (K^{(r-1)})^{(1)}$. If $a_i = \hat{A}_i$ (the barycentre of A_i) for each $A_i \in K$ then the first derived subdivision is called the *first barycentric subdivision* of K. Similarly, we can define the *r-th barycentric subdivision*.

Observe that S_2 has two vertices of degree 5 but S_1 has no degree 5 vertex. So, $S_1 \not\cong S_2$. Now, S_3 is obtained from S_1 by starring at 7 (in the edge 36) and is obtained from S_2 by starring at 7 (in the edge 12). Thus, $S_1 \approx S_2$ ($\approx S_3$).

Bistellar Moves

Let K be a d-dimensional ($d \geq 2$) pure simplicial complex in $\mathbb{R}^N$. Let A be an $(d-k)$-simplex in K such that $\mathrm{lk}_K(A) = \mathrm{Bd}(B)$ for some k-simplex B which is not in K. Let $C = \langle A \cup B \rangle$. If $C \cap |K| = A\overset{\bullet}{B}$ then consider the simplicial complex $L = (K \setminus \{D : A < D \in K\}) \cup \{FB : F < A, F \neq A\}$ (i.e., $L = (K \setminus \mathrm{Cl}(A) * \mathrm{Bd}(B)) \cup (\mathrm{Bd}(A) * \mathrm{Cl}(B))$). We say that L is obtained from K by the *bistellar k-move* $\kappa(A, B)$.

[For $0 < k < d$, C is a polytope of dimension d or $d+1$ with $d+2$ vertices. If $\dim(C) = d$ then $C = A\overset{\bullet}{B} = \overset{\bullet}{A}B$, $\overset{\bullet}{C} = \overset{\bullet}{A}\overset{\bullet}{B}$ and $|L| = |K|$. If $\dim(C) = d+1$ then $C = AB$, $\overset{\bullet}{C} = A\overset{\bullet}{B} \cup \overset{\bullet}{A}B$ and $|L|$ is pl homeomorphic to $|K|$.]

If $k = d$ we take $B = b \in \mathring{A}$ then $\kappa(A, b)$ is equivalent to starring at b. If $k = 0$ then $\kappa(u, B)$ is equivalent to collapsing the vertex u. If $0 < k < d$ then $\kappa(A, B)$ is called a *proper* bistellar move.

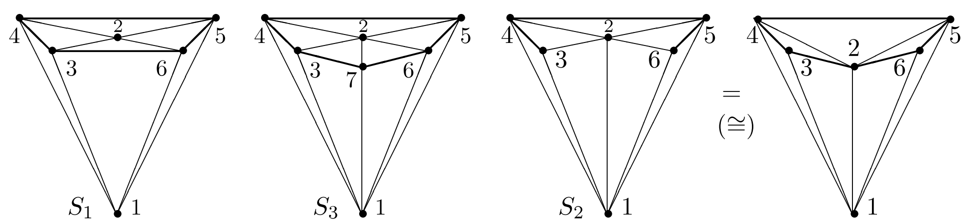
Observe that S_2 (defined above) is obtained from S_1 by the bistellar 1-move $\kappa(36, 12)$.

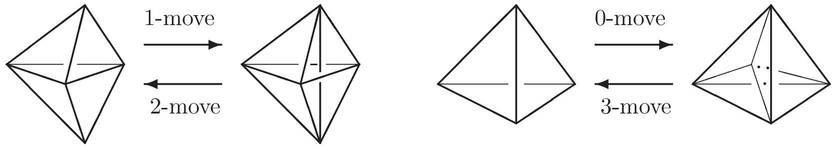
Two pure simplicial complexes M and N are called *bistellar equivalent* (or N is obtained from M by *bistellar flips*) if there exists a finite sequence $M_1, \dots, M_n$ of pure simplicial complexes such that $M_1 = M$, $M_n = N$ and M_{i+1} is obtained from M_i by a bistellar move for $1 \leq i \leq n-1$.

Triangulations

A *triangulation* of a polyhedron P is a pair (K, t), where K is a simplicial complex and $t : |K| \to P$ is a pl homeomorphism. Moreover, if t is linear on each simplex then (K, t) is called a *linear triangulation*. We identify two triangulations of a polyhedron P if they differ by an isomorphism (i.e., we identify (K_1, t_1) and (K_2, t_2) if there is an isomorphism $i : K_1 \to K_2$ such that $t_2 \circ |i| = t_1$). If (K, t) is a triangulation of P and $K' \lhd K$ then (K', t) is called a *subdivision* of (K, t).

For a simplicial complex K, if $|K|$ is homeomorphic to a topological space X then we say that K is a *triangulation* of X.





Bistellar moves in dimension 3

Combinatorial Balls
For $d \geq 0$, let Δ^d be the d-simplex $\{(x_1, x_2, \dots, x_{d+1}) : x_i \geq 0 \text{ for } 1 \leq i \leq d+1, \sum_{i=1}^{d+1} x_i \leq 1\}$ in $\mathbb{R}^{d+1}$ with vertices $v_1 = (1, 0, \dots, 0), \dots, v_{d+1} = (0, \dots, 0, 1)$. Then Δ^d is pl homeomorphic to the pl ball I^d. Let $\mathrm{Cl}(\Delta^d)$ denote the simplicial complex whose simplices are all the faces of Δ^d. Then $|\mathrm{Cl}\Delta^d| = \Delta^d$. The simplicial complex $\mathrm{Cl}(\Delta^d)$ is called the *standard d-ball*. A finite simplicial complex K is called a *combinatorial d-ball*, if $|K|$ is pl homeomorphic to Δ^d (i.e., $K \approx \mathrm{Cl}(\Delta^d)$).

Combinatorial Spheres
Let $\mathrm{Bd}(\Delta^{d+1})$ denote the simplicial complex $\mathrm{Cl}(\Delta^{d+1}) \setminus \{\Delta^{d+1}\}$ $(d \geq 0)$. Then $|\mathrm{Bd}(\Delta^{d+1})|$ is homeomorphic to the sphere S^d. ($|\mathrm{Bd}(\Delta^{d+1})| = \dot{\Delta}^{d+1}$ is pl homeomorphic to the pl sphere ∂I^{d+1}.) The simplicial complex $\mathrm{Bd}(\Delta^{d+1})$ is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ (or simply by S^d_{d+2}), where $V = \{v_1, \dots, v_{d+2}\}$ is the vertex-set of Δ^{d+1}.

A simplicial complex K is called a *combinatorial d-sphere*, if $|K|$ is pl homeomorphic to $|S^d_{d+2}|$ (i.e., by Proposition 1.5, $K \approx S^d_{d+2}$).

If a combinatorial d-sphere is k-neighbourly then, by Corollary 4.10, $k \leq \lfloor \frac{d+1}{2} \rfloor$. Thus, a $\lfloor \frac{d+1}{2} \rfloor$-neighbourly combinatorial d-sphere is called *neighbourly*.

Polytopal Spheres
For $d \geq 0$, let P be a simplicial $(d+1)$-polytope in $\mathbb{R}^{d+1}$. Then the set of proper faces of P form a combinatorial d-sphere and is called the *boundary complex* of the polytope P.

A combinatorial d-sphere S is called a *polytopal sphere* if it is isomorphic to the boundary complex of a simplicial $(d+1)$-polytope.

Stacked Spheres
A combinatorial d-sphere S is called a *stacked* sphere if there is a sequence $S_1, \dots, S_k$ of combinatorial d-spheres such that $S_1 = S^d_{d+2}$, $S_k = S$ and S_{j+1} is obtained from S_j by starring a vertex on a facet of S_j for $1 \leq j \leq k-1$.

It follows from Proposition 1.2 that a *stacked d-sphere* is isomorphic to the boundary complex of a stacked $(d+1)$-polytope. Clearly, the face-vector of an n-vertex stacked d-sphere S is given by

$$\begin{aligned} f_k(S) &= \binom{d+2}{k+1} + (n-d-2)\binom{d+1}{k} \\ &= \binom{d+1}{k} n - \binom{d+2}{k+1} k \ \text{ for } 1 \leq k < d, \\ f_d(S) &= (d+2) + (n-d-2)d \\ &= dn - (d+2)(d-1). \end{aligned} \tag{1}$$

Combinatorial Manifolds
A simplicial complex K is called a *combinatorial d-manifold* if $\mathrm{lk}_K(v)$ is a combinatorial $(d-1)$-sphere (i.e., $\approx S^{d-1}_{d+1}$) for each vertex v in K. Clearly, a two dimensional complex K is a combinatorial 2-manifold if the link of each vertex is a cycle. (A *cycle* is a connected finite graph in which the degree of each vertex is 2. A cycle with n vertices is called an n-cycle and is denoted by S^1_n. An n-cycle with edges $v_1v_2, \dots, v_{n-1}v_n, v_nv_1$ is also denoted by $S^1_n(v_1, \dots, v_n)$.)

Since the link of a vertex in S^k_{k+2} is a standard $(k-1)$-sphere, it follows that (i) S^k_{k+2} (and hence a combinatorial k-sphere) is a combinatorial k-manifold and (ii) if σ is an i-simplex in a combinatorial d-manifold K then $\mathrm{lk}_K(\sigma)$ is a combinatorial $(d-i-1)$-sphere for $0 \leq i \leq d-1$.

A simplicial complex K is called a *combinatorial d-manifold with boundary* if $\mathrm{lk}_K(v)$ is a combinatorial $(d-1)$-sphere or combinatorial $(d-1)$-ball for each vertex v in K and there exists a vertex u whose link is a combinatorial $(d-1)$-ball.

Triangulated Manifolds
If the geometric carrier $|K|$ of a simplicial complex K is a closed topological d-manifold then K is called a *triangulated d-manifold*. So, a combinatorial manifold is triangulated manifold and for $d \leq 3$, a triangulated d-manifold is a combinatorial d-manifold.

Homology Manifolds
A d-dimensional simplicial complex K is called a *homology manifold* if for any $x \in |K|$ and $i < d$, $H_i(|K|, |K| \setminus \{x\}; \mathbb{Z}) = 0$ and $H_d(|K|, |K| \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$. So, a triangulated manifold is a homology manifold.

Eulerian Complexes

A d-dimensional simplicial complex K is called an *Eulerian Complex* if $\chi(\mathrm{lk}_K(\sigma)) = 1 + (-1)^{d-i-1}$ for any i-simplex σ, $0 \leq i < d$. So, a triangulation of a sphere is an Eulerian Complex.

PL Structures on Manifolds

A compact topological manifold M is called *triangulable* if it is homeomorphic to the geometric carrier of a simplicial complex K. Moreover, if K is a combinatorial manifold (i.e., by Proposition 1.6, $|K|$ is a pl manifold) then we say K is a *combinatorial triangulation* of M.

If a combinatorial manifold K triangulates M, then the combinatorial equivalence class $\mathcal{K}$ of combinatorial manifolds containing K is called a *combinatorial structure* or *pl structure* of M.

Pseudomanifolds

A pure d-dimensional simplicial complex K is called a d-dimensional *pseudomanifold* (or *d-pseudomanifold*) if (i) each $(d-1)$-simplex is a face of exactly two facets of K and (ii) for any pair σ_1, σ_2 of facets of K, there exists a sequence $\tau_1, \dots, \tau_n$ of facets of X, such that $\tau_1 = \sigma_1$, $\tau_n = \sigma_2$ and $\tau_i \cap \tau_{i+1}$ is a $(d-1)$-simplex of K for $1 \leq i \leq n-1$. By convention, S^0_2 is the only 0-pseudomanifold.

Normal Pseudomanifolds

A d-pseudomanifold is said to be a *normal pseudomanifold* if the links of all the simplices of dimension $\leq d-2$ are connected. Clearly, the 1-dimensional normal pseudomanifolds are the cycles and the 2-dimensional normal pseudomanifolds are just the connected combinatorial 2-manifolds. But, normal pseudomanifolds of dimension d form a broader class than connected combinatorial d-manifolds for $d \geq 3$. In fact, any connected triangulated manifold is a normal pseudomanifold.

