REVIEWS

Steering Control of Semilinear Discrete Dynamical System

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Abstract | In this paper, we investigate the controllability property of a class of semilinear non-autonomous system described by the difference equation

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \quad t \in N_0 = \{0, 1, 2, \ldots\}$$

under the assumption that its linear part is controllable and the nonlinear function f satisfies Lipschitz condition. We also give an algorithm to compute steering control for the above system. Numerical example is given to illustrate the result.

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1. Introduction

In [1], Krabs studied the controllability of a general difference system of the form

$$x(t+1) = f(x(t), u(t)).$$

Further, they have also obtained a controller that steers a given initial state to a desired final state for the linear system (1.2). In this paper we consider a semi-linear system of difference equation of the form

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + f(t, x(t)), \\ x(0) &= x_0, \quad t \in N_0 \end{aligned} \tag{1.1}$$

and the corresponding linear system:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t), \\ x(0) &= x_0, \quad t \in N_0. \end{aligned} \tag{1.2}$$

Here, $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$ are sequences of real $n \times n$ and $n \times m$ matrices, respectively, and $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively, $f(.,.) : N_0 \times R^n \to R^n$ is a nonlinear function satisfying Lipschitz condition with respect to the second argument.

We introduce a steering controller for system (1.1) and prove that it is well-defined and it steers any initial state x_0 of system (1.1) to a desired final state x_1 in $N \in N_0$ time steps under certain conditions.

We define the problem of controllability and reachability as follows.

Problem of Controllability

Let $x_0, x_1 \in R^n$ be given arbitrarily. We say that the system is controllable if there exists a sequence of control vectors $(u(t) \in R^m, \quad t \in N_0)$, such that for some $N \in N_0$ the the solution $(x(t))_{t\in N_0}$ of equation (1.1) starting from the initial state $x(0) = 0$, also satisfies the end condition $x(N) = x_1$.

Problem of Reachability

We say that the state $x_1 \in R^n$ is reachable in N time steps, if there exist a sequence of control vectors $u(t) \in R^m, \quad t \in N_0$, such that the corresponding solution starting from $x(0) = 0$, also satisfies $x(N) = x_1$.

We now express the solution of (1.1) and (1.2) in terms of the state-transition matrix $\Phi(t, t_0)$

associated with the homogeneous linear part of (1.2).

The state transition matrix $\Phi(t, t_0)$ is given by [2]

$$\Phi(t,t_0) = A(t-1)A(t-2)\ldots A(t_0) \quad \forall \quad t \geq t_0$$

It can be shown that the solution of (1.1) is given by

$$x(t) \;=\; \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)u(j) + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j)) \qquad (1.3)$$

and the solution of (1.2) is given by

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)u(j) \quad (1.4)$$

In this article, we give the computational scheme for the steering control. For $t = 0,1,\ldots,N-1$, we define a controller

$$u(t) \;:=\; B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \left[x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j))\right] \quad (1.5)$$

for the nonlinear system (1.1), where $W_r(0,N)$ is called reachability Grammian defined by

$$W_r(0,N) := \sum_{j=0}^{N-1} \Phi(N,j+1)B(j)B(j)^*\Phi(N,j+1)^* \quad (1.6)$$

We will prove that this control is well-defined and steers the nonlinear system (1.1) from x_0 to x_1. We make the following assumptions to obtain the result.

Assumptions

$[L]$: The linear system (1.2) is controllable.
$[N]$: The nonlinear function $f(t,x)$ is Lipschitz continuous with respect to x. That is, there exists $\alpha > 0$ such that

$$\| f(t,x) - f(t,y) \| \leq \alpha \;\; \| x - y \| \quad \forall x, y \in R^n$$

Under the above assumptions, we prove in Section 2 that the system (1.1) is controllable and also prove that controllability and reachability of the system (1.1) are equivalent. Numerical example for steering control of system (1.1) is provided in Section 3.

Further, let $S_N \equiv S_N(R^n)$ $(N \geq 0)$ be the linear space of terminating sequences $\{x(t)\}_{t=0}^{N}$, $(x(t) \in R^n)$ and denote by $S_N^\infty \equiv S_N^\infty(R^n)$, the corresponding Banach space with norm $\| \,.\, \|_N^\infty$:

$$\| x \|_N^\infty = \sup_{0 \leq t \leq N} \| x(t) \|$$

We denote the linear space of control sequences by

$$U_{[0,N]} \;=\; \{u \in R^{m(N+1)} : \; u := [u(0), u(1), \ldots, u(N)], \; \text{with} \quad u(t) \in R^m,\ 0 \leq t \leq N\}$$

The following propositions will be employed to prove our results.

Proposition 1.1. *(Callier and Desoer [3]). Let $(A(t)), (B(t)), t \in N_0$ be given compatible matrix-sequences. Then the following are equivalent:*

(i) The linear system (1.2) is controllable on $[0,N]$.
(ii) $det W_r(0,N) \neq 0$, where the reachability grammian W_r is as defined as in (1.6).

