On the proof of the Poincaré conjecture

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Abstract | In 2002, Perelman proved the Poincare conjecture, building on the work of Richard Hamilton on the Ricci flow. In this article, we sketch some of the arguments and attempt to place them in a broader dynamical context.

The proof of the Poincaré conjecture, due to Perelman [7–9], is an epic piece of mathematics involving an extraordinary range of ideas, many of which are due to Perelman and many others developed over time by various mathematicians, most notably Richard Hamilton. The goal here is to view these ideas in a more general context. There are no new ideas in this note—it is a collection of loose remarks inspired by the work of Hamilton and Perelman and many related geometric ideas. For a more serious account, we refer the reader to [4–6] and [10].

The Poincaré conjecture is a statement characterizing the three-dimensional sphere among three-dimensional spaces M (more accurately closed three-dimensional manifolds) as the only one that is *simply-connected*. This means that any curve can be deformed to a single point in the space M.

The Poincaré conjecture can be reformulated as saying that a simply-connected, closed, three-dimensional manifold M admits a *Riemannian metric* g that satisfies the *Einstein equation*. For the present, the reader can regard this as simply saying there is a collection of functions on M (more accurately a tensor on M) satisfying a certain equation. The goal is to show such a solution exists, using the hypothesis that M is simply-connected.

Hamilton's approach [2,3], taken to a successful conclusion by Perelman, was to start with an arbitrary metric g and modify this with time using a particular equation, the *Ricci flow*. Einstein metrics are precisely the metrics that are fixed up to scaling by the Ricci flow. Thus, one sets up an *infinte dimensional* dynamical system and hopes that it flows to a fixed point. One of the early results of Hamilton was that the Ricci flow exists for a short time, so we can speak of a dynamical system corresponding to the equation.

Hamilton showed that this does happen in an imporant special case—manifolds of positive Ricci curvature. However, it is too optimistic to hope that a dynamical system flows to a fixed point even in finite dimensions. To understand the subtleties, it is instructive to consider finite dimensional smooth systems. An orbit for such a system may go to infinity, i.e., may not remain in a compact region.

For the Ricci flow, not remaining in a compact region turns out to correspond to unbounded *curvature* or having *injectivity radius* not bounded below. Understanding such regions was the crux of Hamilton's programme.

Even if an orbit remains in a compact region, it may correspond to a periodic orbits, or more generally a compact invariant set. However, we can conclude we have fixed points for an important special class of dynamical systems, namely those given by gradient flows. Such a dynamical system does not have periodic orbits or other compact invariant sets not containing fixed points.

A surprising result of Perelman was that the Ricci flow is closely related to a gradient flow. Namely, we can modify the Ricci flow by symmetries so that it is the gradient flow for what Perelman calls the *entropy* (as pointed out by Topping [10], and independently Mokshay Madiman (personal communication), this is really a Fisher information).

Of the results of Perelman and Hamilton that buildup to the Poincaré conjecture, many are valid for the Ricci flow in all dimensions while others are specific to dimension three. We shall emphasise

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those that are special to dimension three. These can be regarded as the results that prevent complex behaviors. Thus, it seems a worthwhile exercise to try to isolate the features that make a system well-behaved.

The Ricci flow, like General Relativity, has the feature that rather than equations for fields on a fixed space, one has evolution equations for the space itself. This leads to special subtleties. In the case of the Ricci flow, there take the form of non-collapsing conditions that must be satisfied. As these subtleties, and the techniques for dealing with them, are less likely to be relevant in other contexts, we shall avoid much discussion of them. We remark that the techniques invented by Perelman to address these are valid in all dimensions and hence are not those determining regularity.

1. Reaction-diffusion equations

To understand the Ricci flow, it is very useful to work with so called *harmonic co-ordinates*, i.e., co-ordinate functions that are harmonic. These were used extensively by Einstein in general relativity and have recently come to play a major role in Riemannian geometry.

In harmonic co-ordinates, the Ricci flow takes the form of a reaction-diffusion equation, the prototypical equation for complex systems. Thus, the Ricci flow is given by

$$\frac{\partial g}{\partial t} = \Delta g + Q(g, \partial g)$$

where $Q(g, \partial g)$ is a quadratic expression in the metric g and its first derivatives.

Without the quadratic term, the above equation is closely related to the heat equation, which governs the diffusion of a fluid. The extra term is a so-called reaction term. A large part of Hamilton's work on the Ricci flow amounted to controlling the sign of the reaction term to obtain *maximum principles* extending the maximum principle for the heat equation. We illustrate the principles with some simple examples below.

First, consider the case of the heat equation, which we regard as the equation governing the flow of a fluid that is not involved in chemical reactions. In this case, the fluid flows from regions of higher concentration to those of lower concentration. In particular, the *minimum* concentration increases with time. This is the maximum principle in this case.