Irreducible Pseudomanifolds

For $n \geq d+3$, an n-vertex d-pseudomanifold M is called *irreducible* if M can not be written as $S^c_{c+2} * N$ for some pseudomanifold N and $c \geq 0$. M is called completely reducible if it is the join of one or more standard spheres. By Theorem 5.28, $(d+3)$-vertex d-pseudomanifolds are completely reducible.

One-Point Suspension

Let K be an n vertex d-dimensional pseudomanifold in $\mathbb{R}^m \equiv \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^{m+1}$ and $u \in V(K)$. Let $v = (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$. Consider the $(d+1)$-dimensional pseudomanifold $\Sigma_u K$ whose facet-set is $\{u\alpha : \alpha \text{ a facet of } K \text{ and } u \notin \alpha\} \cup \{v\beta : \beta \text{ a facet of } K\}$. Observe that uv is an edge of $\Sigma_u K$ and if w is an interior point in uv then the simplicial complex obtained from $\Sigma_u K$ by starring at w is isomorphic to $K * S^0_2$. So, $|K|$ is homeomorphic to the suspension of $|K|$. The pseudomanifold $\Sigma_u K$ is called the *one-point suspension* of K (see [7] for more).

Complementarity

A simplicial complex K is said to satisfy *complementarity* if $\emptyset \neq U \subseteq V(K)$ and $U \neq V(K)$ imply exactly one of $\langle U \rangle$, $\langle V(K) \setminus U \rangle$ is a simplex of K. The simplicial complexes $\mathbb{R}P^2_6$ and $\mathbb{C}P^2_9$ (in Examples 3.2 and 3.10 respectively) satisfy complementarity.

Abstract Simplicial Complex

An *abstract simplicial complex* is a collection of non-empty finite sets (sets of *vertices*) such that every non-empty subset of a member is also a member. For $i \geq 0$, a member of size $i+1$ is called an *i-simplex* of the complex. For an abstract simplicial complex K, $V(X)$ denotes the vertex-set of K. An abstract simplicial complex is called *pure* if all the maximal simplices contain same number of vertices. For an abstract simplicial complex X, $\mathrm{EG}(X)$ denote edge-graph of X (i.e., $\mathrm{EG}(X)$ consists of vertices and edges of X). If $\mathrm{EG}(X)$ is connected then we say that X is *connected*.

If K is a simplicial complex then $K_a := \{\sigma : \sigma \subseteq V(K),\ \langle\sigma\rangle \in K\}$ is an abstract simplicial complex and is called the *abstract simplicial complex corresponding to K*.

Let X be an abstract simplicial complex. A simplicial complex K is called a *geometric realization* of X (and is denoted by X_{gr}) if K_a is isomorphic to X. Clearly, two geometric realizations of a finite abstract simplicial complex are isomorphic.

Let X be a finite abstract simplicial complex with $V(X) = \{v_1, \dots, v_n\}$. Let A be an $(n-1)$-simplex with vertices $a_1, \dots, a_n$ in $\mathbb{R}^{n-1}$. Let $K = \{a_{i_1} \cdots a_{i_k} : \{v_{i_1}, \dots, v_{i_k}\} \text{ is a simplex of } X\}$. Then K is a subcomplex of the simplicial complex $\mathrm{Cl}(A)$. Clearly, K is a geometric realization of X. So, geometric realizations exist for finite complexes. For infinite case see [68].

Isomorphism and Automorphism

An *isomorphism* between two abstract simplicial complexes X and Y is bijection $\varphi: V(X) \to V(Y)$ such that $A \in X$ if and only if $\varphi(A) \in Y$. Two abstract simplicial complexes are called *isomorphic* if such an isomorphism exists. We identify two isomorphic complexes. An isomorphism from an abstract simplicial complex X to itself is called an *automorphism*. All the automorphisms of X form a group under composition, which is denoted by $\mathrm{Aut}(X)$.

Quotient Complex and Proper Action

Let X be an abstract simplicial complex and G be a group of automorphism (i.e., G is subgroup of $\mathrm{Aut}(X)$). Let $\eta : V(X) \to V(X)/G$ be the natural projection. Let X/G denote the abstract simplicial complex $\{\eta(A) : A \in X\}$. This complex is called the *quotient complex.*

Let X be a connected abstract simplicial complex. For $u, v \in V(X)$, let $d_X(u,v)$ denote the length of a shortest path from u to v in $\mathrm{EG}(X)$. (Then d_X is a metric on $V(X)$.) Let G be a group of automorphism of X. It is easy to see that if X is a pure d-dimensional abstract simplicial complex and $d_X(u, g(u)) \geq 2$ for all $u \in V(X)$ and $1 \neq g \in G$ then X/G is also a pure d-dimensional abstract simplicial complex. We say that G acts *properly* on X if $d_X(u, g(u)) \geq 3$ for all $u \in V(X)$ and $1 \neq g \in G$.

Proposition 1.1. *Let K be a combinatorial d-manifold. Let G be a group of automorphism of K_a. Let K/G denote the geometric realization $(K_a/G)_{gr}$ of the quotient K_a/G. If G acts properly on K_a then K/G is also a combinatorial d-manifold.*

Proof. Let $\eta : V(K_a) \to V(K_a/G)$ be the natural projection. Then η induces an abstract simplicial map η_{gr} from K to K/G. For $v \in V(K_a)$, let $[v] = \eta(v)$. Since the action is proper, it follows that K/G is a pure d-dimensional simplicial complex. Assume that $V(K_a) = V(K)$ and $V(K/G) = V(K_a/G) = \{[v] : v \in K_a\}$.

Let $[u]$ be a vertex of K/G. Let v and w be two vertices in $\mathrm{lk}_K(u)$. Since vu and uw are edges in K, it follows that the length of the shortest path in K between v and w is at most 2. Therefore (since the action of G is proper), $[v] \neq [w]$. This implies that $\eta_{gr}|_{\mathrm{lk}_K(u)} : \mathrm{lk}_K(u) \to \mathrm{lk}_{K/G}([u])$ is injective and hence an isomorphism. Thus the link of each vertex in K/G is a combinatorial sphere. This proves the result. $\square$

Proposition 1.2. *Let M be the boundary complex of a simplicial $(d+1)$-polytope P. Let N be the combinatorial d-sphere obtained from M by starring a vertex in a facet σ of M. Then N is isomorphic to the boundary of the polytope Q which is obtained from P by attaching a $(d+1)$-simplex along the d-face σ of P.*

Proof. Assume that P is in $\mathbb{R}^{d+1}$. Let $\sigma = v_0 \cdots v_{d+1}$ and $a = \frac{1}{d+2}(v_0 + \cdots + v_{d+1}) \in \overset{\circ}{\sigma}$. We may assume that N is obtained from M by starring at a. Let L be the closed half line through a and perpendicular to σ such that $L \cap P = \{a\}$. Then there exists $\varepsilon > 0$ such that $x \in L \setminus \{a\}$ and distance between a and x $\leq \varepsilon$ implies $P \cup (x\sigma)$ is convex. ($x\sigma$ denotes the join of x and σ.) Fix a point v on L at a distance ε from a. Let $Q = P \cup (v\sigma)$. Then Q is a $(d+1)$-polytope. Let φ be the map from N to the boundary of Q given by $\varphi(u) = u$ if u is a vertex of P and $\varphi(a) = v$. It is easy to see that φ is an isomorphism. $\square$

In [7], we have shown the following:

Proposition 1.3. *Let $\Sigma_u K$ be the one-point suspension of a pseudomanifold K. The pseudomanifold $\Sigma_u K$ is a polytopal sphere if and only if K is so.*

Here we present some basic results in pl-topology. See [65] for proofs.

Proposition 1.4. *Any compact polyhedron is the geometric carrier of some simplicial complex.*

Proposition 1.5. *Let K and L be two simplicial complexes. If $f : |K| \to |L|$ is pl, then there are simplicial subdivisions $K' \lhd K$ and $L' \lhd L$ such that $f : |K'| \to |L'|$ is simplicial.*

Proposition 1.6. *Suppose K is a simplicial complex then $|K|$ is a pl n-manifold if and only if* $\mathrm{lk}_K(v) \approx \mathrm{Bd}(\Delta_n)$ *or* $\mathrm{Cl}(\Delta_{n-1})$ *for each $v \in V(K)$.*

Proposition 1.7. *For $p,\ q \geq 1$, let B^p, B^q be combinatorial balls (of dimensions p and q respectively) and S^{p-1}, S^{q-1} be combinatorial spheres (of dimensions $p-1$ and $q-1$ respectively). Then* (i) *$B^p * B^q$ is a combinatorial $(p+q+1)$-ball,* (ii) *$S^{p-1} * B^q$ is a combinatorial $(p+q)$-ball and* (iii) *$S^{p-1} * S^{q-1}$ is a combinatorial $(p+q-1)$-sphere.*

2. A Brief History of Triangulations

- It was shown by Rado in 1924 that all 2-manifolds are triangulable. Since the link of a vertex in a triangulated 2-manifold is a cycle, 2-manifolds have pl structures.
- In 1935, Cairns proved that each closed smooth manifold is triangulable.
- In 1940, Whitehead proved that each closed smooth manifold has a pl structure.
- In 1952, Moise showed that all 3-manifolds are triangulable. Again, the link of a vertex in a triangulated 3-manifold is a triangulation of the 2-sphere and all triangulations of the 2-sphere are combinatorial 2-spheres. So, 3-manifolds have pl structures. Moise also showed that each 3-manifold admits a unique pl structure.
- In 1960, Kervaire gave the example of a pl 10-manifold which is not smoothable.

- In 1961, Eells and Kuiper (independently, Tamura) gave examples of 8 dimensional pl manifolds which are not smoothable.

- In 1964, Lojaciewitz proved that each real algebraic variety is triangulable.

- In 1967, Kuiper obtained algebraic equations for all non-smoothable pl 8-manifolds.

- It is shown by Munkres in 1967 that there is an one to one correspondence between the set of smooth homotopy m-spheres and the set of pl homotopy m-spheres for $3 \leq m \leq 4$. So, by Freedman's classification of 4-manifolds, there is an one to one correspondence between the set of pl structures on S^4 and the set of smooth structures on S^4.

- In 1969, Kirby and Siebenmann (independently, Lashof and Rosenberg) proved that (i) there is exactly one well defined obstruction in $H^4(M;\mathbb{Z}_2)$ to imposing a pl structure on a closed topological m-manifold M, $m \geq 5$ and (ii) given one pl structure, there is a bijection between the class of distinct pl structures and $H^3(M,\mathbb{Z}_2)$. Therefore, S^m has a unique pl structure for $m \geq 5$. [For smooth structures on S^m we know the following: In 1963, Milnor and Kervaire proved that the set $\Theta_m^{\rm DIFF}$ of smooth (homotopy) m-spheres is a finite abelian group under the connected sum operation for $m \geq 5$. For $m = 5, 6, 7, 8, 9, 10, 11$, $\Theta_m^{\rm DIFF} = 0, 0, \mathbb{Z}_{28}, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_6, \mathbb{Z}_{992}$ respectively.]

- In 1970, Siebenmann showed that for each $n \geq 5$ there exists a closed manifold M^n of dimension n which does not admit a pl structure.

- In 1970, Siebenmann gave the example of a triangulable 5-manifold which does not admit any pl structure.

- In 1974, Hirsch and Mazur showed that if the dimension of a closed manifold M with a pl structure is ≤ 7 then M is smoothable. So, for $n \leq 7$, a n-dimensional closed topological manifold is smoothable if and only if it has a pl structure.

- In 1974, Hirsch showed that if $M \times N$ is smoothable, where M and N are closed pl manifolds, then both M and N are smoothable. So, if M is an 8-dimensional non-smoothable pl manifold (by Eells and Kuiper such M exists) then $M \times S^d$ is a non-smoothable pl manifold of dimension $8 + d$ for all $d \geq 1$.

- In 1976, Galewski and Stern (independently, Matumoto) defined an obstruction element $\tau \in H^5(M;\rho)$ such that the closed topological n-manifold M^n, $n \geq 5$, is triangulable if and only if $\tau = 0$. Then they proved that all closed topological manifolds of dimension ≥ 5 are triangulable if and only if there is a homology 3-sphere Σ such that (i) Σ has Rohlin (or Rochlin) invariant 1 (i.e., bounding a parallelizable 4-manifold of index 8) (ii)* the $(n-3)$-fold suspension of Σ is homeomorphic to S^n and (iii) $\Sigma \# \Sigma$ bounds a smooth homology 4-disc. They also proved that each simply connected closed topological 6-manifold is triangulable.