Proposition 1.2. *If the system (1.2) is controllable on* $[0,N]$, *then for all $x_0, x_1 \in R^n$, there exists $u \in U_{[0,N]}$ defined by*

$$u(t) \;:=\; B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \times [x_1 - \Phi(N,0)x_0]$$

that steers the initial state x_0 to the desired final state x_1 in N time-steps.

2. Main Results

Theorem 2.1. *If the the linear system is controllable in N time-steps and the control $u(t)$ defined by (1.5) is well-defined, then it steers the nonlinear system (1.1) from the initial state x_0 to the desired final state x_1 in N time-steps.*

Proof: Since the linear system (1.2) is controllable on $[0, N]$, we have by Proposition 1.1 that $det W_r(0,N) \neq 0$. If we substitute the control given by (1.5) in the solution

$$x(t) \;=\; \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)u(j) + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j))$$

we get,

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1}\Phi(t,j+1)B(j)B(j)^* \times\Phi(N,j+1)^* W_r(0,N)^{-1} \left\{x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right\} + \sum_{j=0}^{t-1}\Phi(t,j+1)f(j,x(j)) \quad (2.1)$$

It can be easily verified that at $t=0$, $x(0)=x_0$ and at $t=N, x(N)=x_1$. Thus, the control u defined in (1.5) steers the non-linear system from the given initial state x_0 to the desired final state x_1.

We now prove that the control defined in (1.5) is meaningful. This control u is well-defined if there is a solution to the equation (2.1) with this control. We will prove existence and uniqueness of solution of (2.1).

We make use of the following notations and definitions:

Let $C = \max_{N\geq t\geq j\geq 0} \parallel \Phi(t,j) \parallel$, $M_1 = \max_{N\geq j\geq 0} \parallel B(j) \parallel$ and

$M_2 = \parallel W_r(0,N)^{-1} \parallel$.

$\beta = C(1+C^2M_1^2M_2(N-1))$

$\eta = \alpha\beta(N-1)$.

Theorem 2.2. *Under Assumptions* [*L*],[*N*] *and* $\eta < 1$ *the steering control defined by*

$$u(t) = B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \times\left[x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right]$$

is well-defined.

Proof: We prove this by showing that the nonlinear system with this control has a unique solution. In Theorem 2.1 we have shown that this control does the required steering. We show that the following nonlinear equation has a unique solution.

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1}\Phi(t,j+1)B(j)B(t)^* \times\Phi(N,t+1)^*W_r(0,N)^{-1} \left\{x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right\} + \sum_{j=0}^{t-1}\Phi(t,j+1)f(j,x(j))$$

To prove the existence of the solution, we define a mapping
$T: S_N^\infty(R^n) \to S_N^\infty(R^n)$ by

$$T(x(t)) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1}\Phi(t,j+1) \times B(j)B(j)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \left\{x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right\} + \sum_{j=0}^{t-1}\Phi(t,j+1)f(j,x(j)),$$
$$t = 0,1,\ldots,N.$$

Since $S_N^\infty(R^n)$ is a complete Banach space, we show that operator T has a fixed point by using Banach Contraction mapping theorem.

Consider

$$\parallel T(x(t)) - T(\tilde{x}(t)) \parallel \leq \sum_{j=0}^{t-1} \parallel \Phi(t,j+1) \times\{f(j,x(j)) - f(j,\tilde{x}(j))\} \parallel + \sum_{j=0}^{t-1}\parallel \Phi(t,j+1)B(j)B(j)^* \Phi(N,j+1)^*W_r(0,N)^{-1} \sum_{i=0}^{N-1}\Phi(N,i+1)\{f(i,x(i)) - f(i,\tilde{x}(i))\} \parallel$$
$$\leq C\sum_{j=0}^{t-1}\parallel f(x(j)) - f(\tilde{x}(j)) \parallel + C^2M_1^2M_2 \sum_{j=0}^{t-1}C\sum_{i=0}^{N-1}\parallel f(i,\tilde{x}(i)) - f(i,x(i)) \parallel$$

$$\leq \alpha C \sum_{j=0}^{t-1} \| x(j) - \tilde{x}(j) \|$$
$$+\alpha C^3 M_1^2 M_2 (t-1)$$
$$\sum_{i=0}^{N-1} \| \tilde{x}(i) - x(i) \|$$
$$\leq \alpha C(1 + C^2 M_1^2 M_2 (t-1))$$
$$\sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \|$$
$$\leq \alpha\beta \sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \|,$$

Thus, $$\sup_{0\leq t\leq N} \| T(x(t)) - T(\tilde{x}(t)) \|$$
$$\leq \alpha\beta(N-1) \sup_{0\leq t\leq N} \| x(t) - \tilde{x}(t) \|$$
$$\| T(x) - T(\tilde{x}) \| \leq \eta \| x - \tilde{x} \| .$$

Since $\eta < 1$, T is a contraction. Hence T has a unique fixed point. Therefore, the nonlinear equation is uniquely solvable. This proves that the control defined in (1.5) is well-defined.

We now give the following computational result for the steering control for the nonlinear system.