Suppose now that we have a mixture of fluids that react (in addition to diffusing). We may nevertheless be able to deduce certain aspects of the behavior from simple facts regarding the reactions. For instance, if the reactions always result in the *total* concentration of all substances increasing, we can deduce that the minimum of the total concentrations increases with time, and even estimate the rate of this increase. Indeed the first of Hamilton's maximum principles, that for *scalar curvature*, is of this form.

For the Ricci flow in dimension three, we have a more subtle result called the Hamilton-Ivey pinching. This is once more analagous to a simple process. Imagine a mixture of two fluids that react in such a way that the relative concentration of the sparser fluid increases, i.e., the ratio of the fluids tends to one (we call this *pinching*). Then the minimum of the ratio of the fluids increases with time.

Note that if concentrations of the chemicals are high, then the reaction term is large compared to the diffusion term. If the rate of pinching is sufficiently high, then it will follow that this ratio goes to one (or an appropriate constant) in regions of high concentration. This means such regions have much simpler behaviour.

2. Riemannian manifolds

A smooth manifold is the natural generalisation of a smooth surface to higher dimensions. As in the case of surfaces, given a point $x \in M$ in a smooth manifold M, we can consider the set of vectors tangent to M at x. This is a vector space called the tangent space T_xM.

A Riemannian metric g assigns an inner product to each tangent space T_xM, $x \in M$, in a smooth fashion. Given a Riemmannian metric, the length of a parametrised curve $\gamma(t)$ can be obtained by integrating the infinitesimal length with respect to γ

$$l(\gamma) = \int g(\dot{\gamma}, \dot{\gamma})^{1/2}$$

Fix a point $x \in M$. For points y in a neighbourhood of x, there is a unique shortest curve (parametrised by arclength) joining x to y. Such a curve is a *geodesic*. These geodesics give polar co-ordinates for a neighbourhood of x. An example is the set of longitudes near the north pole.

In the case of the sphere, the geodesics at a point move towards each other, i.e., the distance between points along two geodesics is less than those between corresponding points in Euclidean space. Such a space is said to be positively curved. In general, we can associate to surfaces with Riemmannian metrics a number at each point, called the *Gaussian curvature*, which measures the rate at which geodesics converge or diverge (relative to the Euclidean case).

For surface, any pair of geodesics converge and diverge at the same rate. However, in higher dimensions this is not the case. The rate of convergence or divergence depends on the plane in the tangent space spanned by the initial velocities of the two geodesics. Hence, one needs to consider such a number, the *sectional curvature*, for each plane in the tangent space.

One can average sectional curvatures over different planes to obtain two other useful quanitities measuring curvature, the *Ricci curvature* Ric and the *scalar curvature* R.

The minimising geodesics do not give polar co-ordinates on all of M. The largest radius for which they do is called the *injectivity radius*.

3. The space of Riemannian metrics

We fix a manifold M. The Ricci flow is a dynamical system on the space $Riem$ of Riemannian metrics on M. The manifold M has a large group of symmetries, the *diffeomorhisms* of M (which we regard as the Gauge transformations). These can be regarded as change of co-ordinates, so metrics that differ by diffeomorphisms should be regarded as esentially the same.

The Ricci flow also induces flows on the space $Riem/Diff$ of Riemmannian metrics up to diffeomorphism. It is the flow on this space that is of interest. Hence a central issue is to understand the dynamics on this space. We also consider a further quotient where we regard two metrics as the same if they differ by scaling. A fixed point on this space, i.e., a solution constant up to diffeomorphisms and rescaling is called a *Ricci soliton*.

To understand the dynamics, it is important to understand when the Ricci flow remains in a compact region of the space $Riem/Diff$. Consider a closed subset X of $Riem/Diff$. The volume of a Riemannian manifold gives a continuous function on $Riem/Diff$, hence on X. If X is compact, then this is bounded, i.e., there is an upper bound on the volumes of the manifolds in X. Similarly, we can consider the injectivity radius, which gives a positive function on X. If X is compact, then the injectivity radius is bounded below. Similarly, we can associate to each closed Riemannian manifold the maximum sectional curvature for the manifold. This is also bounded on X.

Important results in Riemannian geometry tell us that in fact the set X is compact if and only if the volume and maximum absolute value of the curvature are bounded above on X and the injectivity radius is bounded below.

We shall see later that the curvature can be regarded as the energy density. A crucial fact is that rescaling the metric causes the curvature to scale in inverse proportion to the square of the scaling factor. Thus, the curvature can be considered the natural scale for the Riemannian manifold, as we see below.

Perelman introduced an entropy and other related quantities that increase along the Ricci flow. Further, one may view the Ricci flow as a gradient flow of the entropy after making appropriate changes of co-ordinates, i.e., applying diffeomorphisms (which are the *gauge transformations* for the Ricci flow). This is crucial in controlling the dynamics.