- In 1977, Akbulut and King proved that each pl manifold of dimension ≤ 10 is homeomorphic to a real algebraic variety.

- In 1982, Freedman showed that there are closed 4-manifolds which are not smoothable. So, by Hirsch and Mazur's result, there are closed 4-manifolds which have no pl structures.

- In 1985, Casson showed that there exists a closed 4-manifold which is not triangulable.

- For more, see [49], [66] and the following AMS Mathematical Review numbers : 2,73e, 14,72d, 22 #12536, 25: #2608, 25: #2612, 26 #5584, 26: #6978, 26 #6980, 31 #5209, 33 #6641, 35 #3671, 39 #3494, 39 #3500, 40 #895, 42 #6837, 54: #3711, 54 #8650, 54 #11335, 55 #13434, 80e: 57019, 80m: 57014, 81b: 57015, 81f: 57012, 84e: 57006.

3. Examples

- In this section, we present some combinatorial manifolds. Most of these are vertex-minimal triangulations. We will discuss about these in the next section.

- Since the facet-set of a pure simplicial complex determines the simplicial complex, we identify a pure simplicial complex with its facet-set in this section.

- Whenever we say that $v_1 \cdots v_k$ is a simplex then we mean that $v_1 \cdots v_k$ is the convex hull of k affinely independent points $v_1, \ldots, v_k$ in some $\mathbb{R}^N$.

*This condition is now superfluous because of Cannon's result mentioned in Example 3.19.

- In the examples below, $v_1 \cdots v_m$ and $u_1 \cdots u_n$ are two simplices in a simplicial complex K and $\{v_1, \ldots, v_m\} \cap \{u_1, \ldots, u_n\} = \emptyset$ mean we have taken the vertices of K in some $\mathbb{R}^N$ such that $v_1 \cdots v_m \cap u_1 \cdots u_n = \emptyset$. This is possible since K is finite. In fact, if K has r vertices then we can consider K in $\mathbb{R}^{r-1}$ by considering the vertices of K to be affinely independent.

Example 3.1. For $c_1, \ldots, c_n \geq 0$, $S^{c_1}_{c_1+2} * \cdots * S^{c_n}_{c_n+2}$ is a combinatorial $(c_1 + \cdots + c_n + n - 1)$-sphere on $c_1 + \cdots + c_n + 2n$ vertices.

Example 3.2. Two combinatorial 2-manifolds of positive Euler characteristics.

$$\begin{aligned} \mathcal{I} &= \{uu_iu_{i+1}, u_iu_{i+1}v_{i+3}, \\ &\quad v_iv_{i+1}u_{i+3}, vv_iv_{i+1} : 1 \leq i \leq 5\} \text{ and} \\ \mathbb{R}P^2_6 &= \{uu_iu_{i+1}, u_iu_{i+1}u_{i+3} : 1 \leq i \leq 5\}. \end{aligned}$$

Additions in the subscripts are modulo 5. The geometric carrier of $\mathcal{I}$ is the 2-sphere and it corresponds to the boundary of the Platonic solid icosahedron. The geometric carrier of $\mathbb{R}P^2_6$ is the real projective plane. The complex $\mathbb{R}P^2_6$ is called the hemi-icosahedron. Observe that $\mathbb{Z}_2$ $(= \{1, -1\})$ acts properly on the abstract simplicial complex $\mathcal{I}_a$ by $(-1)u = v$, $(-1)v = u$, $(-1)u_i = v_i$, $(-1)v_i = u_i$ and $\mathcal{I}/\mathbb{Z}_2 = \mathbb{R}P^2_6$.

Example 3.3. Two combinatorial 2-manifolds of Euler characteristic 0.

$$\begin{aligned} \mathcal{T} &= \{w_iw_{i+1}w_{i+3}, w_iw_{i+2}w_{i+3} : 1 \leq i \leq 7\} \text{ and} \\ \mathcal{K} &= \{u_1u_2v_1, u_1u_2v_2, u_1u_3v_1, u_1u_3v_3, u_1u_4v_2, \\ &\quad u_1u_4v_4, u_2u_3v_2, u_2u_3v_4, \\ &\quad u_2u_4v_1, u_2u_4v_3, u_3u_4v_3, u_3u_4v_4, u_1v_3v_4, \\ &\quad u_2v_3v_4, u_3v_1v_2, u_4v_1v_2\}. \end{aligned}$$

Additions in the subscripts are modulo 7 in $\mathcal{T}$. The geometric carrier of $\mathcal{T}$ is the torus and the geometric carrier of $\mathcal{K}$ is the Klein bottle.

Example 3.4. Two combinatorial 2-manifolds of negative Euler characteristics.

$$\begin{aligned} M &= \{u_{1+p}u_{4+p}u_{7+p}, u_{i+3p}u_{j+3p}u_{k+3p} : \\ &\quad (i,j,k) \in \{(1,2,5), (1,3,5), (1,3,4), \\ &\quad (1,8,9), (1,6,8), (1,2,6), (2,3,6)\}, \\ &\quad 0 \leq p \leq 2\} \text{ and} \\ N &= \{uu_iu_{i+1}, u_iu_{i+1}u_{i+4}, u_iu_{i+2}u_{i+4}, \\ &\quad u_iu_{i+3}u_{i+6} : 1 \leq i \leq 9\}. \end{aligned}$$

Additions in the subscripts are modulo 9. The geometric carrier of M is the non-orientable surface of Euler characteristic -3 and the geometric carrier of N is the non-orientable surface of Euler characteristic -5.

Example 3.5. Five 8-vertex combinatorial 3-spheres.

$$\begin{aligned} S^3_{8,35} &= \{1234, 1267, 1256, 1245, 2345, 2356, \\ &\quad 2367, 3467, 3456, 4567, 1238, 1278, \\ &\quad 2378, 1348, 3478, 1458, 4578, 1568, \\ &\quad 1678, 5678\}, \\ S^3_{8,36} &= \{1234, 1256, 1245, 1567, 2345, 2356, \\ &\quad 2367, 3467, 3456, 4567, 1268, 1678, \\ &\quad 2678, 1238, 2378, 1348, 3478, 1458, \\ &\quad 1578, 4578\}, \\ S^3_{8,37} &= \{1234, 1256, 1245, 1457, 2345, 2356, \\ &\quad 2367, 3467, 3456, 4567, 1568, 1578, \\ &\quad 5678, 1268, 2678, 1238, 2378, 1348, \\ &\quad 1478, 3478\}, \\ S^3_{8,38} &= \{1234, 1237, 1267, 1347, 1567, 2345, \\ &\quad 2367, 3467, 3456, 4567, 2358, 2368, \\ &\quad 3568, 1268, 1568, 1248, 2458, 1478, \\ &\quad 1578, 4578\} \text{ and} \\ S^3_{8,39} &= \kappa(46, 357)(S^3_{8,4}). \end{aligned}$$

First four of these combinatorial manifolds are 2-neighbourly and were found by Grünbaum and Sreedharan (in [34], these are denoted by P^8_{35}, P^8_{36}, P^8_{37} and $\mathcal{M}$ respectively). They showed that $S^3_{8,35}$, $S^3_{8,36}$, $S^3_{8,37}$ are polytopal spheres and $S^3_{8,38}$ is a non-polytopal sphere (known as the **Brückner-Grünbaum sphere**). The sphere $S^3_{8,39}$ (obtained from $S^3_{8,38}$ by the bistellar 2-move $\kappa(46, 357)$) is a non-polytopal sphere and found by Branette in [16].

Example 3.6. Consider the 11-vertex pure 3-dimensional simplicial complex $\mathbb{R}P^3_{11}$ (on the vertex-set $\{1, \ldots, 9, a, b\}$) whose maximal simplices are

$$\begin{aligned} &1237, 123b, 1269, 126b, 1279, 135a, 135b, 137a, \\ &1479, 147a, 1489, 148a, 1568, 156b, 158a, 1689, \\ &2348, 234b, 2378, 246a, 246b, 248a, 2578, 2579, \\ &258a, 259a, 269a, 3459, 345b, 3489, 359a, 3678, \\ &367a, 3689, 369a, 4567, 456b, 4579, 467a, 5678. \end{aligned}$$

This simplicial complex is a combinatorial 3-manifold and triangulates the 3-dimensional real projective space $\mathbb{R}P^3$. This was first constructed by Walkup in [71]. Theorem 5.14 shows that 11 is the minimal number of vertices required to triangulate $\mathbb{R}P^3$.

Example 3.7. Let L^3_{12} be the 12-vertex pure 3-dimensional simplicial complex (on the vertex set $\{1,\dots,9,a,b,c\}$) whose facets are

$$1234, 123a, 1249, 1256, 1259, 126b, 12ab, 1347, 1378, 138a, 1479, 156c, 1579, 157c, 16bc, 178c, 18ab, 18bc, 234c, 23ac, 2489, 248c, 2568, 2589, 2678, 267b, 278c, 27ab, 27ac, 3456, 345b, 3467, 34bc, 3568, 3589, 359b, 3678, 389a, 39ac, 39bc, 456a, 45ab, 4679, 469a, 489a, 48ab, 48bc, 56ac, 579b, 57ab, 57ac, 679b, 69ac, 69bc.$$

This complex is a combinatorial 3-manifold and triangulates the lens space $L(3,1)$ ([51]). Since $L(3,1)$ is a $\mathbb{Z}_2$-homology 3-sphere ($H_1(L(3,1),\mathbb{Z}) = \mathbb{Z}_3$, $H_2(L(3,1),\mathbb{Z}) = 0$), it follows from Theorem 5.51 that 12 is the least number of vertices required to triangulate $L(3,1)$.

Example 3.8. Consider the 15-vertex 3-dimensional pure simplicial complex

$$T^3_{15} = \{u_i u_{i+p} u_{i+p+q} u_{i+p+q+r} : \{p,q,r\} = \{1,2,4\}, 1 \le i \le 15\}.$$

(Additions in the subscripts are modulo 15.) This simplicial complex is a combinatorial 3-manifold and triangulates $S^1 \times S^1 \times S^1$ ([47]). A generalization of this is presented in Example 3.22.

Example 3.9. Let H^3_{16} be the 16-vertex pure 3-dimensional simplicial complex (on the vertex set $\{1,\dots,9,a,b,c,d,e,f,g\}$) whose facets are

$$1249, 124f, 126e, 126f, 129e, 134c, 134f, 137a, 137c, 13af, 149c, 156d, 156e, 158b, 158d, 15be, 16df, 178a, 178b, 17bc, 18ad, 19bc, 19be, 1adf, 235a, 235b, 237a, 237d, 23bd, 249d, 24bd, 24bf, 258b, 258c, 25ac, 26ac, 26ae, 26cf, 279d, 279e, 27ae, 28bf, 28cf, 345e, 345f, 34ce, 35af, 35be, 37cd, 3bde, 3cde, 4567, 456e, 457f, 467b, 46ab, 46ae, 47bf, 489c, 489d, 48ad, 48ae, 48ce, 4abd, 567d, 579d, 579f, 589c, 589d, 59ac, 59af, 67bc, 67cd, 6abc, 6cdf, 78ae, 78bf, 78ef, 79ef, 8cef, 9abc, 9abg, 9afg, 9beg, 9efg, abdg, adfg, bdeg, cdef, defg.$$

This simplicial complex is a combinatorial 3-manifold and was constructed by Björner and Lutz in [19]. The complex H^3_{16} has f-vector $(16, 106, 180, 90)$ and triangulates the Poincaré homology 3-sphere. It follows from Theorem 5.51 that at least 12 vertices are required to triangulate the Poincaré homology 3-sphere.