Theorem 2.3. *Under the assumptions of Theorem 2.2, the steering control and controlled trajectory of the nonlinear system (1.1) driving the system from $x(0) = x_0$ to $x(N) = x_1$ can be computed by the following iterative scheme:*

$$u^m(t) = B(t)^*\Phi(N, t+1)^* W_r(0, N)^{-1} \times \Big[x_1 - \Phi(N, 0)x_0 - \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x^m(j)) \Big] \quad (2.2)$$

and

$$x^{m+1}(t) = \Phi(t, 0)x_0 \quad (2.3)$$
$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u^m(j)$$
$$+ \sum_{j=0}^{t-1} \Phi(t, j+1) f(j, x^m(j)) \quad (2.4)$$

starting with arbitrary $x^0(t)$, $t = 0, 1, 2, \ldots, N-1$, $m = 0, 1, 2, \ldots$

Proof: The computational scheme follows directly from Banach contraction principle and from Theorem 2.2.

Although for nonlinear systems controllability and reachability notions are not equivalent, we prove in the following theorem that for the semilinear sytem (1.1), the two notions are equivalent.

Theorem 2.4. *The two notions of controllability and reachability are equivalent for the semilinear system (1.1).*

Proof: From definition, it is obvious that for the system (1.1), controllability implies reachability. Conversely, let the system (1.1) is reachable on $[0, N]$. Thus, the 0 state can be steered to any desired state $\tilde{x}_1$.
Now, for arbitrary $x_0, x_1 \in R^n$, choose

$$\tilde{x}_1 = x_1 - \Phi(N, 0)x_0.$$

Since there exists $u(t) \in U_{[0,N]}$ that steers $x_0 = 0$ to $\tilde{x}_1$ for some N. Hence

$$\tilde{x}_1 = \Phi(N, 0)0 + \sum_{j=0}^{N-1} \Phi(N, j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x(j)) \quad (2.5)$$

i.e.

$$\tilde{x}_1 = \sum_{j=0}^{N-1} \Phi(N, j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x(j)) \quad (2.6)$$

i.e.

$$x_1 = \Phi(N, 0)x_0 + \sum_{j=0}^{N-1} \Phi(N, j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x(j)) \quad (2.7)$$

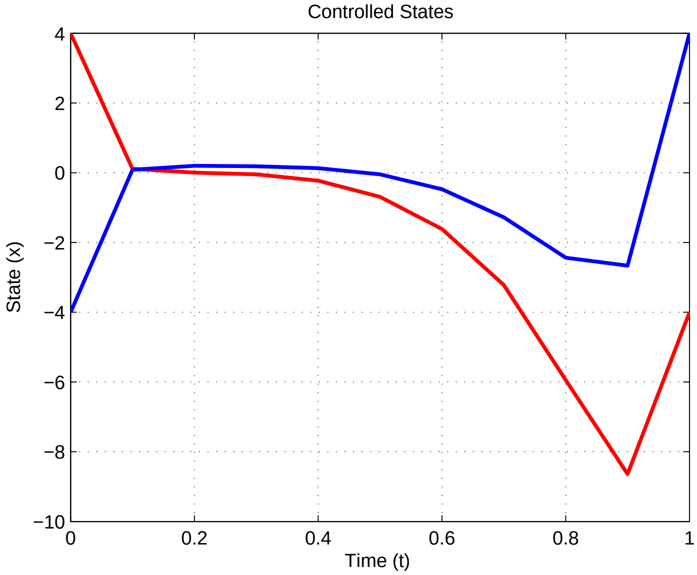
which shows that the same control steers x_0 to x_1. Hence the system (1.1) is controllable.

3. Numerical Example

Example 1. Consider the nonlinear system given by the following equation:

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)) \quad (3.1)$$

Figure 1: Controlled trajectories.



where $A(t) = \frac{1}{4}\begin{pmatrix} \cos(2t) & 1 \\ t^2 & \cos^2(t) \end{pmatrix}$,

$B(t) = \begin{pmatrix} .5 \\ .5t \end{pmatrix}$ and

$f(t,x) = \frac{1}{5}\begin{pmatrix} \sin^2(x_1(t)) \\ \cos^2(x_2(t)) \end{pmatrix}$

Let $N = 10$. Here the reachability Grammian can be computed as

$W_r(0,10) = \begin{pmatrix} 1.0774 & 0.2760 \\ 0.2760 & 0.2078 \end{pmatrix}$

and $det W_r(0,N) = 0.1477 \neq 0$.

Hence the linear system is controllable, and

$$\| f(t,x) - f(t,y)) \| = \frac{1}{5} \| \begin{pmatrix} \sin^2(x_1) - \sin^2(y_1) \\ \cos^2(x_2) - \cos^2(y_2) \end{pmatrix} \| \leq \frac{2}{5} \| x - y \|$$

Hence f is Lipschitz with Lipschitz constant $\frac{2}{5}$. We can easily verify the conditions of Theorem 2.2 to conclude that the system is controllable. Figure 1 shows the controlled trajectory steering the system from the initial state $x_0 = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ to the final state $x_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$.

Received 19 November 2007; revised 28 November 2007.

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