Using these functionals, Perelman was able to show that there is a lower bound on the injectivity radius for the appropriate rescaled manifolds. By a result of Cheeger, this amounts to showing a lower bound for the volume on the natural scale of the manifold given by the curvature. This is a not unexpected consequence of a lower bound on the entropy (though in general the entropy of a system may decrease locally). This is Perelman's non-collapsing result.

4. Energy and scales

As mentioned before, curvature can be viewed an an analogue of the energy density. From this viewpoint, the behaviour of a system depends on the relation between the dimension and the nature of the appropriate energy, with different kinds of behaviour in the case of *elliptic regularity*, in the *critical case* and for dimensions beyond the critical case. One way of viewing a part of Perelman's work is to say that we do have regularity for the Ricci flow in dimension three even though the dimension is greater than the critical case. We attempt to view this in a general context in the hope that this facilitates other applications of this idea.

For comparison, consider smooth functions on a unit ball $B(0, 1)$ in R^k with the energy (assumed to be finite) given by

$$E(f) = \int \|\nabla f\|^2$$

We consider functions with a fixed energy E. A general principle is that the number of possible states of the system (whose log is the entropy) grows exponentially with energy. One can formulate an appropriate continuous analog of this, which holds for this simple system. In particular, we say the system shows non-trivial behaviour if the energy is not small (which we denote $E \gg 0$). Denote the volume of the unit ball by V_0/

Let us now restrict such functions to a small ball $B(\epsilon)$ of radius ϵ (with centre 0 for simplicity), whose volume is $V_\epsilon = (\epsilon)^k V_0$. Let the energy of the restriction f_ϵ of a given function f be E_ϵ. This is simply the integral above restricted to the ball $B(\epsilon)$. We can view these restrictions as a subsystem.

We can view these restrictions as functions on the unit ball by rescaling. Namely consider the function f' on the unit ball given by

$$f'(x) = f(\epsilon x)$$

The function f' has energy given by

$$E' = \epsilon^{2-k} E_\epsilon$$

As with the original system, we say that the subsystem at scale ϵ has non-trivial behaviour if $E' \gg 0$.

Observe that if the dimension $k < 2$, then if ϵ is small so is E'. Hence we cannot have non-trivial behaviour at small scales. This is the domain of elliptic regularity.

For the critical case $k = 2$, $E' = E_\epsilon$. Thus, we do have non-trivial behaviour at the scale ϵ whenever E_ϵ is comparable to E, i.e., a definite proportion of the energy is localised at b_ϵ (so called *energy concentration*). Evidently such energy concentration can only happen at finitely many points. At these points we have the well known phenomenon of bubbling.

Finally, if $k > 2$, then we evidently can have non-trivial behaviour at many scales as E' may be large even when $E_\epsilon \ll E$. This is typical of complex systems. As mentioned earlier, the Ricci flow in dimension three is in this range, yet is well-behaved.

The energy can be interpreted as giving a *natural scale* for the system at a point, i.e., the smallest scale at which one has non-trivial behaviour. First, assume that the energy is given by an energy density

$$\eta(x) = \lim_{\epsilon \to 0} \frac{E_\epsilon}{V_\epsilon}$$

If the energy density is well-defined (i.e., the limit exists), E_ϵ is approximately ηV_ϵ for a ball of radius ϵ (ϵ small enough) centred about the point x. Observe that in this case the energy of the rescaled function is $E' = \epsilon^2 \eta(x)$.

Thus, the natural scale for the system at x can be taken to be ϵ satisfying $\epsilon^2 \eta = 1$. At this scale, one can have non-trivial behaviour near x.

If the energy density oscillates rapidly, there are in general points y close to x with $\eta(y) \gg \eta(x)$. This means that there can be much smaller regions exhibiting non-trivial behaviour in the ball B_ϵ. This makes the system complex for $k > 2$. It is precisely this rapid oscillation of η that Perelman showed does not happen for the Ricci flow in dimension three.

5. Curvature as energy density

Recall that rescaling a manifold decreases the curvature by the square of the rescaling factor. Thus, given a ball of radius ϵ with curvature function κ, on rescaling to a unit ball the curvature function for corresponding points becomes $\epsilon^2 \kappa$. This is the same transformation as for the energy density in the above example.

Further, consider Riemannian manifolds (possibly with boundary) with diameter at most D and injectivity radius bounded below. We first consider the case of closed manifolds, to understand how complexity is governed by curvature. In this case, Cheeger's finiteness theorem asserts that given an upper bound K on the maximum absolute sectional curvaure, there are only finitely many such manifolds up to diffeomorphism with curvature satisying this upper bound. Further, this number grows exponentially with K. Thus, K can be regarded as the total energy, or more accurately the maximum energy density. In the case of both closed manifolds and those with boundary, one can also see that in some sense the volume of the space of Riemannian manifolds satisfying theses conditions grows exponentially with K.