Example 3.10. Consider a 9-vertex abstract simplicial complex $\mathcal{X}$ as follows. The vertices of $\mathcal{X}$ are the points of the affine plane $\mathcal{P}$ over the 3-element field. Fix a set Π of three mutually parallel lines of $\mathcal{P}$ (i.e., Π is a parallel class of lines in $\mathcal{P}$). Let the lines in Π be γ_0, γ_1, γ_2, in a fixed cyclic orientation. The set of maximal simplices of $\mathcal{X}$ is as follows.

$$\{\gamma_{i+1} \cup \gamma_i \setminus \{x\} : x \in \gamma_i, 0 \le i \le 2\} \cup \{\alpha \cup \beta : \alpha \ne \beta \text{ two intersecting lines of } \mathcal{P} \text{ outside } \Pi\}.$$

(Addition in the suffix is modulo 3.) This gives $3 \times 3 + (9 \times 6)/2 = 9 + 27 = 36$ maximal simplices of $\mathcal{X}$. Then the geometric realization $\mathcal{X}_{gr}$ of $\mathcal{X}$ is a combinatorial 4-manifold. This $\mathcal{X}_{gr}$ triangulates the complex projective plane and is denoted by $\mathbb{C}P^2_9$ ([6,43,44,60]). This was first constructed by Kühnel and Banchoff in [45]. Check that, the link of any vertex in $\mathbb{C}P^2_9$ is isomorphic the Brückner-Grünbaum sphere $S^3_{8,38}$.

Example 3.11. Consider the 11-vertex 4-dimensional pure simplicial complex $S^{2,2}_{11}$ whose facets are

$$12346, 12347, 12369, 12379, 12458, 12459, 12468, 12479, 12568, 12569, 13467, 13567, 13569, 1357a, 1359b, 135ab, 1379a, 139ab, 1458a, 1459b, 145ab, 1467b, 1468a, 146ab, 1479b, 15678, 1578a, 1678b, 168ab, 178ab, 179ab, 23468, 23478, 2357a, 2357b, 235ab, 2368a, 2369a, 2378b, 2379a, 238ab, 24589, 24789, 2568b, 2569a, 256ab, 25789, 2578b, 2579a, 268ab, 3467b, 3468a, 3469a, 3469b, 3478b, 3489a, 3489b, 3567b, 3569b, 389ab, 4569a, 4569b, 456ab, 4589a, 4789b, 5678b, 5789a, 789ab.$$

The simplicial complex $S^{2,2}_{11}$ is a combinatorial 4-manifold and triangulates $S^2 \times S^2$. This was constructed by Lutz in [51]. Observe that, by Theorems 5.21 and 5.23, 11 is the minimum number of vertices required to triangulate $S^2 \times S^2$.

Example 3.12. Let G be the subgroup of S_{16} generated by $(2,7)(4,10)(5,6)(11,12)$ and $(1,2,3,4,5,10)(6,8,9)(11,12,13,14,15,16)$. Then $G \cong S_6$ and G acts on the set $V = \{u_1, \dots, u_{16}\}$ by $\alpha(u_i) = u_{\alpha(i)}$ for $\alpha \in G$. This action induces an action on the set of subsets of V, namely, $\alpha(U) = \{\alpha(a) : a \in U\}$ for $U \subseteq V$. Consider the 16-vertex abstract simplicial complex

$$K = \{\alpha(\{u_1, u_2, u_4, u_5, u_{11}\}), \alpha(\{u_1, u_2, u_4, u_{11}, u_{13}\}) : \alpha \in G\}.$$

Let $\mathbb{R}^4_{16} = K_{gr}$ be the geometric realization of K. Then $\mathbb{R}^4_{16}$ is a combinatorial 4-manifold with f-vector $(16, 120, 330, 375, 150)$ and triangulates the 4-dimensional real projective space $\mathbb{R}P^4$. This was constructed by Lutz in [51]. It follows, from Theorem 5.42, that 16 is the minimum number of vertices required to triangulate $\mathbb{R}P^4$.

Example 3.13. Let $\mathbb{F}_4 = \{0, 1, x, y\}$ be the field of order 4. Consider the space $\mathbb{F}_4 \oplus \mathbb{F}_4 =$

$$\Big\{ v_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} x \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} y \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_6 = \begin{pmatrix} x \\ 1 \end{pmatrix}, v_7 = \begin{pmatrix} y \\ 1 \end{pmatrix}, v_8 = \begin{pmatrix} 0 \\ x \end{pmatrix}, v_9 = \begin{pmatrix} 1 \\ x \end{pmatrix}, v_{10} = \begin{pmatrix} x \\ x \end{pmatrix}, v_{11} = \begin{pmatrix} y \\ x \end{pmatrix}, v_{12} = \begin{pmatrix} 0 \\ y \end{pmatrix}, v_{13} = \begin{pmatrix} 1 \\ y \end{pmatrix}, v_{14} = \begin{pmatrix} x \\ y \end{pmatrix}, v_{15} = \begin{pmatrix} y \\ y \end{pmatrix} \Big\}.$$

Let G be the group generated by all the translations in $\mathbb{F}_4 \oplus \mathbb{F}_4$ and the matrix $A = \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix}$. Then the order of G is 240 and G acts transitively on $\mathbb{F}_4 \oplus \mathbb{F}_4$. This action induces an action on the set of subsets of $\mathbb{F}_4 \oplus \mathbb{F}_4$, namely, $g(U) := \{g(a) : a \in U\}$ for $U \subseteq \mathbb{F}_4 \oplus \mathbb{F}_4$.

Consider the abstract simplicial complex $\mathcal{K}$ on the vertex-set $\mathbb{F}_4 \oplus \mathbb{F}_4$ as

$$\mathcal{K} = \{g(\{v_1, v_2, v_3, v_4, v_8\}), g(\{v_1, v_4, v_6, v_9, v_{10}\}) : g \in G\}.$$

(One orbit of 4-simplices of length 240 and one orbit of 4-simplices of length 48.) Let $\mathrm{K3}_{16} = \mathcal{K}_{gr}$ be the geometric realization of $\mathcal{K}$. Then $\mathrm{K3}_{16}$ is a combinatorial 4-manifold and triangulates a K3 surface. This was constructed by Casella and Kühnel in [23]. Since the Euler characteristic of a K3 surface is 24, by Theorem 5.21, 16 is the minimum number of vertices required to triangulate a K3 surface.

Example 3.14. Consider the 12-vertex 5-dimensional pure simplicial complex $S^{3,2}_{12}$ (on the vertex set $\{1, \dots, 9, a, b, c\}$) whose facets are

$12346a, 12346b, 123478, 12347b, 12348a, 12357b, 12357c, 12359b, 12359c, 1236ab, 12378c, 1238ac, 1239ab, 1239ac, 124678, 12467b, 124689, 12469a, 12489a, 1257bc, 1259bc, 12678c, 1267bc, 12689c, 1269ab, 1269bc, 1289ac, 134678, 13467b, 13468a, 13579b, 13579c, 13678c, 1367bc, 1368ac, 136abc, 1379ab, 1379ac, 137abc, 145689, 14568a, 14569a, 14589c, 1458ac, 1459ac, 1489ac, 15689b, 1568ab, 1569ab, 1579ab, 1579ac, 157abc, 1589bc, 158abc, 1689bc, 168abc, 23456a, 23456c, 23458a, 23458b, 2345bc, 2346bc, 23478b, 235678, 23567c, 23568a, 23578b, 2359bc, 23678c, 2368ac, 236abc, 239abc, 24567a, 24567c, 24578a, 24578b, 2457bc, 246789, 24679a, 2467bc, 24789a, 25678a, 26789a, 2689ac, 269abc, 345679, 34567c, 345689, 34568a, 34579c, 34589b, 3459bc, 346789, 3467bc, 34789b, 3479bc, 356789, 35789b, 379abc, 45679a, 4578ab, 4579ac, 457abc, 4589bc, 458abc, 4789ab, 479abc, 489abc, 56789a, 5689ab, 5789ab, 689abc.$

The simplicial complex $S^{3,2}_{12}$ is a combinatorial 5-manifold and triangulates $S^3 \times S^2$. This was constructed by Lutz in [51]. Observe that, by Theorem 5.47, 12 is the minimum number of vertices required to triangulate $S^3 \times S^2$.

Example 3.15. In $\mathbb{R}^{d+1}$ consider the moment curve M_{d+1} defined parametrically by $x(t) = (t, t^2, \dots, t^{d+1})$. Let $t_1 < t_2 < \dots < t_n$ and $v_i = x(t_i)$ for $1 \le i \le n$. For $n \ge d+2$, let $V = \{v_1, \dots, v_n\}$. Let $C(n, d+1)$ be the convex hull of V. Then $C(n, d+1)$ is a simplicial convex $(d+1)$-polytope. The boundary complex of $C(n, d+1)$ is called the *cyclic d-sphere* and is denoted by C^d_n. Then (i) $C(n, d+1)$ (and hence C^d_n) is $\lfloor \frac{d+1}{2} \rfloor$-neighbourly and (ii) a set U ($\subseteq V$) of $d+1$ vertices spans a d-face of $C(n, d+1)$ if and only if any two points of $V \setminus U$ are separated on M_{d+1} by even number of points of U (see [33], Pages 61–63). Observe that the link of a vertex in C^{2c+1}_{m+1} is isomorphic to C^{2c}_m.

If d is odd then, by (ii), $v_1v_3\dots v_{d+2}$ is not a simplex of C_n^d. So, C_n^d is not $(\frac{d+1}{2}+1)$-neighbourly. If d is even then, by (ii), $v_2v_4\dots v_{d+2}$ is not a simplex of C_n^d. So, C_n^d is not $(\frac{d}{2}+1)$-neighbourly. Thus, C_n^d is not $\lfloor\frac{d+1}{2}+1\rfloor$-neighbourly for all $d\geq 1$.

For odd $d\geq 1$, consider the following pure abstract simplicial complex $\mathcal{C}$. The vertices of $\mathcal{C}$ are the vertices of the n-vertex circle S_n^1 and a set of $d+1$ vertices is a maximal simplex of $\mathcal{C}$ if and only if the induced subgraph of S_n^1 on these $d+1$ vertices has no connected component of odd size. By (ii), it follows that C_n^d is the geometric realization of this $\mathcal{C}$.

Example 3.16. For $d\geq 2$ and $n\geq 2d+3$, let $v_1,\dots,v_n$ be n affinely independent points (in $\mathbb{R}^{n-1}$). Consider the $(d+1)$-dimensional pure simplicial complex X on the vertex set $\{v_1,\dots,v_n\}$ given by:

$$X=\{v_iv_{i+1}\cdots v_{i+d+1}\,:\,1\leq i\leq n\}.$$

(Addition in the suffix is modulo n.) Then $|X|$ is a pl manifold with boundary. Let K_n^d be the boundary complex of $|X|$. More explicitly, the facet-set of K_n^d is

$$\{\alpha:\alpha \text{ is a } d\text{-simplex in } X \text{ and } \alpha \text{ is in a unique facet of } X\}.$$

Then K_n^d is a combinatorial d-manifold. It was shown in [43] the following: (i) K_{2d+3}^d triangulates $S^{d-1}\times S^1$ for d even, and triangulates the twisted product $S^{d-1}\mathbin{\underline{\times}} S^1$ (the twisted S^{d-1}-bundle over S^1) for d odd. (ii) K_{2d+4}^d triangulates $S^{d-1}\times S^1$ for all d. In particular, K_9^3 triangulates the twisted product $S^2\mathbin{\underline{\times}}S^1$ (often called the *3-dimensional Klein bottle*) and K_{10}^3 triangulates the product $S^2\times S^1$. The combinatorial 3-manifolds K_9^3 and K_{10}^3 were first constructed by Walkup in [71].

From the definition, it follows that the abstract simplicial complex $(K_n^d)_a$ corresponding to K_n^d is the pure abstract simplicial complex whose vertices are the vertices of the n-cycle $S_n^1(v_1,\dots,v_n)$ and the d-simplices are the sets of $d+1$ vertices obtained by deleting an interior vertex from the $(d+2)$-paths in the cycle. (In fact, the maximal simplices of X_a are the $(d+2)$-paths in $S_n^1(v_1,\dots,v_n)$.)

Example 3.17. For $d\geq 2$, consider the $(3d+5)$-vertex abstract simplicial complexes $\mathcal{M}$ and $\mathcal{N}$ on the vertex set $V=\{1,\dots,3d+5\}$ given by:

$$\begin{aligned}\mathcal{N} &= \{\{i,\dots,j-1,j+1,i+d+1\,:\\ &\qquad i+1\leq j\leq i+d,\ 1\leq i\leq 2d+4\},\\ \mathcal{M} &= \mathcal{N}\cup\{\{1,\dots,d+1\},\{2d+5,\dots,3d+5\}\}.\end{aligned}$$

Then $\mathcal{M}_{gr}$ is a stacked d-sphere and (hence) $\mathcal{N}_{gr}$ triangulates $S^{d-1}\times[0,1]$.

Let $p=(p_1,\dots,p_k)$ be a partition of $d+1$. Put $s_0=0$ and $s_j=\sum_{i=1}^{j}p_i$ for $1\leq j\leq k$. Let π_p be the permutation of $\{1,2,\dots,d+1\}$ which is the product of k disjoint cycles $(s_{j-1}+1,s_{j-1}+2,\dots,s_j)$, $1\leq j\leq k$. Since $\pi_p(i)\leq i+1$ for $1\leq i\leq d+1$, it follows that $d_{\mathcal{M}}(2d+4+i,\pi_p(i))\geq 3$. Consider the equivalence relation $\rho(p)$ on V given by: $j\rho(p)j$ for $j\in V$ and $(2d+4+i)\rho(p)\pi_p(i)$ for $1\leq i\leq d+1$. Let $\eta_p\colon V\to V/\rho(p)$ be the canonical surjection.

Let $\mathcal{K}(p)=\mathcal{N}/\rho(p)$ denote the abstract simplicial complex whose vertex set is $V/\rho(p)$ and simplices are $\eta_p(\sigma)$, where $\sigma\in\mathcal{N}$. Let $K_{2d+4}^d(p)=\mathcal{K}(p)_{gr}$ be the geometric realization of $\mathcal{K}(p)$. It was shown in [12] that $K_{2d+4}^d(p)$ is a combinatorial d-manifold and triangulates $S^{d-1}\times S^1$ (respectively, the twisted product $S^{d-1}\mathbin{\underline{\times}}S^1$) if p is an even (respectively, odd) partition of $d+1$. If $p_0=(1,1,\dots,1)$ then $\pi_{p_0}=\mathrm{Id}$ and the corresponding combinatorial d-manifold $K_{2d+4}^d(p_0)$ is same as K_{2d+4}^d defined in Example 3.16.

[Recall that for any positive integer n, a *partition* of n is a finite weakly increasing sequence of positive integers adding to n. The terms of the sequence are called the *parts* of the partition. A partition of n is *even* (respectively, *odd*) if it has an even (respectively, odd) number of even parts. It was shown in [12] that the number of odd (respectively, even) permutations of n is $\geq\frac{1}{2}\times$(the number of partitions of $n-1$).]

Observe that the number of edges in $K_{2d+4}^d(p)$ is $\binom{2d+4}{2}-(d+2)$. We would like to make the following:

Conjecture 3.18. *Let K be a non-simply connected $(2d+4)$-vertex combinatorial d-manifold. If $d\geq 3$ then the number of edges in K is at least $\binom{2d+4}{2}-(d+2)$.*

Example 3.19. Let H_{16}^3 be the combinatorial 3-manifold defined in Example 3.9 and u be a vertex of H_{16}^3. Let $\Sigma^1H_{16}^3:=\Sigma_uH_{16}^1$ be the one-point suspension of H_{16}^3 and for $n\geq 2$, let $\Sigma^nH_{16}^3:=\Sigma_u(\Sigma^{n-1}H_{16}^3)$. Let $U=V(\Sigma^nH_{16}^3)\setminus V(H_{16}^3)$. Then $\sigma:=\langle U\rangle$ is an $(n-1)$-simplex of $\Sigma^nH_{16}^3$ and $\mathrm{lk}_{\Sigma^nH_{16}^3}(\sigma)=H_{16}^3$. Thus, $\Sigma^nH_{16}^3$ is not a combinatorial manifold. Since $|H_{16}^3|$ is a homology 3-sphere and $|\Sigma^nH_{16}^3|$ is the n-th suspension of $|M|$, by Cannon's theorem (which states that the double suspension of any homology d-sphere is homeomorphic to S^{d+2}) [22], $|\Sigma^nH_{16}^3|$ is homeomorphic to S^{3+n} for $n\geq 2$. So, $\Sigma^nH_{16}^3$ is a triangulation of S^{3+n} for $n\geq 2$. Clearly, $\Sigma^nH_{16}^3$ has $3+n$ vertices. So, for $d\geq 5$, S^d has a $(d+13)$-vertex non-combinatorial (non-pl) triangulation.

Example 3.20. Let $S^3_{8,38}$ be the Brückner-Grünbaum 3-sphere defined in Example 3.5 and u be a vertex of $S^3_{8,38}$. Let $\Sigma^1 S^3_{8,38} := \Sigma_u S^3_{8,38}$ be the one-point suspension of $S^3_{8,38}$ and for $n \geq 2$, let $\Sigma^n S^3_{8,38} := \Sigma_u(\Sigma^{n-1} S^3_{8,38})$. Then $\Sigma^n S^3_{8,38}$ is an $(n+8)$-vertex combinatorial $(n+3)$-sphere. Since $S^3_{8,38}$ is a non-polytopal sphere, by Proposition 1.3, $\Sigma^n S^3_{8,38}$ is a non-polytopal sphere. So, for $d \geq 3$, there exists an $(d+5)$-vertex non-polytopal combinatorial d-sphere. Note that, by Theorem 5.30, a $(d+k)$-vertex combinatorial d-sphere is a polytopal sphere for $2 \leq k \leq 4$.

Example 3.21. For $d \geq 1$, let Δ^{d+1} be the d-simplex with vertices $v_1 = (1, 0, \dots, 0) - \frac{1}{d+2}(1, 1, \dots, 1)$, $\dots, v_{d+2} = (0, \dots, 0, 1) - \frac{1}{d+2}(1, 1, \dots, 1)$ in $\mathbb{R}^{d+2}$. We know that the standard d-sphere S^d_{d+2} is the boundary complex of Δ^{d+1}. Let $\varphi : |S^d_{d+2}| \to S^{d+1} \subseteq \mathbb{R}^{d+2}$ be the radial projection. Then $S := \varphi(|S^d_{d+2}|)$ is a d-sphere in $\mathbb{R}^{d+2}$ and $\varphi : |S^d_{d+2}| \to S$ is a homeomorphism. Let $\alpha : S \to S$ be the antipodal map. Then the quotient space S/α is the d-dimensional real projective space $\mathbb{R}P^d$.

Let $\mathcal{S}^d := (S^d_{d+2})^{(1)}$ be the first barycentric subdivision of S^d_{d+2}. Let $V = \{v_1, \dots, v_{d+2}\}$. Let $\mathcal{S}^d_a$ be the abstract simplicial complex corresponding to $\mathcal{S}^d$. We can identify $V(\mathcal{S}^d_a)$ with the set of proper subsets of V by $\widehat{\langle U \rangle} \mapsto U$. Let $\eta : V(\mathcal{S}^d_a) \to V(\mathcal{S}^d_a)$ be given by $\eta(U) = V \setminus U$. Then η is an automorphism of $\mathcal{S}^d_a$ and $\eta \circ \eta = \mathrm{Id}$. Observe that $d_{\mathcal{S}^d_a}(U, V \setminus U) = 3$ for any proper subset U of V. (Since $d \geq 1$, $\#(U)$ or $\#(V \setminus U) \geq 2$. Assume that $\#(U) \geq 2$ and $u \in U$. Then $U\{u\}((V \setminus U) \cup \{u\})(V \setminus U)$ is a path of length 3 from U to $\eta(U) = V \setminus U$.) So, $G = \{\mathrm{Id}, \eta\}$ acts properly on $\mathcal{S}^d_a$. Thus, by Proposition 1.1, $\mathcal{P}^d := \mathcal{S}^d/G$ is a combinatorial d-manifold. Since the number of vertices in $\mathcal{S}^d$ is $2^{d+2} - 2$, the number of vertices in $\mathcal{P}^d$ is $2^{d+1} - 1$.

Let $\eta_{gr} : V(\mathcal{S}^d) \to V(\mathcal{S}^d)$ be the simplicial map induced by η (i.e., $\eta(\widehat{\langle U \rangle}) = \widehat{\langle V \setminus U \rangle}$). Then η_{gr} is an automorphism and $\alpha \circ \varphi = \varphi \circ |\eta_{gr}|$. This implies that $|\mathcal{S}^d/G| = |\mathcal{S}^d|/\eta_{gr}$ is homeomorphic to S/α. Thus, $\mathcal{P}^d$ is a $(2^{d+1}-1)$-vertex triangulation of $\mathbb{R}P^d$. An explicit description of $\mathcal{P}^d$ is given by Mukherjea in [61]. (In [42], Kühnel has given another description of $\mathcal{S}^d_a$ and the abstract simplicial complex corresponding to $\mathcal{P}^d$.)

Example 3.22. Consider the isometry group $G := \mathbb{Z}^d : S_d$ of $\mathbb{R}^d$, where the symmetric group S_d acts on $\mathbb{R}^d$ by $(g(x))_i = x_{g(i)}$ for $1 \leq i \leq d$ and $\mathbb{Z}^d$ acts by translations. Let $\sigma := \{(x_1, \dots, x_d) : 0 \leq x_d \leq x_{d-1} \leq \cdots \leq x_1 \leq 1\} \subseteq \mathbb{R}^d$. Then σ is a d-simplex with vertices $(0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, \dots, 1, 0), (1, \dots, 1)$. Observe that $\mathring{\sigma} \cap g(\sigma) = \emptyset$ for $1 \neq g \in G$. Consider the pure d-dimensional simplicial complexes

$$Y = \{g(\sigma) : g \in S_d\} \text{ and } X = \{g(\sigma) : g \in G\}.$$

Then Y triangulates the d-cube $[0, 1]^d$ and hence X triangulates $\mathbb{R}^d$. Clearly, G is a group of automorphisms of X_a (the abstract simplicial complex corresponding to X).

Let $\mathcal{L}_d := \{(x_1, \dots, x_d) : \sum_{i=1}^{d} 2^{i-1} x_i \equiv 0 \bmod 2^{d+1} - 1\} \subseteq \mathbb{Z}^d$. Then $\mathcal{L}_d$ is a subgroup of G. ($\mathcal{L}_d$ is a sub lattice of $\mathbb{Z}^d$ and $\{(2^{d+1} - 1, 0, \dots, 0), \dots, (-1, 0, \dots, 0, 2^{d+1-i}, 0, \dots, 0), \dots, (-1, 0, \dots, 0, 4)\}$ is a basis of $\mathcal{L}_d$.) Then $\mathcal{L}_d$ acts properly on X_a. Therefore, by Proposition 1.1, $X/\mathcal{L}_d$ is a combinatorial d-manifold. Since $[\mathbb{Z}^d : \mathcal{L}_d] = 2^{d+1} - 1$, it follows that the number of vertices in $X/\mathcal{L}_d$ is $2^{d+1} - 1$. The combinatorial d-manifold $X/\mathcal{L}_d$ was constructed by Kühnel and Laßmann in [48]. They have shown that $X/\mathcal{L}_d$ triangulates the d-dimensional torus $S^1 \times \cdots \times S^1$.

In [52], Lutz conjectured the following:

Conjecture 3.23. *The combinatorial manifold $X/\mathcal{L}_d$ is a vertex-minimal triangulation of the d-dimensional torus $S^1 \times \cdots \times S^1$ for all $d \geq 3$.*

Conjecture 3.24. *The combinatorial manifold $X/\mathcal{L}_3$ is the unique 2-neighbourly triangulation of $S^1 \times S^1 \times S^1$.*

4. Some General Results on Triangulations

In this section, we are presenting some results on triangulations. Some of them are interesting and classical and some of them are very useful.

Theorem 4.1. ***(Dehn-Sommerville Equations.)*** *If M is a combinatorial d-manifold then the f-vector and the h-vector of M satisfy the following.*

(i) $\sum_{i=0}^{d}(-1)^i f_i(M) = \chi(M)$ *($= 0$ if d is odd).*

(ii) *If d is even then* $\sum_{i=2j-1}^{d}(-1)^i \binom{i+1}{2j-1} f_i(M) = 0$ *for* $1 \leq j \leq \frac{d}{2}$.