These considerations make it clear that one should regard the curvature as analogous to the energy density, and the natural scale at a point is a scale ϵ so that rescaling an ϵ ball to a unit ball makes the maximum curvature 1 (or any universal constant).

6. The blow-up analysis

We now return to the case of the Ricci flow. In view of Perelman's non-collapsing result, the main remaining issue is to understand regions of high curvature. As indicated in the previous section, a standard method of doing this is to rescale so that the curvature (more generally energy) is bounded, and to rescale time correspondingly. Such a rescaling gives a new solution on the Ricci flow which is *ancient*, i.e., is defined on $(-\infty, 0]$.

In the case of the Ricci flow, there are a few special features which lead to strong results from the blow-up analysis. Firstly, the curvature pinching results mean that the resulting ancient solutions are manifolds of non-negative curvature. Such manifolds are very special.

Secondly, we consider a sequence of manifolds backward in time. As the Ricci flow is a gradient, a subsequence has a limit which is a fixed point for the flow on $Riem/Diff$, i.e., a Ricci soliton (in fact a gradient soliton). As the manifolds have non-negative curvature, the limit is a gradient shrinking soliton of non-negative curvature. There are very few such solitons.

Further, the solitons obtained above are stable (in the dynamical sense) under the Ricci flow, so that as the solution is close to the asymptotic soliton at very negative times, it remains close at all times. Hence the Riemannian manifolds at all times of the ancient solution are close to the soliton. From this one can deduce a *canonical neighbourhood theorem* for points of maximum curvature, using the corresponding result for asymptotic solitons.

The blow-up analysis is a common method used in mathematics. However, it normally yields results only near points of maximum curvature at the natural scale for these points. At such points, on rescaling according to the natural scale, we get solutions with bounded energy (curvature in our case), which can be analysed using appropriate compactness results.

In general we have points where the energy density (curvature) does tend to infinity but slower than the maximum energy density. Furthermore, such points can be the limits of points where the energy density grows much faster than the given point. In this case, rescaling using the natural scale does not give solutions with bounded energy. Hence blow-up analysis does not suffice to obtain an understanding of the solution near such points.

Observe that the above situation does not arise if the curvature does not oscillate, i.e., the gradient of the curvature is bounded after rescaling so that the curvature is bounded. In the case of the Ricci flow, one can deduce from the nature of the blow-up solutions (using curvature pinching) that this is the case at the points of maximum curvature. A clever inductive argument of Perelman uses this to establish the canonical neighbourhood theorem for all points of high curvature.

Thus, we can view Perelman's argument in two steps. Fistly, blow-up limits are close to appropriate solitons. This is something which one may obtain in many complex systems. However, the second and special feature is that a soliton that is ancient is of a very special form, so that, by adding constraints from the topology, there are very few possibilities for points close to the blow-up limit. Looked at in another way, the special cases are positive Einstein metrics. Negative Einstein metrics, which are far more common, do not give ancient solutions to the Ricci flow. As a result, if we consider a Ricci flow for a time $\epsilon > 0$ with any initial metric, the result cannot be a negative Einstein metric with very high curvature.

7. An overview of the proof

We conclude with some general remarks about the proof of the Poincaré conjecture.

- We consider a manifold with a field on it—a Riemannian metric in our situation.
- There is a group of Gauge transformations for the field—in our case the group of diffeomorphisms. We consider the flow both on fields, and modulo Gauge transformations. We sometimes also consider the fields modulo scaling and Gauge symmetries.
- We wish to find fixed points of the flow. Fixed points modulo symmetries are solitons.
- Perelman considers the flow modified by Gauge transformations. In this case the flow is a gradient and entropy increases along the flow.
- One wishes to understand points where the energy density (maximum curvature) blows up. We can rescale to obtain bounds at the point of maximum energy. The energy density gives the natural scale for the system.
- The maximum principles say that at points where the energy blows up, the curvature is of a special form. This is a condition on the germ of the field at each point.
- As a result, blowing up at the maximum points gives an ancient solution, with the field at each time of a special form (of non-negative curvature)
- One can take times backward to $-\infty$ and extract a limit. This is a soliton with nonnegative curvature.
- These can be classified and are of a special form. In particular the *energy does not oscillate rapidly* for these.
- The special solitons we obtain are the *only maxima for the entropy* on a subspace of $Riem/Diff$ that is stable under the Ricci flow. Hence these solitons are attractors. It follows that the entire ancient solution is close to the solitons.
- Thus, we have regularity for points of maximum energy.
- The control in oscillations also allows blow-up analysis at high but not maximum curvature.

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