(iii) *If d is odd then* $\sum_{i=2j}^{d}(-1)^i \binom{i+1}{2j} f_i(M) = 0$ *for* $1 \leq j \leq \frac{d-1}{2}$.

(iv) *If* $d = 2k$ *then* $h_j(M) - h_{d+1-j}(M) = (-1)^{d+1-j}\binom{d+1}{j}(\chi(M) - 2)$ *for* $0 \leq j \leq k$.

(v) *If* $d = 2k-1$ *then* $h_j(M) - h_{d+1-j}(M) = 0$ *for* $0 \leq j \leq k-1$.

Proof. The first equation is the Euler equation.

If d is even then the link of a $(2j-2)$-simplex is a triangulation of the odd dimensional sphere S^{d-2j-1} and hence the Euler characteristic of the link of a $(2j-2)$-simplex is 0. If we take the sum of the Euler equations of the links of all $(2j-2)$-simplices, then we get the second equation.

Similarly, we get the third equation by taking the sum of the Euler equations of the links of all the $(2j-1)$-simplices if d is odd.

The last two equations follow from first three and the definition of h-vector. □

Thus, for a combinatorial d-manifold M, $f_d(M), \dots, f_{(d+1)/2}(M)$ can be express in terms of $f_0(M), \dots, f_{(d-1)/2}(M)$ if d is odd and $f_d(M), \dots, f_{d/2}(M)$ can be express in terms of $\chi(M), f_0(M), \dots, f_{d/2-1}(M)$ if d is even. Since, by the last equation in Theorem 4.1, $(h_0(M), \dots, h_{\lfloor (d+1)/2 \rfloor}(M))$ determines the h-vector of M, it follows that the f-vector of M is determined by $(h_0(M), \dots, h_{\lfloor (d+1)/2 \rfloor}(M))$. See [33,41] for more.

For $n \geq d+2$ and $d \geq 1$, let $\varphi_d(n, d+1) := dn - (d+2)(d-1)$ and $\varphi_k(n, d+1) := \binom{d+1}{k} n - \binom{d+2}{k+1} k$ for $1 \leq k \leq d-1$. In the definition of stacked sphere, we have seen that $f_k(S) = \varphi_k(n, d+1)$ for any n-vertex stacked d-sphere S and $k \geq 1$. In [14,15], Barnette proved the following:

Theorem 4.2. ***(Lower Bound Theorem for Polytopal Spheres.)*** *If M is an n-vertex polytopal d-sphere $(d \geq 2)$ then*

(i) $f_k(M) \geq \varphi_k(n, d+1)$ for $1 \leq k \leq d$ and
(ii) for $d \geq 3$, $f_d(M) = \varphi_d(n, d+1)$ if and only if M is a stacked sphere.

In [17], Barnette proved the following generalization of Theorem 4.2 (i).

Theorem 4.3. *If M is an n-vertex connected closed triangulated manifold of dimension $d \geq 2$, then $f_d(M) \geq \varphi_d(n, d+1)$.*

Towards the classification of all the n-vertex triangulated d-manifolds M for which $f_k(M) = \varphi_k(n, d+1)$, McMullen, Perles and Walkup observed the following independently (see [15,41, 59]).

Theorem 4.4. *Let M be an n-vertex d-dimensional $(d \geq 1)$ simplicial complex, such that $f_1(\mathrm{lk}_M(\sigma)) \geq \varphi_1(\deg_M(\sigma), d-i)$ for any i-simplex σ in M $(0 \leq i \leq d-2)$.*

(i) Then $f_k(M) \geq \varphi_k(n, d+1)$ for $1 \leq k \leq d$.
(ii) Moreover, if $f_k(M) = \varphi_k(n, d+1)$ for some $k \geq 1$ then $f_1(M) = \varphi_1(n, d+1)$.

In [39], Kalai showed that for $d \geq 3$, the edge graph of any connected triangulated d-manifold is "generically $(d+1)$-rigid" in the sense of rigidity of frameworks. The case $k = 1$ of Theorem 4.3 is an immediate consequence of Kalai's rigidity theorem. Kalai also succeeds in using his rigidity theorem to prove the following:

Theorem 4.5. ***(LBT for Triangulated Manifolds).*** *If M is an n-vertex triangulated closed manifold of dimension $d \geq 2$ then*

(i) $f_1(M) \geq \varphi_1(n, d+1)$ and
(ii) for $d \geq 3$, $f_1(M) = \varphi_1(n, d+1)$ if and only if M is a stacked sphere.

In [71], Walkup proved Theorem 4.3 for $d = 3, 4$ and Theorem 4.5 for $d = 3$. For $d = 2$ one observes the following: If M is an n-vertex connected combinatorial 2-manifold of Euler characteristic $\chi(M)$ then $f_1(M) = 3n - 3\chi(M)$ and $f_2(M) = 2n - 2\chi(M)$. For every connected combinatorial 2-manifold M, $\chi(M) \leq 2$ and $\chi(M) = 2$ if and only if M is a (polytopal) 2-sphere. Thus, (i) $f_i(M) \geq \varphi_i(n, 3)$ for $1 \leq i \leq 2$ and (ii) $f_i(M) = \varphi_i(n, 3)$ for $i = 1$ or 2 if and only if M is a (polytopal) combinatorial 2-sphere. From Theorems 4.4 and 4.5 one gets:

Theorem 4.6. *Let M be an n-vertex triangulated closed manifold of dimension $d \geq 3$. If $f_k(M) = \varphi_k(n, d+1)$ for some $k \geq 1$ then M is a stacked d-sphere.*

In [70], Tay generalized Theorem 4.5 to normal pseudomanifolds to prove:

Theorem 4.7. ***(LBT for Normal Pseudomanifolds).*** *If M is an n-vertex normal pseudomanifold of dimension $d \geq 2$ then*

(i) $f_k(M) \geq \varphi_k(n, d+1)$ for $1 \leq k \leq d$ and
(ii) for $d \geq 3$, if $f_k(M) = \varphi_k(n, d+1)$ for some k, $1 \leq k \leq d$ then M is a stacked sphere.

In [13], we have presented a self-contained combinatorial proof of Theorem 4.7.

Let C_n^d be the polytopal d-sphere as in Example 3.15. Then C_n^d is a $\lfloor \frac{d+1}{2} \rfloor$-neighbourly combinatorial d-manifold and hence $h_j(C_n^d) = \binom{n-d-2+j}{j}$ for all $j = 0, \dots, \lfloor \frac{d+1}{2} \rfloor$. In [58], McMullen proved the following:

Theorem 4.8. *Let X be a triangulation of the sphere S^d with n vertices. Then*

(i) *If $h_j(X) \leq \binom{n-d-2+j}{j}$ for all $j = 0, \dots, \lfloor \frac{d+1}{2} \rfloor$ then $f_i(X) \leq f_i(C_n^d)$ for all $i = 0, \dots, d$.*

(ii) *If X is a polytopal d-sphere then $h_j(X) \leq \binom{n-d-2+j}{j}$ for all $j = 0, \ldots, \lfloor \frac{d+1}{2} \rfloor$.*

In [67], Stanley proved the following 'Upper Bound Conjecture' by showing that $h_j(X) \leq \binom{n-d-2+j}{j}$ for all $j = 0, \ldots, \lfloor \frac{d+1}{2} \rfloor$ whenever X triangulates S^d.

Theorem 4.9. ***(Upper Bound Theorem for Spheres).*** *Let X be an n-vertex simplicial complex. If X triangulates S^d then $f_i(X) \leq f_i(C_n^d)$ for $1 \leq i \leq d$.*

For a combinatorial d-sphere, we get the following from Theorem 4.9 :

Corollary 4.10. *Let M be an n-vertex k-neighbourly d-dimensional pseudomanifold. If M triangulates the d-sphere S^d and $n \geq d+3$ then $k \leq \lfloor \frac{d+1}{2} \rfloor$.*

Proof. Since C_n^d is not $(\lfloor \frac{d+1}{2} \rfloor + 1)$-neighbourly for all d with $n \geq d+3$, the corollary follows from Theorem 4.9. □

For a combinatorial d-sphere, Corollary 4.10 also follows from Theorem 4.1.

Let $\mathcal{T}$ and $\mathbb{C}P_9^2$ be as in Examples 3.3 and 3.10 respectively. Then their f-vectors are as follows: $f(\mathcal{T}) = (7, 21, 14)$, $f(\mathbb{C}P_9^2) = (9, 36, 84, 90, 36)$. Since the f-vectors of any S_7^2 and C_9^4 are $(7, 15, 10)$ and $(9, 36, 74, 75, 30)$ respectively, it follows that the Upper bound theorem is not true for all manifolds. In [62], Novik prove proved the following generalizations of Theorem 4.9.

Theorem 4.11. ***(UBT for odd-dimensional Homology Manifolds).*** *Let X be an n-vertex $(2k-1)$-dimensional homology manifold. Then $f_i(X) \leq f_i(C_n^{2k-1})$ for $1 \leq i \leq 2k-1$.*

Theorem 4.12. *For d even, let X be an n-vertex d-dimensional homology manifold. If either*

(i) *$d \equiv 0$ (mod 4) and $\chi(X) \leq 2$, or*
(ii) *$d \equiv 2$ (mod 4), $\chi(X) \geq 2$ and $H_{d/2}(|X|; \mathbb{Z}) = 0$*

then $f_i(X) \leq f_i(C_n^d)$ for $1 \leq i \leq d$.

In [50], Lickorish presented a proof of the following:

Theorem 4.13. *Two simplicial complexes are combinatorially equivalent if and only if they are stellar equivalent.*

Clearly, if two pseudomanifolds are bistellar equivalent then they are combinatorially equivalent. In [63], Pachner proved the following (see [50] for a proof).

Theorem 4.14. *Two combinatorial manifolds are combinatorially equivalent if and only if they are bistellar equivalent.*

5. Minimal Triangulations

In this section, we are presenting some results without proofs. Proofs are available in the references given. We sometime identify a simplicial complex X with the abstract simplicial X_a corresponding to X.

Let K be an n-vertex combinatorial 2-manifold. If (n, f_1, f_2) is the f-vector then $2f_1 = 3f_2$ and $f_1 \leq \binom{n}{2}$. Thus, $\chi(K) = n - f_1 + f_2 = n - \frac{1}{3}f_1 \geq n - \frac{1}{3}\binom{n}{2} = \frac{7n-n^2}{6}$. This implies that $n \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(K)})$. It is known that the Klein bottle (whose Euler characteristic is 0) has an 8-vertex triangulation and has no 7-vertex triangulation (Theorem 5.2 below). From the classification of 8-vertex combinatorial 2-manifolds (Theorem 5.3 below), we know that there is no 8-vertex combinatorial 2-manifold of Euler characteristic -1. In two articles ([64,38]), Ringel and Jungerman proved the following:

Theorem 5.1. *Let M be a closed surface which is not the Klein bottle, the double torus or the non-orientable surface of Euler characteristic -1. Then M has an n-vertex triangulation if and only if $n \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$. In each of those three cases, one needs one more vertex for triangulations.*

It is known that the only combinatorial 2-manifolds on at most 6 vertices are S_4^2, $S_2^0 * S_3^1$, $S_2^0 * S_2^0 * S_2^0$, $\Sigma^1(S_5^1)$ and $\mathbb{R}P_6^2$. In [26], we have shown the following:

Theorem 5.2. *There are exactly nine 7-vertex combinatorial 2-manifolds, five of which triangulate the 2-sphere S^2, three of which triangulate $\mathbb{R}P^2$ and one triangulate $S^1 \times S^1$.*

In [29], we have proved the following:

Theorem 5.3. *There are exactly 44 distinct combinatorial 2-manifolds on 8 vertices. One of these combinatorial 2-manifolds consists of two copies of S_4^2's, 14 of these triangulate S^2, 16 triangulate $\mathbb{R}P^2$, seven triangulate $S^1 \times S^1$ and six triangulate the Klein bottle.*

For $g \geq 0$, let $M(g, +)$ denote the orientable surface of genus g and let $M(g, -)$ denote the non-orientable surface of genus g. (So, $M(1, +) = S^1 \times S^1$ and $M(2, -)$ is the Klein bottle.) Thus, $\chi(M(g, +)) = 2 - 2g$ and $\chi(M(g, -)) = 2 - g$. In [55,69], Lutz and Sulanke have enumerated (via computer search) all the triangulated 2-manifolds with at most 12 vertices. They have shown the following:

Theorem 5.4. *There are precisely 655 combinatorial 2-manifold with 9 vertices: 50 of these triangulate S^2, 112 triangulate $S^1 \times S^1$, 134 triangulate $\mathbb{R}P^2$, 187 triangulate the Klein bottle, 133 triangulate $M(3, -)$, 37 triangulate $M(4, -)$ and 2 triangulate $M(5, -)$.*

Theorem 5.5. *There are precisely* 42426 *combinatorial* 2*-manifold with* 10 *vertices:* 233 *of these triangulate* S^2, 2109 *triangulate* $S^1 \times S^1$, 865 *triangulate* $M(2,+)$, 20 *triangulate* $M(3,+)$, 1210 *triangulate* $\mathbb{R}P^2$, 4462 *triangulate the Klein bottle,* 11784 *triangulate* $M(3,-)$, 13657 *triangulate* $M(4,-)$, 7050 *triangulate* $M(5,-)$, 1022 *triangulate* $M(6,-)$ *and* 14 *triangulate* $M(7,-)$.

Theorem 5.6. *There are precisely* 11590894 *combinatorial* 2*-manifold with* 11 *vertices:* 1249 *of these triangulate* S^2, 37867 *triangulate* $S^1 \times S^1$, 113506 *triangulate* $M(2,+)$, 65876 *triangulate* $M(3,+)$, 821 *triangulate* $M(4,+)$, 11719 *triangulate* $\mathbb{R}P^2$, 86968 *triangulate the Klein bottle,* 530278 *triangulate* $M(3,-)$, 1628504 *triangulate* $M(4,-)$, 3355250 *triangulate* $M(5,-)$, 3623421 *triangulate* $M(6,-)$, 1834160 *triangulate* $M(7,-)$, 295291 *triangulate* $M(7,-)$ *and* 5982 *triangulate* $M(9,-)$.

Theorem 5.7. *There are precisely* 12561206794 *combinatorial* 2*-manifold with* 12 *vertices:* 7,595 *of these triangulate* S^2, 605496 *triangulate* $S^1 \times S^1$, 7085444 *triangulate* $M(2,+)$, 25608643 *triangulate* $M(3,+)$, 14846522 *triangulate* $M(4,+)$, 751593 *triangulate* $M(5,+)$, 59 *triangulate* $M(6,+)$, 114478 *triangulate* $\mathbb{R}P^2$, 1448516 *triangulate the Klein bottle,* 16306649 *triangulate* $M(3,-)$, 99694693 *triangulate* $M(4,-)$, 473864807 *triangulate* $M(5,-)$, 1479135833 *triangulate* $M(6,-)$, 3117091975 *triangulate* $M(7,-)$, 3935668832 *triangulate* $M(8,-)$, 2627619810 *triangulate* $M(9,-)$, 711868010 *triangulate* $M(10,-)$, 49305639 *triangulate* $M(11,-)$ *and* 182200 *triangulate* $M(12,-)$.

We know (see Theorem 5.30 below) that a combinatorial 3-manifold on at most 7 vertices is a polytopal 3-sphere. In [1], Altshuler proved the following:

Theorem 5.8. *Every combinatorial* 3*-manifold with at most* 8 *vertices is a combinatorial* 3*-sphere.*

In [34], Grünbaum and Sreedharan shown the following:

Theorem 5.9. *There are exactly* 37 *polytopal* 3*-spheres on* 8 *vertices.*

Grünbaum and Sreedharan have also constructed the 8-vertex non-polytopal 3-sphere $S^3_{8,38}$ (see Example 3.5). In [16], Barnette have proved the following:

Theorem 5.10. *There are exactly two non-polytopal combinatorial* 3*-sphere on* 8 *vertices, namely,* $S^3_{8,38}$ *and* $S^3_{8,39}$ *(given in Example* 3.5).

So, there are exactly 39 combinatorial 3-manifolds with 8 vertices. We got a different proof of this. This follows from the next two theorems which we have proved in [30].

Theorem 5.11. *Every* 8*-vertex* 3*-pseudomanifold is obtained from a* 2*-neighbourly* 8*-vertex* 3*-pseudomanifold by a sequence of bistellar* 2*-moves.*

Theorem 5.12. *If* M *is an* 8*-vertex* 2*-neighbourly combinatorial* 3*-manifold then* M *is isomorphic to one of* $S^3_{8,35}, \dots, S^3_{8,38}$ *(given in Example* 3.5).

In Example 3.16, we have seen that there exists a 9-vertex triangulation (namely, K^3_9) of the twisted product $S^2 \underline{\times} S^1$ and there exists a 10-vertex triangulation (namely, K^3_{10}) of $S^2 \times S^1$. In [71], Walkup proved the following:

Theorem 5.13. *There exists an* n*-vertex triangulation of* $S^2 \times S^1$ *only if* $n \geq 10$.

Theorem 5.14. *If* K *is a combinatorial* 3*-manifold and* $|K|$ *is not homeomorphic to* S^3, $S^2 \underline{\times} S^1$ *or* $S^2 \times S^1$ *then* $f_1(K) \geq 4f_0(K) + 8$ *and hence* $f_0(K) \geq 11$.

Thus, for a combinatorial triangulation of $\mathbb{R}P^3$ one needs at least 11 vertices. Therefore, from Example 3.6 and Theorem 5.14 one gets:

Corollary 5.15. *There exists an* n*-vertex triangulation of* $\mathbb{R}P^3$ *if and only if* $n \geq 11$.

In [3,4], Altshuler and Steinberg showed (via a computer search) the following:

Theorem 5.16. *There are exactly* 1297 *combinatorial* 3*-manifolds on nine vertices. One of these is* K^3_9 *and other* 1296 *are combinatorial* 3*-spheres. Among these* 1296 *combinatorial spheres,* 50 *are* 2*-neighbourly. Among these* 50 2*-neighbourly combinatorial spheres,* 23 *are polytopal and* 27 *are non-polytopal.*

Altshuler and Steinberg also showed (using computer) that any two of these 1296 spheres are bistellar equivalent via a finite sequence of proper bistellar moves. In [11], we have presented computer-free proofs of the following:

Theorem 5.17. *Every* 9*-vertex combinatorial* 3*-manifold is obtained from a* 2*-neighbourly* 9*-vertex combinatorial* 3*-manifold by a sequence of (at most* 10) *bistellar* 2*-moves.*

Theorem 5.18. *Up to isomorphism, there is a unique* 9*-vertex non-sphere combinatorial* 3*-manifold, namely* K^3_9.

In [54,69], Lutz and Sulanke have enumerated (via computer search) all the triangulated 3-manifolds with 10 and 11 vertices. They have shown the following:

Theorem 5.19. *There are precisely* 249015 *combinatorial* 3*-manifold with* 10 *vertices:* 247882 *of these triangulate the* 3*-sphere* S^3, 615 *triangulate the twisted product* $S^2 \underline{\times} S^1$ *and* 518 *triangulate the sphere product* $S^2 \times S^1$.

Theorem 5.20. *There are precisely* 172638650 *combinatorial* 3*-manifolds with* 11 *vertices:* 166564303 *of these triangulate the* 3*-sphere* S^3, 3116818 *triangulate the twisted sphere product* $S^2 \underline{\times} S^1$, 2957499 *triangulate the sphere product* $S^2 \times S^1$ *and* 30 *triangulate the real projective* 3*-space* $\mathbb{R}P^3$.

To get an estimate of the minimal number of vertices for a triangulation of a 4-manifold in terms of the Euler characteristic, Kühnel has proved the following (in [43]):

Theorem 5.21. *If M is a combinatorial* 4*-manifold with n vertices then* $10(\chi(M)-2) \leq \binom{n-4}{3}$. *Equality holds if and only if M is* 3*-neighbourly.*

Since the Euler characteristic of any K3 surface is 24, by Theorem 5.21, any combinatorial triangulation of a K3 surface requires at least 16 vertices. In [23], Casella and Kühnel have constructed a 16-vertex triangulation of a K3 surface ($K3_{16}$ in Example 3.13). It follows from Theorem 5.21 that any combinatorial triangulation of $(S^2 \times S^2)\#(S^2 \times S^2)$ requires at least 12 vertices. In [51], Lutz has proved the following:

Theorem 5.22. *There are at least two* 12*-vertex combinatorial triangulations of* $(S^2 \times S^2)\#(S^2 \times S^2)$.

Observe that the equality holds in Theorem 5.21 for S^4_6, $\mathbb{C}P^2_9$ and $K3_{16}$. In [46], Kühnel and Laßmann showed (by the help of a computer) the following:

Theorem 5.23. *Let M be an n-vertex combinatorial* 4*-manifold. If* $n \leq 13$ *and M is* 3*-neighbourly then* $M = S^4_6$ *or* $\mathbb{C}P^2_9$.

For negative Euler characteristic, we get a lower bound of number of vertices from the following result of Walkup [71]:

Theorem 5.24. *If M is an n-vertex combinatorial* 4*-manifold then* $f_1(M) \geq 5n - \frac{15}{2}\chi(M)$. *Equality holds if and only if the links of all the vertices are stacked* 3*-spheres.*

Since $f_1(M) \leq \binom{n}{2}$ for any n-vertex simplicial complex M, from Theorem 5.24, one gets the following (cf. [44]):

Corollary 5.25. *If M is an n-vertex combinatorial* 4*-manifold then* $-15\chi(M) \leq n(n-11)$. *Equality implies M is* 2*-neighbourly.*

In [37], Januszkiewicz has proved the following:

Theorem 5.26. *If M is an n-vertex combinatorial* 4*-manifold then* $n - f_1(M) + f_3(M) \geq 0$.

If M is a combinatorial 4-manifold then, from Theorem 4.1, $15n - 5f_1(M) + f_3(M) = 15\chi(M)$. Thus, from Theorem 5.26, one gets the following:

Corollary 5.27. *If M is an n-vertex combinatorial* 4*-manifold then* $4f_1(M) \geq 14n - 15\chi(M)$.

A d-dimensional pseudomanifold has at least $d+2$ vertices. It is easy to see that the only d-pseudomanifold with $d+2$ vertices is S^d_{d+2}. It is also known that a combinatorial d-sphere on $d+3$ vertices is a join of standard spheres. In [7], we have seen the following:

Theorem 5.28. *If M is a d-dimensional* $(d \geq 1)$ *pseudomanifold with* $d+3$ *vertices then M is a polytopal sphere and is isomorphic to* $S^c_{c+2} * S^{d-c-1}_{d-c+1}$ *for some* $c < d$.

Thus, $(d+3)$-vertex d-dimensional pseudomanifolds are completely reducible. In [57], Mani has proved the following:

Theorem 5.29. *Every combinatorial d-spheres on at most* $d+4$ *vertices is polytopal.*

In [7], we have classified all the d-dimensional pseudomanifold on $d+4$ vertices. In particular, we have proved the following:

Theorem 5.30. *For* $n \geq 6$, *the n-vertex combinatorial* $(n-4)$*-manifolds consist of:*

(a) *The* 6*-vertex combinatorial* 2*-manifold* $\mathbb{R}P^2_6$ *(defined in Example* 3.2*),*

(b) *completely reducible polytopal spheres; their number is* $\left\lfloor \frac{n(n-6)}{12} \right\rfloor + 1$, *and*

(c) *irreducible polytopal spheres; their number is the integer nearest to*

$$2^{\lfloor (n-3)/2 \rfloor} - \frac{1}{12}n^2 - 1 + \frac{1}{4n}\sum_r \varphi(r)2^{n/r}\ .$$

Here φ is Euler's totient function and the sum is over all the odd divisors r of n.

In [18], Barnette and Gannon proved the following:

Theorem 5.31. *Let M be an n-vertex combinatorial d-manifold, where $d \geq 3$ and $n \leq d+5$. If $d \neq 4$ then M is a combinatorial d-sphere.*

In [20], Brehm and Kühnel proved the following more general results:

Theorem 5.32. *Let M^d_n be an n-vertex combinatorial d-manifold ($d > 0$).*

(a) *If $n < 3\lceil d/2 \rceil + 3$ then $M^d_n \approx S^d_{d+2}$.*

(b) *If $n = 3d/2 + 3$ and $M^d_n \not\approx S^d_{d+2}$ then $d = 2, 4, 8$ or 16. Moreover, $M^2_6 = \mathbb{R}P^2_6$, M^4_9 triangulates $\mathbb{C}P^2$ and for $d = 8$ or 16, $|M^d_n|$ is a simply connected cohomology projective plane over quaternions or Cayley numbers, respectively.*

In [45], Kühnel and Banchoff constructed a 9-vertex triangulation of $\mathbb{C}P^2$ (see Example 3.10). In [46], Kühnel and Laßmann showed (by the help of a computer) the following:

Theorem 5.33. *Up to isomorphism there is a unique 9-vertex triangulation of $\mathbb{C}P^2$.*

Computer-free proofs of the uniqueness of $\mathbb{C}P^2_9$ have appeared in [5] and [6]. In [8], we have presented a very short (theoretical) proof of the uniqueness of $\mathbb{C}P^2_9$.

In [21], Brehm and Kühnel constructed three 15-vertex combinatorial 8-manifolds of Euler characteristic 3. They also showed that these three triangulate the same pl manifold, say $\sim\mathbb{H}P^2$. So, we have:

Theorem 5.34. *There exist at least three different 15-vertex combinatorial 8-manifolds which are not combinatorial spheres.*

All these 3 triangulations are 5-neighbourly and hence do not allow any proper bistellar moves. Using bistellar flips (from the three constructed by Brehm and Kühnel), Lutz has found three more 15-vertex combinatorial triangulations of $\sim\mathbb{H}P^2$.

Question 5.35. *Is there a 27-vertex combinatorial manifold of Euler characteristic 3?*

In [5], Arnoux and Marin proved the following:

Theorem 5.36. *If M is a non-sphere combinatorial d-manifold on $3d/2+3$ vertices then M satisfies complementarity.*

In [25], we have proved the following converse:

Theorem 5.37. *Let M be an n-vertex combinatorial d-manifold. If M satisfies complementarity then $d = 2, 4, 8$ or 16 with $n = 3d/2+3$ and M is a non-sphere.*

In [27,9], we have shown the following:

Theorem 5.38. *Let M be an n-vertex d-dimensional pseudomanifold with complementarity. If $n \leq d+6$ or $d \leq 6$ then M is either $\mathbb{R}P^2_6$ or $\mathbb{C}P^2_9$.*

As a consequence of Theorems 5.1, 5.21, 5.32, 5.36 and Corollary 4.10, we get.

Theorem 5.39. *Let M be an n-vertex combinatorial $2k$-manifold. If either $k \leq 2$ or $n \leq 3k+3$ then $(-1)^k(\chi(M)-2) \leq \binom{n-k-2}{k+1}/\binom{2k+1}{k}$ with equality if and only if M is $(k+1)$-neighbourly.*

In [44], Kühnel conjectured that Theorem 5.39 holds for any combinatorial $2k$-manifold with n vertices. In [62], Novik has proved the following:

Theorem 5.40. *Let M be an n-vertex $2k$-dimensional homology manifold. If either $n \leq 3k+3$ or $n \geq 4k+3$ then $(-1)^k(\chi(M)-2) \leq \binom{n-k-2}{k+1}/\binom{2k+1}{k}$ with equality if and only if M is $(k+1)$-neighbourly.*

Let M be a homology manifold. Let F be a field such that M is orientable with respect to F. Let $\beta_i = \dim_F \widetilde{H}_i(M;F)$. If the dimension of M is $2k$ then by Poincaré duality $(-1)^k(\chi(M)-2) = \beta_k - (\beta_{k+1}+\beta_{k-1}) + (\beta_{k+2}+\beta_{k-2}) - \cdots = \beta_k + 2\sum_{j=1}^{k}(-1)^j\beta_{k-j} = \beta_k + 2\sum_{i=0}^{k-1}(-1)^{k-i}\beta_i$. In [62], Novik has shown that if M is $(k+1)$-neighbourly then $\beta_i = 0$ for $i \neq k, 2k$, $\beta_k = \binom{n-k-2}{k+1}/\binom{2k+1}{k}$ and Kühnel's conjecture holds. She has also proved the following theorem (which is stronger than Theorem 5.40):

Theorem 5.41. *Let M be an n-vertex d-dimensional homology manifold.*

(a) *If $d = 2k$ and either $n \leq 3k+3$ or $n \geq 4k+3$*
$$\text{then } \beta_k + 2\sum_{i=0}^{k-2}\beta_i \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

(b) *If $d = 2k$ and either $n \leq 3k+3$ or $n \geq 7k+4$*
$$\text{then } \sum_{i=1}^{2k-1}\beta_i = \beta_k + 2\sum_{i=0}^{k-1}\beta_i \leq \frac{\binom{n-k-2}{k+1}}{\binom{2k+1}{k}}.$$

(c) *If* $d = 2k-1$ *and either* $n \le 3k+2$ *or* $n \ge 4k+1$ *then* $\sum_{i=1}^{2k-2} \beta_i = 2\sum_{i=1}^{k-1} \beta_i \le \frac{\binom{n-k-2}{k}}{\binom{2k-1}{k}} \cdot \frac{2n}{n+k+2}$.

We know from Theorem 5.14 that the minimal number of vertices required for a triangulation of $\mathbb{R}P^3$ is 11. We have seen in Example 3.12 that there exists a 16-vertex triangulation of $\mathbb{R}P^4$. In [5], Arnoux and Marin proved the following:

Theorem 5.42. *Let M be an n-vertex combinatorial d-manifold. If the cohomology ring of* $|M|$ *is same as that of* $\mathbb{R}P^d$ *then* $n \ge \binom{d+2}{2}$. *Moreover, equality is possible only for* $d = 1$ *and* $d = 2$.

Theorem 5.43. *Let M be an n-vertex combinatorial* $2d$*-manifold. If the cohomology ring of* $|M|$ *is same as that of* $\mathbb{C}P^d$ *then* $n \ge (d+1)^2$. *Moreover, equality is possible only for* $d = 1$ *and* $d = 2$.

Question 5.44. *Is there a* 17*-vertex triangulation of* $\mathbb{C}P^3$*?*

From Examples 3.6 & 3.12, we know that there is an 11-vertex triangulation of $\mathbb{R}P^3$ and a 16-vertex triangulation of $\mathbb{R}P^4$. Let $\mathcal{P}_d$ be as in Example 3.21. Using bistellar flip, Lutz found a 24-vertex triangulation of $\mathbb{R}P^5$ from $\mathcal{P}_5$. We would like to make the following:

Conjecture 5.45. *The combinatorial manifold* $\mathbb{R}P^4_{16}$ *defined in Example* 3.12 *is the unique* 16*-vertex triangulation of* $\mathbb{R}P^4$.

Conjecture 5.46. *For* $d \ge 5$*, if there is an n-vertex triangulation of* $\mathbb{R}P^d$ *then* $n > \binom{d+2}{2} + 1$.

In [20], Brehm and Kühnel proved the following:

Theorem 5.47. *Let M be a combinatorial d-manifold with n vertices. If* $n \le 2d+3-i$ *for some i with* $1 \le i < d/2$ *then* $|M|$ *is i-connected.*

Corollary 5.48. *Let M be a combinatorial d-manifold with n vertices. If* $|M|$ *has the same homology as* $S^{d-i} \times S^i$ *then* $n \ge 2d+4-i$.

Thus, if $m \ge n \ge 1$ then for a combinatorial triangulation of $S^m \times S^n$ we need at least $2m+n+4$ vertices. In Example 3.14, we have seen that there exists a combinatorial triangulation of $S^3 \times S^2$ with 12 vertices. In [51], Lutz has proved the following:

Theorem 5.49. *There are at least two* 13*-vertex combinatorial triangulations of* $S^3 \times S^3$.

In [12], we have improved the case $i = 1$ of Theorem 5.47. We proved the following:

Theorem 5.50. *Let X be a non-simply connected n-vertex triangulated manifold of dimension d. If* $d \ge 3$ *then* $n \ge 2d+3$.

In [10], we have proved the following:

Theorem 5.51. *Let M be an n-vertex combinatorial d-manifold. If* $|M|$ *is a* $\mathbb{Z}_2$*-homology sphere and* $n \le d+8$ *then M is a combinatorial sphere.*

Theorem 5.52. *Let M be a* $(d+9)$*-vertex combinatorial triangulation of a* $\mathbb{Z}_2$*-homology d-sphere. If M is not a combinatorial sphere then M can not admit any bistellar i-move for* $i < d$.

We have seen in Example 3.7 that there exists a 12-vertex combinatorial triangulation of the lens space $L(3,1)$. Since $L(3,1)$ is a $\mathbb{Z}_2$-homology 3-sphere, Theorem 5.51 is sharp for $d = 3$. It follows from Theorem 5.52 that a 12-vertex combinatorial triangulation of $L(3,1)$ can not admit any bistellar i-move for $0 \le i \le 2$. We would like to make the following:

Conjecture 5.53. *The combinatorial manifold* L^3_{12} *defined in Example* 3.7 *is the unique* 12*-vertex triangulation of* $L(3,1)$.

From Theorem 5.47, we know that a triangulation of a non-simply connected closed pl manifold of dimension $d \ge 3$ requires at least $2d+3$ vertices. We also know that there exist such triangulations (namely, Kühnel's combinatorial d-manifold K^d_{2d+3} in Example 3.16) with $(2d+3)$ vertices. In [12] we have proved the following:

Theorem 5.54. *For* $d \ge 3$*, Kühnel's complex* K^d_{2d+3} *is the only non-simply connected* $(2d+3)$*-vertex triangulated manifold of dimension d.*

This result has provided the only known infinite family of closed manifolds (other than spheres) of dimensions more than 2 for which the minimal triangulation is unique.

In [24], Chestnut, Sapir and Swartz have proved the uniqueness of K^d_{2d+3} in the broader class of homology d-manifolds but with a much more restrictive topological condition. They have proved the following:

Theorem 5.55. *For* $d \ge 4$*, Kühnel's complex* K^d_{2d+3} *is the only* $(2d+3)$*-vertex homology manifold of dimension d with first Betti number nonzero and second Betti number zero.*

From Examples 3.17 and Theorem 5.54 we get:

Corollary 5.56. *The minimum number of vertices for a triangulation of* $S^{d-1} \times S^1$ *is* $2d+3$ *for d even and* $2d+4$ *if d is odd.*

Corollary 5.57. *The minimum number of vertices for a triangulation of $S^{d-1} \times S^1$ is $2d+3$ for d odd and $2d+4$ if d is even.*

These results have provided the only known infinite families of closed manifolds (other than spheres) of dimensions more than 2 with vertex-minimal triangulations. Other than these there are ten exceptional examples of manifolds for which we know vertex-minimal triangulations ([52]); see the table below.

Ten known pl-manifolds which have vertex-minimal triangulations:

Dimensions	Manifolds	No. of vertices	Minimality follows from
3	$\mathbb{R}P^3 = L(2,1)$	11	Corollary 5.15 & Example 3.6
3	$L(3,1)$	12	Theorem 5.51 & Example 3.7
4	$\mathbb{C}P^2$	9	Theorem 5.32 & Example 3.10
4	$S^2 \times S^2$	11	Theorems 5.21, 5.23 & Example 3.11
4	$(S^2 \times S^2)\#(S^2 \times S^2)$	12	Theorems 5.21 & 5.22
4	$\mathbb{R}P^4$	16	Theorem 5.42 & Example 3.12
4	a K3 surface	16	Theorem 5.21 & Example 3.13
5	$S^3 \times S^2$	12	Corollary 5.48 & Example 3.14
6	$S^3 \times S^3$	13	Corollary 5.48 & Theorem 5.49
8	$\sim\mathbb{H}P^2$	15	Theorems 5.32 & 5.34

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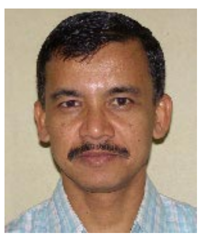

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